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REFINED GEOMETRICALLY NONLINEAR THEORIES OF ANISOTROPIC LAMINATED SHELLS*

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1. Introduction. The foundation of consistent theories of laminated anisotropic shells has attracted increasing attention during recent years. This interest was stimulated by the advent of new materials such as pyrolitic graphite, fiber-reinforced composites (glass/epoxy, boron/epoxy, graphite/epoxy), etc., as well as by the interesting properties characterizing these structures which are more and more used in various fields of modern technology. The available monographs [1-6], as well as the bibliographical papers [7-10] reviewing in depth the literature in the field, illustrate in a best way the great interest afforded to the analysis of laminated anisotropic shells. As it was fully outlined (see e.g. [1, 4, 5]) the classical theory of multilayered shells (based on the Love-Kirchhoff (L.K.) assumptions), in spite of its successes, is no longer applicable in many important cases. They occur whenever the multilayered shell (or plate) is composed of anisotropic materials characterized by high degrees of anisotropy, even if the classical thinness requirement is fulfilled (or in other words, even if the composite shell is "geometrically thin").

Such a property is typical for fiber-reinforced composite and pyrolitic graphite material systems (see e.g. [4]). In their case, the ratios of in-plane Young's moduli to transverse shear moduli vary between 20 and 50 while the coefficient of thermal expansion in the thickness direction is many times greater than the one in the isotropy-plane.

It has been shown conclusively that in these cases, more refined theories are needed to describe in an accurate manner the static and dynamic behavior of geometrically thin/thick/anisotropic multilayered (and single layered) shells. Such refined theories should include transverse shear deformation and transverse normal strain effects and should account for the high-order effects.

There are a number of methods used to model the refined theory of multilayered shells (and plates).[1]

In one of them appropriate assumptions for each layer, separately, are to be stipulated[2] (see e.g. [12-14]).

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[1] For an excellent account of the available methods in the field see [11] where, in addition, a theory of multilayered plates based on the Cosserat continuum concept is substantiated.

[2] This possibility is also appropriate in the modeling of the theory of sandwich/multisandwich type shells (see e.g. [5, Chap. VI, 15, 16]).

In this case, however, we are confronted with a cumbersome system of governing equations involving an increased number of unknown functions (dependent on the number of the constituent layers). Other methods utilize one expansion for the displacement field throughout the entire laminated thickness. It was largely used in the foundation of anisotropic multilayered plate [17–20] and shell [21, 22] theories and will be employed in the present work, too.

In the following development, a Lagrangian formulation of a refined geometrically nonlinear theory of anisotropic laminated shells of arbitrary shape will be given. Toward this end, use is made of a modified variational principle of the 3-D nonlinear elasticity theory together with a high-order representation of the displacement field throughout the laminate thickness.

The resulting field equations are expressed in terms of the high-order stress-couples and strain measures (including those of the transverse shear and transverse normal strains).

The theory is general in its character in the sense that it incorporates the effects of the material anisotropy, structural lamination, high-order dynamical effects as well as the presence of a steady temperature field. In addition, the theory is free of any further assumptions beyond those that are initially stipulated (namely the statement of the elasticity of the material of the layers as well as the one which concerns the representation of the displacement field across the shell thickness.)

Attention is also given to the problems of the continuity conditions at the surfaces between the contiguous layers, as well as to some simplified variants of the general theory developed in the first part of the work. Owing to the inherent complexities which appear when the L–K assumptions are discarded, the literature in the field has dealt with the modeling of the theories of laminated flat plates (mainly) and shallow shells; references to the pertinent contributions are appropriately traced in the paper. However, substantiation of a general theory of laminated composite shells, free of the traditional assumptions and incorporating a number of important effects (such as those mentioned previously), should be of a great practical and heuristic importance. As far as the author of the present paper is aware, no attempt to substantiate such a theory has been given in the specialized literature. It is in fact the basic object of the following developments.

2. The geometry of the laminated shell. Assumptions. Let us consider the shell composed of a finite number N of individually homogeneous layers (sometimes called laminae). Let $2^{(k)}h$ denote the uniform thickness of the kth layer ($k = \overline{1, N}$). Let us assume in addition that:

(i) Each constituent layer has its own geometrical and physico-mechanical characteristics.

(ii) The material of each constituent layer is linearly elastic and anisotropic.

(iii) The layers are in perfect bond; no slip between two adjacent laminae may occur. The points of the 3-D space of the shell in its undeformed state will be referred to the set of curvilinear normal coordinates x^i; $x^3 = 0$ defines the undeformed reference surface ${}_0\sigma$ (which is chosen to coincide with the mid-surface of the bottom layer), while x^α denotes curvilinear coordinates on ${}_0\sigma$.

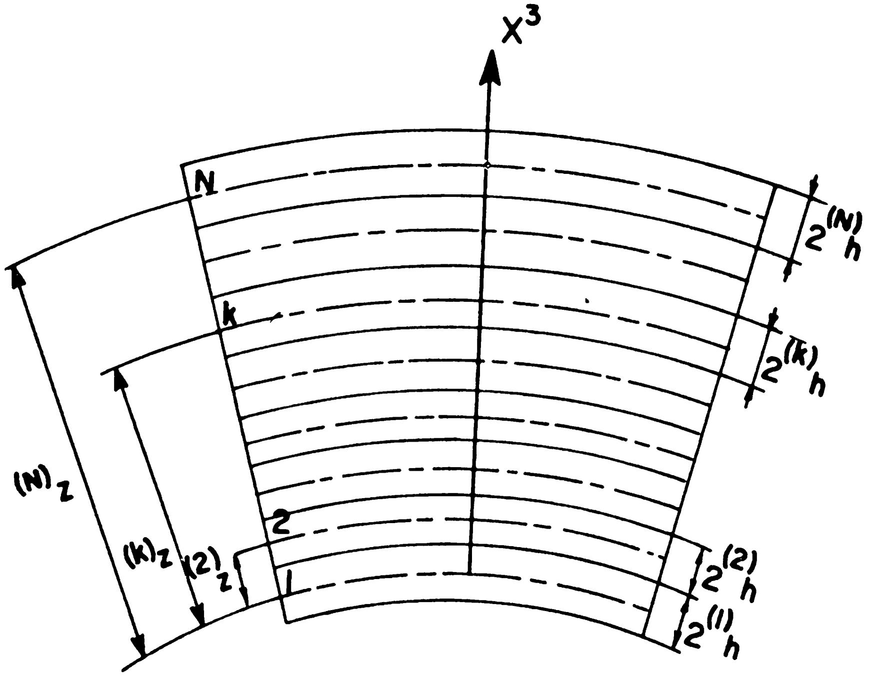

FIG. 1.

The distance (measured along x^3) between the reference surface ${}_0\sigma$ and the mid-surface of a generic kth layer (see Fig. 1) constitutes another important parameter of the composite shell. It will be denoted by ${}^{(k)}z(k = \overline{1, N})$, where ${}^{(1)}z \equiv 0$.

The corresponding spatial metric tensor of the undeformed shell-space reads:

$$g_{\alpha\beta} = \mu^{\lambda}_{\alpha}\mu^{\omega}_{\beta}a_{\lambda\omega}; \qquad g^{\alpha\beta} = (\mu^{-1})^{\alpha}_{\lambda}(\mu^{-1})^{\beta}_{\omega}a^{\lambda\omega};$$

$$g_{\alpha 3} = g^{\alpha 3} = 0; \qquad g^{33} = g_{33} = 1. \tag{1}$$

Here $a_{\lambda\alpha}$ denotes the metric tensor of the undeformed reference surface, while μ^{α}_{β} defined by

$$\mu^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - x^3 b^{\alpha}_{\beta} \tag{2}$$

is referred to as the "shell tensor", where δ^{α}_{β} is Kronecker's symbol; b^{α}_{β} is the mixed curvature tensor of the undeformed reference surface.

As shown in [23] μ^{α}_{β} is nonsingular. Its unique inverse $(\mu^{-1})^{\alpha}_{\beta}$ satisfying $(\mu^{-1})^{\lambda}_{\beta}\mu^{\alpha}_{\lambda} = \delta^{\alpha}_{\beta}$ is expressible in convergent series of x^3 as:

$$(\mu^{-1})^{\alpha}_{\beta} = \sum_{n=0}^{\infty} (b^n)^{\alpha}_{\beta}(x^3)^n \tag{3}$$

where

$$(b^n)^{\alpha}_{\beta} = b^{\lambda}_{\beta}(b^{n-1})^{\alpha}_{\lambda} = b^{\alpha}_{\lambda}(b^{n-1})^{\lambda}_{\beta} \tag{4}$$

and in addition $(b^0)^{\alpha}_{\beta} \equiv \delta^{\alpha}_{\beta}$ and $(b^n)^{\alpha}_{\beta} \equiv 0$ for $n < 0$. By virtue of (1) and (3), $g^{\alpha\beta}$ may be expressed as follows:

$$g^{\alpha\beta} = a^{\alpha\lambda} \sum_{p=0}^{\infty} (1+p)(b^p)^\beta_\lambda (x^3)^p. \tag{5}$$

Concerning the relationships between covariant derivatives of space and surface tensors (see [23] and also [5]), the ensuing ones turn out to be useful in the following developments;

$$T_{\alpha||\beta} = \mu^\gamma_\alpha\left(\bar{T}_{\gamma|\beta} - b_{\gamma\beta}\bar{T}_3\right);\ T_{\alpha||3} = \mu^\gamma_\alpha \bar{T}_{\gamma,3}; \tag{6}$$

$$T_{3||\alpha} = \bar{T}_{3,\alpha} + b^\sigma_\alpha \bar{T}_\sigma;\ T_{3||3} = \bar{T}_{3,3};$$

$$T^{\alpha\beta}||_\gamma = (\mu^{-1})^\beta_\delta (\mu^{-1})^\alpha_\lambda \left(\bar{T}^{\lambda\delta}|_\gamma - b^\delta_\gamma \bar{T}^{\lambda 3} - b^\lambda_\gamma \bar{T}^{3\delta}\right). \tag{7}$$

The shifted components are denoted by an upper bar; the double and single strokes are used to identify the covariant differentiation with respect to the space and surface undeformed metrics, respectively, while a comma denotes partial differentiation. Throughout the paper the Einsteinian summation convention is adopted for tensor quantities; Greek indices run over the range 1, 2 while the Latin ones run over the range 1, 2, 3. *Superscript* (k) in brackets attached on the right (or left) of any quantity identifies affiliation to the kth layer.

3. Preliminaries concerning the derivation of the field equations. A modified version of the Hellinger-Reissner variational principle (see [24, 25]) (referred to as the Hu-Washizu variational theorem) will be used in order to derive the basic field equation of the geometrically nonlinear theory of laminated shells. In its terms, the stationary condition applied to the functional

$$\begin{aligned} J &\equiv J\left(s^{ij}, e_{ij}, V_i\right) \\ &= \int_{0^\tau} {}_0\rho\left({}_0\vec{H} - {}_0\vec{h}\right)\vec{V}\,d\tau + \int_{0^\tau}\left(s^{ij}e_{ij} - W\right)d\tau \\ &\quad - \int_{0^\tau}\frac{1}{2}\left(V_{i||j} + V_{j||i} + V^r||_i V_{r||j}\right)s^{ij}\,d\tau \\ &\quad + \int_{0^{\Omega_s}} \mathbf{s}^i V_i\,d\Omega + \int_{0^{\Omega_V}} s^i\left(V_i - \mathbf{V}_i\right)d\Omega \end{aligned} \tag{8}$$

where the stress tensors s^{ij}, the strain tensor e_{ij} and the displacements V_i are allowed to vary independently throughout the volume and on the boundary of the body, yields the basic field equations and the boundary conditions of the nonlinear elasticity in terms of a reference state as presented, e.g., in [26].

In Eq. (8), s^{ij} stands for the second Piola-Kirchhoff stress tensor correlated with the Cauchy stress tensor σ^{ij} as $s^{ij} = (G/g)^{1/2}\sigma^{ij}$, where G and g denote the determinants of the metric tensor of the deformed and undeformed body, respectively. In the nonpolar case (as considered in the present paper) s^{ij} is a symmetric tensor, thus fulfilling the equation

$$\varepsilon_{ijk}s^{ij} = 0. \tag{9}$$

In addition, e_{ij} stands for the Lagrangian strain tensor; ${}_0h^i$ and ${}_0H^i$ are components of the acceleration and body force vectors per unit volume of the undeformed body $({}_0\vec{h} = {}_0h^i\vec{g}_i;\ {}_0\vec{H} = {}_0H^i\vec{g}_i)$; V_i are the components of the displacement vector $\vec{V}(\equiv V_i\vec{g}^i)$; ${}_0\Omega_s$, ${}_0\Omega_V$ denote the two parts of the total undeformed boundary surface ${}_0\Omega$ where the stress and displacement vectors, respectively, are prescribed $({}_0\Omega = {}_0\Omega_s \cup {}_0\Omega_V)$; ${}_0\tau$ denotes the volume of the undeformed body; s^i stands for the stress vector referred to base vectors in the undeformed body while W denotes the strain energy function measured per unit volume of the undeformed body; the bold letters denote a prescribed quantity; δ denotes the sign of variation.

As it may be remarked, the functional (8) involves integrals over the undeformed configuration of the body. The importance of this feature in the general context of the nonlinear solid mechanics was underlined in [27]. Operating in (8) the variation on the nonprescribed quantities, assuming $\delta_0\vec{H} = \delta_0\vec{h} = 0$, and using Green's theorem correlating surface and volume integrals, all these yield the variational equation expressed under a convenient form as:

$$\delta J = \sum_{j=1}^{5} \delta J_i = 0, \tag{10}$$

where

$$\begin{aligned}
\delta J_1 &= \int_{{}_0\tau} \left\{ \left[s^{jr}\left(\delta_r^i + V^i||_r\right)||_j + {}_0\rho\left({}_0H^i - {}_0h^i\right)\right] \right\} \delta V_i \, d\tau, \\
\delta J_2 &= \int_{{}_0\tau} \left[e_{ij} - \frac{1}{2}\left(V_i||_j + V_j||_i + V^r||_i V_r||_j\right)\right] \delta s^{ij} \, d\tau, \\
\delta J_3 &\equiv \int_{{}_0\tau} \left(s^{ij} - \frac{1}{2}\left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}}\right)\right) \delta e_{ij} \, d\tau, \\
\delta J_4 &\equiv \int_{{}_0\Omega_s} \left[\mathbf{s}^i - {}_0n_j s^{jr}\left(\delta_r^i + V^i||_r\right)\right] \delta V_i \, d\Omega, \\
J_5 &\equiv \int_{{}_0\Omega_V} \left(V_i - \mathbf{V}_i\right) \delta s^i \, d\Omega,
\end{aligned} \tag{11}$$

n_i denoting the components of the outward unit vector normal to the external bounding surface of the undeformed body. By invoking the arbitrary character of the variations δV_i, δe_{ij}, δs^{ij} (throughout ${}_0\tau$ and on ${}_0\Omega_V$ and ${}_0\Omega_s$), the coefficients in the five integrands appearing in δJ_i $(i = \overline{1,5})$ must vanish independently, thus yielding the basic field equations of the nonlinear elasticity theory in terms of a reference state.

The full nonlinear form of (8) and its linearized counterpart have been used in the substantiation of refined shell theories in [5, 25] and [23], respectively. In the next developments, the geometrically nonlinear theory of anisotropic laminated shells will be substantiated by using the variational principle (8).

4. Displacement field in the shell. Let us consider the displacement vector $\vec{V}(x^\omega, x^3)$ of the 3-D points of the laminated shell expressed in terms of the spatial and their shifted components as

$$\vec{V} = \overline{V}_\alpha \vec{a}^\alpha + \overline{V}_3 \vec{a}^3 = V^\alpha \vec{g}_\alpha + V^3 \vec{g}_3, \tag{12}$$

$\vec{g}_\alpha$ and $\vec{a}_\alpha$ denoting the space and surface base vectors in the undeformed body (related by $\vec{g}_\alpha = \mu_\alpha^\lambda \vec{a}_\lambda$), where $\vec{g}_3$ ($\equiv \vec{a}_3$) stands for the unit normal vector to the undeformed reference surface ${}_0\sigma$. It is easily seen from (12) that:

$$V_\alpha = \mu_\alpha^\beta \overline{V}_\beta; \qquad V_3 = \overline{V}_3 \tag{13}$$

where $\overline{V}_\alpha \equiv \overline{V}_\alpha(x^\omega, x^3, t)$; $\overline{V}_3 \equiv \overline{V}_3(x^\omega, x^3, t)$ denote the shifted displacements (with respect to the reference surface). They will be represented across the entire laminate thickness as

$$\begin{aligned} \overline{V}_\alpha(x^\omega, x^3; t) &= \sum_{r=0}^{R} (x^3)^r \overset{(r)}{V}_\alpha(x^\omega, t), \\ \overline{V}_3(x^\omega, x^3; t) &= \sum_{s=0}^{S} (x^3)^s \overset{(s)}{V}_3(x^\omega, t), \end{aligned} \tag{14}$$

where R and S ($R \gtrless S$) are two natural numbers defining the level of truncation in the series expansion, while r and s are summation indices. For the sake of simplicity we shall assume $R = S \equiv \mathscr{R}$. As it may readily be inferred, the results could easily be modified when such an equality is not invoked *a priori*.

In the general case $R \gtrless S$, the unknown functions of the problem ($2R + S + 3$ in number) are: $\{V_\alpha^{(r)}, V_3^{(s)};\ r = \overline{0, R};\ s = \overline{0, S}\}$. In the case $R = S \equiv \mathscr{R}$, the unknown functions reduce to $\{V_i^{(r)},\ r = \overline{0, \mathscr{R}}\}$, being $3\mathscr{R} + 3$ in number.

A large diversity of high-order theories of multilayered plates [17–20] and shells [21–22], relies upon the various selection of the degree of approximation of (14).

5. Modified expression of δJ. The variational equation (10) of the 3-D elasticity theory will be modified to include the peculiarities of the laminated shell.

5.1 *Modified form of* δJ_1. The volume element $d\tau$ will be expressed in terms of the area element $d\sigma$ of the undeformed reference surface as $d\tau = \mu\, d\sigma dx^3$, where $\mu = (g/a)^{1/2}$, $d\sigma = a^{1/2} dx^1 dx^2$; $a = \det(a_{\alpha\beta})$; $g = \det(g_{ij})$. At this point it is more convenient to use in $(11)_1$ the Lagrangian stress tensor t^{ij} connected with the second Piola-Kirchhoff stress tensor as

$$t^{ij} = s^{ir}\left(\delta_r^j + V^j||_r\right). \tag{15}$$

Employment further of the relationships between space and surface derivative of tensors (see [23, 5]) all yield the following modified expression of δJ_1;

$$\begin{aligned} \delta J_1 = \int_{\sigma_0} \sum_{k=1}^{N} \sum_{n=0}^{\mathscr{R}} \int_{{}^{(k)}z - {}^{(k)}h}^{{}^{(k)}z + {}^{(k)}h} \Big\{ &[(\mu t^{\alpha\omega} \mu_\omega^\rho)|_\alpha - b_\alpha^\rho(\mu t^{\alpha 3}) + (\mu \mu_\psi^\rho t^{3\psi})_{,3} \\ &+ {}_0\rho \mu \mu_\alpha^\rho ({}_0H^\alpha - {}_0h^\alpha)] \delta \overset{(n)}{V}_\rho (x^3)^n\, dx^3 \\ + [(\mu t^{\alpha 3})|_\alpha + (\mu t^{33})_{,3} &+ b_{\rho\alpha}(\mu \mu_\psi^\rho t^{\alpha\psi}) + {}_0\rho\mu({}_0H^3 - {}_0h^3)] \delta \overset{(n)}{V}_3 (x^3)^n\, dx^3 \Big\}\, d\sigma \end{aligned} \tag{16}$$

On the other hand, employment of (6) into (15) yields the more explicit expression for the various components of t^{ij} as follows

$$\begin{aligned} t^{\alpha\omega} &= s^{\alpha\delta}\left[\delta_{\delta}^{\omega} + (\mu^{-1})_{\lambda}^{\omega}\bar{\phi}^{\lambda}_{\ .\delta}\right] + s^{\alpha 3}(\mu^{-1})_{\delta}^{\omega}\bar{V}_{,3}^{\delta};\\ t^{3\psi} &= s^{3\beta}\left[\delta_{\beta}^{\psi} + (\mu^{-1})_{\delta}^{\psi}\bar{\phi}^{\delta}_{\ .\beta}\right] + s^{33}(\mu^{-1})_{\beta}^{\psi}\bar{V}_{,3}^{\beta};\\ t^{\alpha 3} &= s^{\alpha\beta}\bar{\phi}_{\beta}^{3} + s^{\alpha 3}(1 + \bar{V}_{3,3});\\ t^{33} &= s^{3\alpha}\bar{\phi}_{\alpha}^{3} + s^{33}(1 + \bar{V}_{3,3}) \end{aligned} \tag{17}$$

where

$$\bar{\phi}^{\lambda}_{\cdot\beta} = \bar{V}^{\lambda}|_{\beta} - b_{\beta}^{\lambda}\bar{V}_{3}; \qquad \bar{\phi}_{\omega}^{3} = \bar{V}_{3,\omega} + b_{\omega}^{\lambda}\bar{V}_{\lambda}. \tag{18}$$

The complete reduction of δJ_1 to 2-D quantities will be accomplished further by replacing (17) and (18) into (16), by considering therein the representation (14) for the shifted displacements as well as the following definitions:

a) The nth order stress couples afferent to the kth layer:

$$\begin{aligned} {}^{(k)}L_{\langle n\rangle}^{\omega\pi} &\equiv \int_{^{(k)}z-{}^{(k)}h}^{^{(k)}z+{}^{(k)}h} (x^3)^n \mu\mu_{\sigma}^{\omega}s^{\sigma\pi}\,dx^3,\\ {}^{(k)}\hat{L}_{\langle n\rangle}^{\omega\pi} = {}^{(k)}\hat{L}_{\langle n\rangle}^{\pi\omega} &\equiv \int_{^{(k)}z-{}^{(k)}h}^{^{(k)}z+{}^{(k)}h} (x^3)^n \mu s^{\omega\pi}\,dx^3,\\ {}^{(k)}L_{\langle n\rangle}^{33} &= \int_{^{(k)}z-{}^{(k)}h}^{^{(k)}z+{}^{(k)}h} (x^3)^n \mu s^{33}\,dx^3\\ {}^{(k)}N_{\langle n\rangle}^{\omega 3} &= \int_{^{(k)}z-{}^{(k)}h}^{^{(k)}z+{}^{(k)}h} (x^3)^n \mu s^{\omega 3}\,dx^3; \end{aligned} \tag{19}$$

b) The nth order body couples of the kth layer:

$$\begin{aligned} {}^{(k)}\mathcal{F}_{\langle n\rangle}^{\omega} &= \int_{^{(k)}z-{}^{(k)}h}^{^{(k)}z+{}^{(k)}h} (x^3)_0^{n}\,{}^{(k)}H^{\alpha}\mu_{\alpha}^{\omega(k)}{}_0\rho\mu\,dx^3\\ {}^{(k)}\mathcal{F}_{\langle n\rangle}^{3} &= \int_{^{(k)}z-{}^{(k)}h}^{^{(k)}z+{}^{(k)}h} (x^3)_0^{n}\,{}^{(k)}H^{3(k)}{}_0\rho\mu\,dx^3. \end{aligned} \tag{20}$$

c) The nth order inertial couples:

$$\begin{aligned} {}^{(k)}f_{\langle n\rangle}^{\alpha} &= \sum_{q=0}^{\mathcal{R}} {}^{(k)}m_{q+n}\overset{(q)}{\ddot{V}}{}^{\alpha},\\ {}^{(k)}f_{\langle n\rangle}^{3} &= \sum_{q=0}^{\mathcal{R}} {}^{(k)}m_{q+n}\overset{(q)}{\ddot{V}}{}^{3} \end{aligned} \tag{21}$$

where

$$\begin{gathered} {}^{(k)}m_{q+n} \equiv {}_0\rho^{(k)}\left[{}^{(k)}\eta(n+q+1) - 2H^{(k)}\eta(n+q+2) + K^{(k)}\eta(n+q+3)\right]\\ \left(k = \overline{1, N};\ n, q = \overline{0, \mathcal{R}}\right) \end{gathered} \tag{22}$$

denotes the mass term afferent to the kth layer, while ${}^{(k)}\eta(r)$ is defined by

$${}^{(k)}\eta(r) = \frac{1}{r}\left[(z^{(k)} + h^{(k)})^r - (z^{(k)} - h^{(k)})^r\right]. \tag{23}$$

In (22) H and K denote the mean and the Gaussian curvatures of the undeformed reference surface; an overdot indicates differentiation with respect to time t. Furthermore, by defining the gross nth order stress, body and inertia couples as

$$\begin{matrix} L^{\omega\pi}_{\langle n\rangle} \\ L^{33}_{\langle n\rangle} \\ N^{\omega 3}_{\langle n\rangle} \end{matrix} = \sum_{k=1}^{N} \begin{cases} {}^{(k)}L^{\omega\pi}_{\langle n\rangle} \\ {}^{(k)}L^{33}_{\langle n\rangle} \\ {}^{(k)}N^{\omega 3}_{\langle n\rangle} \end{cases} \tag{24}$$

$$\begin{matrix} \mathscr{F}^{\,i}_{\langle n\rangle} \\ f^{\,i}_{\langle n\rangle} \end{matrix} = \sum_{k=1}^{N} \begin{cases} {}^{(k)}\mathscr{F}^{\,i}_{\langle n\rangle} \\ {}^{(k)}f^{\,i}_{\langle n\rangle}, \end{cases} \tag{25}$$

δJ_1 turns out to be expressed in the form:

$$\delta J_1 = \int_{0\sigma} \sum_{n=0}^{\ell} \left\{ S^{\rho}_{\langle n\rangle} \delta \overset{(n)}{V}_{\rho} + S^{3}_{\langle n\rangle} \delta \overset{(n)}{V}_{3} \right\} d\sigma \tag{26}$$

where $S^{\rho}{}_{\langle n\rangle}$ and $S^{3}{}_{\langle n\rangle}$ will be rendered explicit later.

5.2 *Modified form of* δJ_2. Employment in $(11)_2$ of the relationships (6); of the definitions (19), (24) and of the representation for e_{ij} given by

$$e_{ij} = \sum_{n} (x^3)^n \overset{(n)}{e}_{ij} \tag{27}$$

yields the following expression of δJ_2:

$$\delta J_2 = \int_{0\sigma} \left[\sum_{n=0}^{2\ell} \left(\overset{(n)}{e}_{\alpha\beta} - \overset{(n)}{A}_{\alpha\beta} \right) \delta \hat{L}^{\alpha\beta}_{\langle n\rangle} + \sum_{n=0}^{2\ell-2} \left[\overset{(n)}{e}_{33} - \overset{(n)}{B}_{33} \right] \delta L^{33}_{\langle n\rangle} + \sum_{n=0}^{2\ell-1} \left[\overset{(n)}{e}_{\alpha 3} - \overset{(n)}{C}_{\alpha 3} \right] \delta N^{\alpha 3}_{\langle n\rangle} \right] d\sigma. \tag{28}$$

$\overset{(n)}{A}_{\alpha\beta}$, $\overset{(n)}{B}_{33}$, $\overset{(n)}{C}_{\alpha 3}$ will be given explicitly later.

5.3 *Modified form of* δJ_3. Employment in $(11)_3$ of (27), (19) and of (24) yields the following modified expression for δJ_3:

$$\delta J_3 = \int_{_0\sigma} \left\{ \sum_{n=0}^{2\mathcal{R}} \left[\hat{L}^{\alpha\beta}_{\langle n\rangle} - \frac{1}{2} \sum_{k=1}^{N} \left(\frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{e}_{\alpha\beta}} + \frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{e}_{\beta\alpha}} \right) \right] \delta \overset{(n)}{e}_{\alpha\beta} \right.$$

$$+ \sum_{n=0}^{2\mathcal{R}-2} \left[L^{33}_{\langle n\rangle} - \sum_{k=1}^{N} \frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{e}_{33}} \right] \delta \overset{(n)}{e}_{33} \tag{29}$$

$$\left. + \sum_{n=0}^{2\mathcal{R}-1} \left[N^{\alpha 3}_{\langle n\rangle} - \frac{1}{2} \sum_{k=1}^{N} \left(\frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{e}_{\alpha 3}} + \frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{e}_{3\alpha}} \right) \right] \delta \overset{(n)}{e}_{\alpha 3} \right\} d\sigma$$

where $\overline{W}^{(k)}$ expressed as

$$\overline{W}^{(k)} = \int_{^{(k)}z - {}^{(k)}h}^{^{(k)}z + {}^{(k)}h} \mu W \, dx^3 \tag{30}$$

stands for the strain energy function of the kth layer per unit area of the undeformed reference surface.

5.4 *Modified form of* δJ_4. In order to perform the necessary transformations of δJ_4 as given by $(11)_4$ we shall consider the undeformed surface ${}_0\Omega_s$ consisting of the edge boundary surface ${}_0\hat{\Omega}_s$ and of the upper and lower boundary surfaces $S^{\pm}$ of the shell body. Making use of the relationships:

$$ {}_0n_\alpha \, d\Omega = {}_0\nu_\alpha \mu \, ds \, dx^3, \qquad dS^{\pm} = \mu^{\pm} \, d\sigma, \tag{31}$$

where ${}_0\nu_\alpha$ denotes the components of the outward unit vector normal to the undeformed edge surface at $x^3 = 0$, ds denotes the line element along Γ_s as resulting from the intersection of ${}_0\hat{\Omega}$ with the shell-reference surface: $\mu^+ = \mu|_{x^3 = {}^{(N)}z - {}^{(N)}h}$; $\mu^- = \mu|_{x^3 = {}^{(1)}z - {}^{(1)}h}$, and employment of (6) and (14), all these yield the following expression of δJ_4:

$$\delta J_4 = \delta \hat{J}_4 + \delta \hat{\hat{J}}_4, \tag{32$_1$}$$

where

$$\delta \hat{J}_4 = \int_{_0\Gamma_s} \left\{ \sum_{n=0}^{\mathcal{R}} \left[\left({}_0\nu_\alpha \mathbf{L}^{\alpha\rho}_{\langle n\rangle} - {}_0\nu_\alpha \mathcal{L}^{\alpha\rho}_{\langle n\rangle} \right) \delta \overset{(n)}{V}_\rho \right. \right.$$

$$\left. \left. + \left({}_0\nu_\alpha \mathbf{N}^{\alpha 3}_{\langle n\rangle} - {}_0\nu_\alpha \mathcal{N}^{\alpha 3}_{\langle n\rangle} \right) \delta \overset{(n)}{V}_3 \right] \right\} ds; \tag{32$_2$}$$

$$\delta \hat{\hat{J}}_4 = \int_{_0\sigma} \left\{ \sum_{n=0}^{\mathcal{R}} \left[\left(p^{\rho}_{\langle n\rangle} - \Pi^{\rho}_{\langle n\rangle} \right) \overset{(n)}{V}_\rho + \left(p^{3}_{\langle n\rangle} - \Pi^{3}_{\langle n\rangle} \right) \delta \overset{(n)}{V}_3 \right] \right\} d\sigma. \tag{32$_3$}$$

$\mathcal{L}^{\alpha\beta}{}_{\langle n\rangle}$, $\mathcal{N}^{\alpha 3}_{\langle n\rangle}$, $\Pi^{i}_{\langle n\rangle}$ will be rendered explicitly later.

5.5 *Modified form of* δJ_5. Employment in $(11)_5$ of (14) and (19) yields:

$$\delta J_5 = \int_{{}_0\Gamma_\nu} \left\{ \sum_{n=0}^{\mathcal{R}} \left[\left(\overset{(n)}{\mathbf{V}}_\rho - \overset{(n)}{V}_\rho \right) {}_0\nu_\alpha \delta \hat{L}^{\rho\alpha}_{\langle n \rangle} + \left(\overset{(n)}{\mathbf{V}}_3 - \overset{(n)}{V}_3 \right) {}_0\nu_\alpha N^{\alpha 3}_{\langle n \rangle} \right] \right\} ds. \tag{33}$$

6. The field equations of the composite shell. At this stage it should be remarked that the variational equation (10) of the 3-D elasticity theory $\delta J(s^{ij}, e_{ij}, V_i) = 0$ was converted in terms of 2-D quantities characterizing the state of stress and strain of the multilayered shell as

$$\delta J \left(L^{\alpha\beta}_{\langle n \rangle}, L^{33}_{\langle n \rangle}, N^{\alpha 3}_{\langle n \rangle}, \overset{(n)}{e}_{ij}, \overset{(n)}{V}_i \right) \equiv \sum_{i=1}^{5} \delta J_i = 0,$$

where δJ_i are expressed by (26)–(33). Considering the variations of indicated quantities as arbitrary throughout the surface ${}_0\sigma$ and on ${}_0\Gamma_s$ and ${}_0\Gamma_v$, their coefficients in the five integrands entering (26)–(33) must vanish independently. They yield in succession the field equations of the refined geometrically nonlinear theory of laminated shells as given by:

6.1 *The equations of motion.*

$$\begin{aligned} &\mathcal{L}^{\alpha\rho}_{\langle n \rangle}|_\alpha - b^\rho_\alpha \mathcal{N}^\alpha_{\langle n \rangle} + n b^\rho_\alpha N^{\alpha 3}_{\langle n \rangle} - n N^{\rho 3}_{\langle n-1 \rangle} \\ &\qquad - n \sum_{m=0}^{\mathcal{R}} N^{\omega 3}_{\langle n-1+m \rangle} \overset{(m)}{\phi}{}_{\cdot\omega}^{\rho} - n \sum_{m=0}^{\mathcal{R}-1} (m+1) L^{33}_{\langle n-1+m \rangle} \overset{(n+1)}{V}{}^{\rho} \\ &\qquad\qquad + \mathcal{F}^\rho_{\langle n \rangle} - f^\rho_{\langle n \rangle} + p^\rho_{\langle n \rangle} \left(\equiv S^\rho_{\langle n \rangle} \right) = 0, \end{aligned} \tag{34$_1$}$$

$$\begin{aligned} &b_{\alpha\rho} \mathcal{L}^{\rho\alpha}_{\langle n \rangle} + \mathcal{N}^\rho_{\langle n \rangle}|_\rho - n L^{33}_{\langle n-1 \rangle} - n \sum_{n=0}^{\mathcal{R}-1} (m+1) L^{33}_{\langle n-1+m \rangle} \overset{(m+1)}{V}_3 \\ &\qquad - n \sum_{m=0}^{\mathcal{R}} N^{\alpha 3}_{\langle n-1+m \rangle} \overset{(m)}{\phi}{}_\alpha^{\ 3} + \mathcal{F}^3_{\langle n \rangle} - f^3_{\langle n \rangle} + p^3_{\langle n \rangle} (\equiv S^3_{\langle n \rangle}) = 0. \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}^{\alpha\rho}_{\langle n \rangle} &= L^{\alpha\rho}_{\langle n \rangle} + \sum_{m=0}^{\mathcal{R}} \hat{L}^{\alpha\delta}_{\langle m+n \rangle} \overset{(m)}{\phi}{}_{\cdot\delta}^{\rho} + \sum_{m=0}^{\mathcal{R}-1} (m+1) N^{\alpha 3}_{\langle m+n \rangle} \overset{(m+1)}{V}{}^{\rho}, \\ \mathcal{N}^\alpha_{\langle n \rangle} &= N^{\alpha 3}_{\langle n \rangle} + \sum_{m=0}^{\mathcal{R}} \hat{L}^{\alpha\omega}_{\langle m+n \rangle} \overset{(m)}{\phi}{}_\omega^{\ 3} + \sum_{m=0}^{\mathcal{R}-1} (m+1) N^{\alpha 3}_{\langle m+n \rangle} \overset{(m+1)}{V}_3 \qquad (n = \overline{0, \mathcal{R}}). \end{aligned} \tag{34$_2$}$$

The equations of motion $(34)_1$ must be supplemented with the *nondifferential equilibrium equations*

$$\varepsilon_{\alpha\beta} \left(L^{\beta\alpha}_{\langle n-1 \rangle} - b^\beta_\gamma L^{\gamma\alpha}_{\langle n \rangle} \right) = 0 \qquad (n = \overline{1, \mathcal{R}}), \tag{34$_3$}$$

which represent the macroscopic equivalent of Eqs. (8) expressing the symmetry of the stress tensor s^{ij}. They are obtained in a similar way as their counterpart in the classical shell theory (see [5, 23]).

6.2 *The strain-displacement equations.* The strain-displacement relationships result in the form:

$$2\overset{(n)}{e}_{\alpha\beta} = \overset{(n)}{\phi}_{\alpha\beta} + \overset{(n)}{\phi}_{\beta\alpha} - b^{\gamma}_{\alpha}\overset{(n-1)}{\phi}_{\gamma\beta} - b^{\gamma}_{\beta}\overset{(n-1)}{\phi}_{\gamma\alpha} + \sum_{p=0}^{n}\left(\overset{(p)}{\phi}_{\cdot\alpha}{}^{\sigma}\overset{(n-p)}{\phi}_{\cdot\beta}{}^{\sigma} + \overset{(p)}{\phi}_{\alpha}{}^{3}\overset{(n-p)}{\phi}_{\beta}{}^{3}\right) \quad \left(\equiv 2\overset{(n)}{A}_{\alpha\beta}\right) \qquad \left(n = \overline{0, 2\mathcal{R}}\right);$$

$$2\overset{(n)}{e}_{33} = 2(n+1)\overset{(n+1)}{V}_{3} + \sum_{p=1}^{n} p(n-p+2)\overset{(p)}{V}{}^{\rho}\overset{(n-p+2)}{V}_{\rho} + \sum_{p=1}^{n} p(n-p+2)\overset{(p)}{V}_{3}\overset{(n-p+2)}{V}_{3} \quad \left(\equiv 2\overset{(n)}{B}_{33}\right) \qquad \left(n = \overline{0, 2\mathcal{R}-2}\right).$$

$$2\overset{(n)}{e}_{\alpha 3} = (n+1)\overset{(n+1)}{V}_{\alpha} - nb^{\gamma}_{\alpha}\overset{(n)}{V}_{\alpha} + \overset{(n)}{\phi}_{\alpha}{}^{3} + \sum_{p=0}^{n}\left[(p+1)\overset{(n-p)}{\phi}_{\cdot\alpha}{}^{\gamma}\overset{(p+1)}{V}_{\gamma} + (p+1)\overset{(n-p)}{\phi}_{\alpha}{}^{3}\overset{(p+1)}{V}_{3}\right] \quad \left(\equiv 2\overset{(n)}{C}_{\alpha 3}\right) \qquad \left(n = \overline{0, 2\mathcal{R}-1}\right). \tag{35}$$

In (35), $\overset{(m)}{\phi}_{\cdot\alpha}{}^{\omega}$ and $\overset{(m)}{\phi}_{\omega}{}^{3}$ are defined as

$$\begin{aligned} \overset{(m)}{\phi}_{\cdot\alpha}{}^{\rho} &\equiv \overset{(m)}{V}{}^{\rho}|_{\alpha} - b^{\rho}_{\alpha}\overset{(m)}{V}_{3}; \\ \overset{(m)}{\phi}_{\omega}{}^{3} &\equiv \overset{(m)}{V}_{3,\omega} + b^{\lambda}_{\omega}\overset{(m)}{V}_{\lambda}. \end{aligned} \tag{36}$$

In all the previous equations $\overset{(r)}{V}_{i}$, $\overset{(r)}{\phi}_{\alpha\beta}$, $\overset{(r)}{\phi}_{\alpha}{}^{3}$ are to be considered zero whether: (i) $r < 0$ or (ii) $r > \mathcal{R}$.

6.3 *Constitutive equations.* From (29), the constitutive equations result in the form:

$$\hat{L}^{\alpha\beta}_{\langle n\rangle} = \frac{1}{2}\sum_{k=1}^{N}\left(\frac{\partial\overline{W}^{(k)}}{\partial\overset{(n)}{e}_{\alpha\beta}} + \frac{\partial\overline{W}^{(k)}}{\partial\overset{(n)}{e}_{\beta\alpha}}\right), \qquad \left(n = \overline{0, 2\mathcal{R}}\right),$$

$$L^{33}_{\langle n\rangle} = \sum_{k=1}^{N}\frac{\partial\overline{W}^{(k)}}{\partial\overset{(n)}{e}_{33}}, \qquad \left(n = \overline{0, 2\mathcal{R}-2}\right)$$

$$N^{\alpha 3}_{\langle n\rangle} = \frac{1}{2}\sum_{k=1}^{N}\left(\frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{e}_{\alpha 3}} + \frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{e}_{3\alpha}}\right) \qquad (n = \overline{0, 2\mathcal{R} - 1}) \tag{37}$$

Owing to the evident relationship between the asymmetric and the symmetric stress-couple measures,

$$L^{\alpha\beta}_{\langle n\rangle} = \hat{L}^{\alpha\beta}_{\langle n\rangle} - b^{\alpha}_{\delta}\hat{L}^{\delta\beta}_{\langle n+1\rangle}, \tag{38}$$

the constitutive equations for $L^{\alpha\beta}_{\langle n\rangle}$ result as

$$L^{\alpha\beta}_{\langle n\rangle} = \frac{1}{2}\sum_{k=1}^{N}\left[\frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{e}_{\alpha\beta}} + \frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{e}_{\beta\alpha}} - b^{\beta}_{\delta}\left(\frac{\partial \overline{W}^{(k)}}{\partial \overset{(n+1)}{e}_{\alpha\delta}} + \frac{\partial \overline{W}^{(k)}}{\partial \overset{(n+1)}{e}_{\delta\alpha}}\right)\right] \qquad (n = \overline{0, 2\mathcal{R}}). \tag{39}$$

Equations (37)–(39) constitute the generalized counterpart of the ones obtained in [28, 29, 5].

It should be remarked that by virtue of (37), $L^{\beta\alpha}_{\langle n-1\rangle} - b^{\beta}_{\gamma}L^{\gamma\alpha}_{\langle n\rangle}$ may be expressed as

$$\begin{aligned} L^{\beta\alpha}_{\langle n-1\rangle} - b^{\beta}_{\gamma}L^{\gamma\alpha}_{\langle n\rangle} = \frac{1}{2}\sum_{k=1}^{N}\Bigg[&\frac{\partial \overline{W}^{(k)}}{\partial \overset{(n-1)}{e}_{\alpha\beta}} + \frac{\partial \overline{W}^{(k)}}{\partial \overset{(n-1)}{e}_{\beta\alpha}} - b^{\beta}_{\gamma}\left(\frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{e}_{\alpha\gamma}} + \frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{e}_{\gamma\alpha}}\right) \\ &- b^{\alpha}_{\gamma}\left(\frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{e}_{\gamma\beta}} + \frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{e}_{\beta\gamma}}\right) \\ &+ b^{\alpha}_{\gamma}b^{\beta}_{\rho}\left(\frac{\partial \overline{W}^{(k)}}{\partial \overset{(n+1)}{e}_{\gamma\rho}} + \frac{\partial \overline{W}^{(k)}}{\partial \overset{(n+1)}{e}_{\rho\gamma}}\right)\Bigg] \qquad (n = \overline{1, 2\mathcal{R}}) \end{aligned} \tag{40}$$

It is easily seen that the right-hand side of (40) is symmetric and satisfies identically the nondifferential equilibrium equations $(34)_3$. In this case the equations $(34)_3$ may be suppressed. The constitutive equation (37) may be rendered more explicitly. In this sense, we shall make use of the well-known fact (see e.g. [30]) that the theory implying small strains but large displacement gradients and rotations may be described by a linear constitutive equation provided second Piola-Kirchhoff stress and Lagrange strain measures be used. Based on this fundamental observation and by considering the case of an elastic anisotropic body (of the elastic-symmetry type with respect to the surface $x^3 = 0$—see, e.g., [31] and [5]) the appropriate strain energy function expresses as:

$$W = \frac{1}{2} E^{\omega\lambda\rho\theta} e_{\rho\theta} e_{\omega\lambda} + E^{\omega\lambda 33} e_{\omega\lambda} e_{33}$$
$$+ 2 E^{\omega 3 \sigma 3} e_{\omega 3} e_{\sigma 3} + \frac{1}{2} E^{3333} (e_{33})^2$$
$$+ \frac{1}{2} \lambda^{\omega\lambda} e_{\omega\lambda} T + \frac{1}{2} \lambda^{33} e_{33} T \tag{41}$$

where $T \equiv T(x^\omega, x^3)$ denotes the temperature excess with respect to the reference temperature T_r; E^{ijkl} and λ^{ij} stand for the spatial tensors of elasticity and of the thermal expansion coefficients, assumed to fulfill the well-known symmetry properties. For some special cases of anisotropy (e.g., orthotropy and transverse-isotropy), their appropriate expressions are presented in [5]. On the basis of (37) and (41), and by representing the temperature field as $T(x^\omega, x^3) = \sum_{n=0}^{\mathscr{R}} (x^3)^n T(x^\omega)$, the constitutive equations appropriate to a kth layer write under an explicit form as

$$\hat{L}^{\gamma\theta}_{\langle n \rangle} = \sum_{q=n}^{n+2\mathcal{R}} \left({}_q B^{\gamma\theta\rho\sigma} \overset{(q-n)}{e_{\rho\sigma}} + {}_q B^{\gamma\theta 33} \overset{(q-n-2)}{e_{33}} + {}_q \Lambda^{\gamma\theta} \overset{(q-n)}{T} \right),$$
$$N^{\omega 3}_{\langle n \rangle} = 2 \sum_{q=n}^{n+2\mathcal{R}-1} {}_q B^{\omega 3 \alpha 3} \overset{(q-n)}{e_{\alpha 3}}. \tag{42}$$
$$L^{33}_{\langle n \rangle} = \sum_{q=n}^{n+2\mathcal{R}} \left({}_q B^{\omega\lambda 33} \overset{(q-n)}{e_{\omega\lambda}} + {}_q B^{3333} \overset{(q-n-2)}{e_{33}} + {}_q \Lambda^{33} \overset{(q-n)}{T} \right).$$

where $\overset{(r)}{e_{ij}}$ and $\overset{(r)}{T}$ are to be considered zero whenever $r < 0$.

The layered shell stiffness tensors intervening in (42) are defined by:

$${}_n B^{ijml} = \sum_{k=1}^{N} \int_{^{(k)}z - ^{(k)}h}^{^{(k)}z + ^{(k)}h} \mu (x^3)^{n (k)} E^{ijml} \, dx^3,$$
$${}_n \Lambda^{ij} = \frac{1}{2} \sum_{k=1}^{N} \int_{^{(k)}z - ^{(k)}h}^{^{(k)}z + ^{(k)}h} \mu (x^3)^{n (k)} \lambda^{ij} \, dx^3. \tag{43}$$

Having in view that both E^{ijml} and λ^{ij} are space tensors, expressible in terms of corresponding surface quantities as

$${}^{(k)}E^{\alpha\beta\gamma\delta} = (\mu^{-1})^{\alpha}_{\omega} (\mu^{-1})^{\beta}_{\rho} (\mu^{-1})^{\gamma}_{\lambda} (\mu^{-1})^{\delta}_{\sigma} {}^{(k)}\bar{E}^{\omega\rho\lambda\sigma},$$
$${}^{(k)}\lambda^{\alpha\beta} = (\mu^{-1})^{\alpha}_{\nu} (\mu^{-1})^{\beta}_{\delta} {}^{(k)}\bar{\lambda}^{\nu\delta}, \tag{44}$$
$${}^{(k)}E^{\alpha 3 \sigma 3} = (\mu^{-1})^{\alpha}_{\nu} (\mu^{-1})^{\sigma}_{\omega} {}^{(k)}\bar{E}^{\nu 3 \omega 3},$$

where the surface tensors ${}^{(k)}E^{ijml}$ and ${}^{(k)}\lambda^{ij}$ are piecewise constant through the laminated wall thickness, it results (see [5]):

$$
{}_nB^{\omega\lambda\rho\beta} = \sum_{k=1}^{N} {}^{(k)}\bar{E}^{\sigma\pi\theta\gamma}\Bigg[\delta_\sigma^\omega\delta_\pi^\lambda\delta_\gamma^\beta\delta_\theta^{\rho(k)}\eta(n+1) + \sum_{m=0}^{\infty}\sum_{t=0}^{m}\sum_{q=0}^{m} K^{\omega\lambda\rho\beta}_{\sigma\kappa\mu\psi}(b^t)^\psi_\gamma (b^q)^\mu_\theta \times (b^{(m-t-q)})^{\kappa(k)}_\pi \eta(m+n+1)\Bigg];
$$

$$
\begin{bmatrix} {}_nB^{\alpha3\sigma3} \\ {}_nB^{\alpha\sigma33} \\ {}_n\Lambda^{\alpha\sigma} \end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} {}^{(k)}\bar{E}^{\pi3\gamma3} \\ {}^{(k)}\bar{E}^{\pi\gamma33} \\ {}^{(k)}\bar{\lambda}^{\pi\gamma} \end{bmatrix} \Bigg[\delta_\pi^\alpha\delta_\gamma^{\sigma(k)}\eta(n+1) + \sum_{m=0}^{\infty} I^{\alpha\sigma}_{\pi\psi}(b^m)^{\psi(k)}_\gamma\eta(m+n+1)\Bigg] \tag{45}
$$

where $K^{\omega\lambda\rho\beta}_{\sigma\kappa\mu\psi}$, $I^{\omega\sigma}_{\pi\psi}$ are defined in [5]; while $\eta(r)$ is expressed by (23).

In the case of symmetrically laminated shells (S.L.S.) (the envisaged symmetry with respect to the medium layer being both geometrically and physically), an odd number of constituent layers ($N = 2l + 1$, $l = 1, 2 \ldots$) may be taken into consideration (see [5]). For this case the reference surface will be selected as to coincide with the mid-surface of the mid-layer. This entails the following modification in (45) consisting of the replacement: $\Sigma_{k=1}^{N} \to 2\Sigma_{k=1}^{l+1}$ as well as of redefinition of ${}^{(k)}\eta(r)$ as

$$
{}^{(k)}\eta(r) = \frac{1}{r}\left[\left({}^{(k)}z + {}^{(k)}h\right)^r - \left({}^{(k)}z - {}^{(k)}h\right)^r\right]\left(1^r - (-1)^r\right) \tag{46}
$$

which results in

$$
{}^{(k)}\eta(r) = \begin{cases} \dfrac{2}{r}\left[\left({}^{(k)}z + {}^{(k)}h\right)^r - \left({}^{(k)}z - {}^{(k)}h\right)^r\right] & \text{for } r \text{ odd}, \\ 0 & \text{for } r \text{ even}. \end{cases}
$$

In light of (16) and (44), in the case of flat (or very shallow) symmetrically laminated panels, *the constitutive equations* split into two independent groups afferent to the bending and stretching states of stress (under the same conditions a similar splitting arises in the expression of inertia forces (21), (22)).

In contrast to this case, for arbitrary laminated panels, such a splitting in the constitutive equations does not occur.

6.4 *The static and geometrical boundary conditions*. Having in view $(32)_2$, the static boundary conditions on ${}_0\Gamma_s$ result as

$$
{}_0\nu_\alpha \mathbf{L}^{\alpha\rho}_{\langle n\rangle} = {}_0\nu_\alpha\left[L^{\alpha\rho}_{\langle n\rangle} + \sum_{m=0}^{\mathcal{R}} \hat{L}^{\alpha\delta}_{\langle m+n\rangle} \overset{(m)}{\phi}{}_{.\delta}{}^{\rho} + \sum_{m=0}^{\mathcal{R}-1}(m+1)N^{\alpha3}_{\langle m+n\rangle}\overset{(m+1)}{V}{}^{\rho}\right] \quad \left(\equiv {}_0\nu_\alpha \mathcal{L}^{\alpha\rho}_{\langle n\rangle}\right);
$$

$$
{}_0\nu_\alpha \mathbf{N}^{\alpha 3}_{\langle n\rangle} = {}_0\nu_\alpha \left[\left(N^{\alpha 3}_{\langle n\rangle} + \sum_{m=0}^{\mathcal{R}} + \hat{L}^{\alpha\omega}_{\langle m+n\rangle} \overset{(m)}{\phi_\omega}{}^3 + \sum_{m=0}^{\mathcal{R}-1}(m+1) N^{\alpha 3}_{\langle m+n\rangle} \overset{(m+1)}{V_3}\right)\right] \qquad \left(\equiv {}_0\nu_\alpha \mathcal{N}^{\alpha 3}_{\langle n\rangle}\right). \tag{47}
$$

From $(32)_3$, the B. C. on $S^\pm$ are:

$$
p^\rho_{\langle n\rangle} = \left[\mu\mu^\rho_\psi s^{3\psi}(x^3)^n + \mu s^{3\omega} \sum_{m=0}^{\mathcal{R}} \overset{(m)}{\phi_\omega}{}^\rho (x^3)^{m+n} + \mu s^{33} \sum_{m=0}^{\mathcal{R}-1} (m+1) \overset{(m+1)}{V}{}^\rho (x^3)^{m+n}\right]\Bigg|^{{}^{(N)}z - {}^{(N)}h}_{{}^{(1)}z - {}^{(1)}h} \qquad \left(\equiv \Pi^\rho_{\langle n\rangle}\right). \tag{48}
$$

$$
p^3_{\langle n\rangle} = \left[\mu s^{33}(x^3)^n + \mu s^{33} \sum_{m=0}^{\mathcal{R}-1} (m+1) \overset{(m+1)}{V_3} (x^3)^{m+n} + \mu s^{3\alpha} \sum_{m=0}^{\mathcal{R}} \overset{(m)}{\phi_\alpha}{}^3 (x^3)^{m+n}\right]\Bigg|^{{}^{(N)}z - {}^{(N)}h}_{{}^{(1)}z - {}^{(1)}h} \qquad \left(\equiv \Pi^3_{\langle n\rangle}\right)
$$

while from (33), the geometrical B. C. on ${}_0\Gamma_V$ write as:

$$
\overset{(n)}{V_\rho} = \overset{(n)}{\mathbf{V}_\rho}; \quad \overset{(n)}{V_3} = \overset{(n)}{\mathbf{V}_3} \qquad (n = \overline{0, \mathcal{R}}). \tag{49}
$$

7. *The continuity conditions at the interfaces.* In the case of laminated shells, there are some specific requirements (of a geometric and static character) which are to be fulfilled. They are referred to as the continuity conditions at the contact surfaces between two consecutive layers.

The geometrical continuity conditions (GCC) require:

$$
\overline{V}_i(x^\omega, x^3)\big|_{x^3 = {}^{(k)}z + {}^{(k)}h} = \overline{V}_i(x^\omega, x^3)\big|_{x^3 = {}^{(k+1)}z - {}^{(k+1)}h} \tag{50}
$$

while the statical continuity conditions (SCC) require:

$$
\begin{aligned}
\left[\mu\mu^\rho_\psi t^{3\psi}\right]\big|_{x^3 = {}^{(k)}z + {}^{(k)}h} &= \left[\mu\mu^\rho_\psi t^{3\psi}\right]\big|_{{}^{(k+1)}z - {}^{(k+1)}h}, \\
\left[\mu t^{33}\right]\big|_{x^3 = {}^{(k)}z + {}^{(k)}h} &= \left[\mu t^{33}\right]\big|_{x^3 = {}^{(k+1)}z - {}^{(k+1)}h}
\end{aligned} \tag{51}
$$

$$
\left(k = \overline{1, N-1}\right)
$$

where $x^3 = {}^{(k)}z + {}^{(k)}h$ and $x^3 = {}^{(k+1)}z - {}^{(k+1)}h$ identify the two contact surfaces between the layers k and $k+1$, respectively.

In light of (14) it is readily seen that (50) are identically satisfied. Concerning the S.C.C., they are to be analyzed in more details.

Towards this end we consider the expressions $[\mu\mu^\rho_\psi t^{3\psi}]$ and $[\mu t^{33}]$, which will be multiplied by $(x^3)^n$, so resulting in $[\mu\mu^\rho_\psi t^{3\psi}(x^3)^n]$ and $[\mu t^{33}(x^3)^n]$, respectively. On the other hand, from the 3-D equations of motion (determined by equating to zero the coefficients of $\delta \overset{(n)}{V}_\alpha$ and $\delta \overset{(n)}{V}_3$ in the expression (16) of δJ_1), the expressions of $[\mu\mu^\rho_\psi t^{3\psi}(x^3)^n]$ and $[\mu t^{33}(x^3)^n]$ available in an arbitrary point x^3, $({}^{(k)}z - {}^{(k)}h \leqslant x^3 \leqslant {}^{(k)}z + {}^{(k)}h)$, of the kth layer, result as

$$\begin{aligned}\left[\mu\mu^\rho_\psi t^{3\psi}(x^3)^n\right] &= -\int_{{}^{(k)}z - {}^{(k)}h}^{x^3} A^\rho \, dx^3 + {}^{(k)}\Phi^\rho, \\ \left[\mu t^{33}(x^3)^n\right] &= -\int_{{}^{(k)}z - {}^{(k)}h}^{x^3} A^3 \, dx^3 + {}^{(k)}\Phi^3,\end{aligned} \tag{52}$$

where,

$$\begin{aligned} A^\rho \equiv{}& \left[(\mu t^{\alpha\omega}\mu^\rho_\omega)|_\alpha - b^\rho_\alpha(\mu t^{\alpha 3}) + {}_0\rho\mu\mu^\rho_\alpha({}_0H^\alpha - {}_0h^\alpha)\right](x^3)^n \\ & - n\mu\mu^\rho_\psi t^{3\psi}(x^3)^{n-1}, \\ A^3 \equiv{}& \left[(\mu t^{\alpha 3})|_\alpha + b_{\rho\alpha}\left(\mu\mu^\rho_\psi t^{\alpha\psi}\right) + \rho_0\mu({}_0H^3 - {}_0h^3)\right](x^3)^n - n(x^3)^{n-1}\mu t^{33} \end{aligned} \tag{53}$$

where ${}^{(k)}\Phi^\rho \equiv {}^{(k)}\Phi^\rho(x^\omega, t)$ and ${}^{(k)}\Phi^3 \equiv {}^{(k)}\Phi^3(x^\omega, t)$ are defined by:

$$\begin{aligned} {}^{(k)}\Phi^\rho &= \left[\mu\mu^\rho_\psi t^{3\psi}(x^3)^n\right]\Big|_{x^3 = {}^{(k)}z - {}^{(k)}h}, \\ {}^{(k)}\Phi^3 &= \left[\mu t^{33}(x^3)^n\right]\Big|_{x^3 = {}^{(k)}z - {}^{(k)}h}. \end{aligned} \tag{54}$$

Expressing the B.C. on $S^\pm$ as under the form:

$$\begin{aligned} [\mu\mu^\rho_\psi t^{3\psi}(x^3)^n]|_{x^3 = {}^{(N)}z + {}^{(N)}h} &= \overset{+}{p}{}^\rho_{\langle n\rangle}, \\ [\mu t^{33}(x^3)^n]|_{x^3 = {}^{(N)}z + {}^{(N)}h} &= \overset{+}{p}{}^3_{\langle n\rangle} \end{aligned} \tag{55$_1$}$$

and

$$\begin{aligned} \left[\mu\mu^\rho_\psi t^{3\psi}(x^3)^n\right]\Big|_{x^3 = {}^{(1)}z - {}^{(1)}h} &= \bar{p}^\rho_{\langle n\rangle}, \\ \left[\mu t^{33}(x^3)^n\right]\Big|_{x^3 = {}^{(1)}z - {}^{(1)}h} &= \bar{p}^3_{\langle n\rangle}, \end{aligned} \tag{55$_2$}$$

and employing them in (52), one obtains:

$$\begin{aligned} \overset{+}{p}{}^\rho_{\langle n\rangle} + \int_{{}^{(N)}z - {}^{(N)}h}^{{}^{(N)}z + {}^{(N)}h} A^\rho \, dx^3 &= {}^{(N)}\Phi^\rho(x^\omega, t), \\ \overset{+}{p}{}^3_{\langle n\rangle} + \int_{{}^{(N)}z - {}^{(N)}h}^{{}^{(N)}z + {}^{(N)}h} A^3 \, dx^3 &= {}^{(N)}\Phi^3(x^\omega, t) \end{aligned} \tag{56$_1$}$$

$$\begin{aligned} \bar{p}^\rho_{\langle n\rangle} &= {}^{(1)}\Phi^\rho(x^\rho, t), \\ \bar{p}^3_{\langle n\rangle} &= {}^{(1)}\Phi^3(x^\omega, t). \end{aligned} \tag{56$_2$}$$

Employment of (52) and (54), considered in conjuction with the S.C.C. given by (51), yields the result

$$
\begin{aligned}
&{}^{(k)}\Phi^{\rho} - {}^{(k-1)}\Phi^{\rho} + \int_{^{(k-1)}z - ^{(k-1)}h}^{^{(k)}z - ^{(k)}h} A^{\rho}\, dx^3 = 0, \\
&{}^{(k)}\Phi^{3} - {}^{(k-1)}\Phi^{3} + \int_{^{(k-1)}z - ^{(k-1)}h}^{^{(k)}z - ^{(k)}h} A^{3}\, dx^3 = 0 \\
&\qquad\qquad (k = \overline{2, N}).
\end{aligned}
\tag{57}
$$

Specialization of (57) for all $k, (k = \overline{2, N})$, followed by the successive addition of the resulting expressions, yields:

$$
\begin{aligned}
&{}^{(N)}\Phi^{\rho} - {}^{(1)}\Phi^{\rho} + \sum_{k=2}^{N} \int_{^{(k-1)}z - ^{(k-1)}h}^{^{(k)}z - ^{(k)}h} A^{\rho}\, dx^3 = 0, \\
&{}^{(N)}\Phi^{3} - {}^{(1)}\Phi^{3} + \sum_{k=2}^{N} \int_{^{(k-1)}z - ^{(k-1)}h}^{^{(k)}z - ^{(k)}h} A^{3}\, dx^3 = 0.
\end{aligned}
\tag{58}
$$

In light of (56), Eqs. (58) may be expressed, after a convenient redefinition of the limits of integration, as

$$
\begin{aligned}
&p^{\rho}_{\langle n\rangle} + \sum_{k=1}^{N} \int_{^{(k)}z - ^{(k)}h}^{^{(k)}z + ^{(k)}h} A^{\rho}\, dx^3 = 0, \\
&p^{3}_{\langle n\rangle} + \sum_{k=1}^{N} \int_{^{(k)}z - ^{(k)}h}^{^{(k)}z + ^{(k)}h} A^{3}\, dx^3 = 0,
\end{aligned}
\tag{59}
$$

where $p^i_{\langle n\rangle} \equiv \overset{+}{p}{}^i_{\langle n\rangle} - \bar{p}^i_{\langle n\rangle}$. The equations (59) represent nothing but a crude form of the macroscopic equations of motion of the geometrically nonlinear theory of laminated shells. In light of preceding developments it may be concluded that the fulfillment of the macroscopic equations of motion (given explicitly by (35)) and of Eqs. (48) implies automatically the fulfillment of static continuity conditions.

Remark. It should be remembered that all the developments are based on the fulfillment of the relationship $R = S = \mathcal{R}$. However, it may be shown that the obtained results maintain their form even for $R \neq S$.

In this instance, when $R > S$, $\mathcal{R}$ is to be assimilated with R and in the summation process occurring in the field equations only the appropriate nonvanishing terms are to be retained.

Conversely, when $S > R$, then $\mathcal{R}$ will be assimilated with S and further the rejection of appropriate vanishing terms is to be applied.

8. Special cases. The results previously obtained encompass a series of special cases involving the nonlinear and the linearized theories of multilayered shells. In the following, several specialized versions of the previous general results are considered and the similarity with certain results encountered in the field literature are pointed out. In addition, the concept of a *refined* approximate theory accounting for small displacement gradients and moderate rotations is briefly discussed.

8.1 *Linearized high-order theory of multilayered shells*. Complete linearization of the nonlinear field equations yields:[3]

The equations of motion:

$$
\begin{aligned}
L^{\rho\omega}_{\langle n\rangle}\big|_{\omega} + (n-1)b^{\rho}_{\omega}N^{\omega 3}_{\langle n\rangle} - nN^{\rho 3}_{\langle n-1\rangle} + p^{\rho}_{\langle n\rangle} + \mathscr{F}^{\rho}_{\langle n\rangle} - f^{\rho}_{\langle n\rangle} = 0,\\
L^{\alpha\rho}_{\langle n\rangle}b_{\rho\alpha} + N^{\alpha 3}_{\langle n\rangle}\big|_{\alpha} - nL^{33}_{\langle n-1\rangle} + p^{3}_{\langle n\rangle} + \mathscr{F}^{3}_{\langle n\rangle} - f^{3}_{\langle n\rangle} = 0,\\
\varepsilon_{\alpha\beta}\left(L^{\beta\alpha}_{\langle n-1\rangle} - b^{\beta}_{\gamma}L^{\gamma\alpha}_{\langle n\rangle}\right) = 0 \qquad (n = \overline{0, \mathcal{R}}).
\end{aligned}
\tag{60}
$$

The strain-displacement equations:

$$
\begin{aligned}
2e_{\alpha\beta} &= \mu^{\mu}_{\alpha}\sum_{r=0}^{\mathcal{R}} \overset{(r)}{\gamma}_{\mu\beta}(x^3)^n + \mu^{\mu}_{\beta}\sum_{r=0}^{\mathcal{R}} \overset{(r)}{\gamma}_{\mu\alpha}(x^3)^n,\\
2e_{\alpha 3} &= \sum_{r=0}^{\mathcal{R}} \overset{(r)}{\gamma}_{\alpha 3}(x^3)^r; e_{33} = \sum_{r=0}^{\mathcal{R}} \overset{(r)}{\gamma}_{33}(x^3)^r,
\end{aligned}
\tag{61}
$$

where

$$
\begin{aligned}
\overset{(r)}{\gamma}_{\alpha\beta} &= \overset{(r)}{V}_{\alpha|\beta} - b_{\alpha\beta}\overset{(r)}{V}_{3},\\
\overset{(r)}{\gamma}_{\alpha 3} &= (r+1)\overset{(r+1)}{V}_{\alpha} - (r-1)b^{\nu}_{\alpha}\overset{(r)}{V}_{\nu} + \overset{(r)}{V}_{3,\alpha}.\\
\overset{(r)}{\gamma}_{33} &= (r+1)\overset{(r+1)}{V}_{3} \qquad (r = \overline{0, \mathcal{R}})
\end{aligned}
\tag{62}
$$

constitute the nth-order strain measures. However, alongside with (62), several other variants of the strain-measures may be defined (see in this sense [32]).

The constitutive equations. Starting with the linearized form of the variational equation (29) and by adopting $\overset{(r)}{\gamma}_{ij}$ (as defined by (62)) as the strain-measures, one obtains the constitutive equations under the form:

$$
\begin{aligned}
L^{\alpha\beta}_{\langle n\rangle} &= \frac{1}{2}\sum_{k=1}^{N}\left(\frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{\gamma}_{\alpha\beta}} + \frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{\gamma}_{\beta\alpha}}\right),\\
L^{33}_{\langle n\rangle} &= \sum_{k=1}^{N}\frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{\gamma}_{33}},\\
N^{\alpha 3}_{\langle n\rangle} &= \frac{1}{2}\sum_{k=1}^{N}\left(\frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{\gamma}_{\alpha 3}} + \frac{\partial \overline{W}^{(k)}}{\partial \overset{(n)}{\gamma}_{3\alpha}}\right).
\end{aligned}
\tag{63}
$$

[3] As in the special case considered in [5, pp. 479–480], some precautions are to be exercised during the linearization process.

The boundary conditions.

$$\nu_\alpha L^{\alpha\rho}_{\langle n\rangle} = \nu_\alpha \mathbf{L}^{\alpha\rho}_{\langle n\rangle} \quad \text{on } \Gamma_s,$$

$$\nu_\alpha N^{\alpha 3}_{\langle n\rangle} = \nu_\alpha \mathbf{N}^{\alpha 3}_{\langle n\rangle} \qquad (n = \overline{0, \mathcal{R}})$$

and

$$\overset{(r)}{V}_\rho = \overset{(r)}{\mathbf{V}}_\rho; \ \overset{(r)}{V}_3 = \overset{(r)}{\mathbf{V}}_3 \text{ on } \Gamma_V \qquad (r = \overline{0, \mathcal{R}}). \tag{64}$$

$$p^\rho_{\langle n\rangle} = \left[\mu\mu^\rho_\psi \sigma^{3\psi}(x^3)^n\right]^{{}^{(N)}z = {}^{(N)}h}_{{}^{(1)}z = {}^{(1)}h},$$

$$p^3_{\langle n\rangle} = \left[\mu\sigma^{33}(x^3)^n\right]^{{}^{(N)}z = {}^{(N)}h}_{{}^{(1)}z = {}^{(1)}h} \text{ on } S^\pm.$$

As it will be shown in a separate memoir, the linearized field equations exhibited before fulfill a series of necessary requirements which are reminiscent of the ones formulated within the classical theory of single and multilayered shells (see in this respect [23, 33, 34] and [5], respectively).

The results include as special cases some linearized refined variants of single and multilayered shell [21–23, 35] and plate [17–19, 36] theories. In addition, the equations in Sec. 8.1 may be used to model the high-order theory of isotropic shells undergoing infinitesimal displacement gradients and finite strains. Towards this end, $\overline{W}$ in the constitutive equations (63) must be expressed appropriately (see, e.g., [37–39]).

It is also to point out that for the case of *symmetrically laminated flat plates*, the appropriate linearized field equations split exactly into two groups defining the high-order bending and stretching theories. For single-layered anisotropic flat plates the problem of splitting of the high-order state of stress was discussed in [5, chap. III].

8.2 *Specialized variants of the geometrically nonlinear theory of multilayered shells.* A special refined variant may be obtained upon the following representation of the displacement field:

$$\overline{V}_\alpha = \overset{(0)}{V}_\alpha + x^3 \overset{(1)}{V}_\alpha, \qquad \overline{V}_3 = \overset{(0)}{V}_3 + x^3 \overset{(1)}{V}_3 \tag{65}$$

(which corresponds to $\mathcal{R} = 1$ in (14)). The basic field equations derived in Sections 6.1–6.5 may easily be specialized for this case. Some results pertaining to this theory are reported in [25, 5, 32, 40–43]. Moreover, a large diversity of high-order theories may result through various selection of $\mathcal{R}$ in (14).

Employment in (11) of the partially nonlinear strain-displacement relationship

$$2e_{ij} = V_{i\|j} + V_{j\|i} + V_{3\|i} V_{3\|j} \tag{66}$$

considered in conjunction with (65) and the further suppression in the ensuing developments of all nonlinear terms depending on V_α, all these yield the field equations appropriate to a refined theory of multilayered shells of a von-Kármán type.

Several results belonging to such a theory (appropriate to single and multilayered shells and plates) are reported in [25, 5, 44, 45] and [4, 5, 20, 46], respectively.

For the L. K. theory of multilayered shells, consistent with its kinematical constraints, it results (see [29, 47–79]) that:

$$\overset{(n)}{V}_\alpha = \begin{cases} \overset{0}{V}_\alpha & \text{for } n = 0, \\ -\overset{0}{\phi}{}^3_\alpha + \overset{0}{\phi}{}^\lambda_{\cdot\alpha}\overset{0}{\phi}{}^3_\lambda & \text{for } n = 1, \\ 0 & \text{for } n \geq 2, \end{cases}$$
$$\overset{(n)}{V}_3 = \begin{cases} \overset{0}{V}_3 & \text{for } n = 0, \\ -\tfrac{1}{2}\overset{0}{\phi}{}^3_\alpha \overset{0}{\phi}{}^{3\alpha} & \text{for } n = 1, \\ 0 & \text{for } n \geq 2, \end{cases} \tag{67}$$

On this basis, the field equations of the classical geometrically nonlinear theory of multilayered shells may be obtained by paralleling the preceding developments. For the case of homogeneous shells, some results, which concern the L. K. theory approached along these lines, are reported in [47, 48].

The approximate shell theory developed within the classical framework and referred to in [50] as *small finite deflections approximation* and in [51] as *approximation of small strains and moderately small rotations* may be extended to the refined shell theory as well.

Towards this end we define the tensors of 3-D small strains and elastic rotations:

$$\eta_{ij} = \tfrac{1}{2}\left(V_{i||j} + V_{j||i}\right),$$
$$\Omega_{ij} = \tfrac{1}{2}\left(V_{i||j} - V_{j||i}\right). \tag{68}$$

By postulating further the following orders of magnitude for the linearized strains and rotations

$$\eta_{ij} = O(\varepsilon^2); \quad \Omega_{3\alpha} = O(\varepsilon); \quad \Omega_{12} = O(\varepsilon^2), \tag{69}$$

where ε^2 is a small number compared to unity ($\varepsilon^2 \ll 1$), the exact 3-D strain-displacement relationship:

$$e_{ij} = \eta_{ij} + \tfrac{1}{2}\eta_{ri}\eta^r_{\cdot j} + \tfrac{1}{2}\left(\eta_{rj}\Omega^r_{\cdot i} + \eta_{ri}\Omega^r_{\cdot j}\right) + \tfrac{1}{2}\Omega_{rj}\Omega^r_{\cdot i} \tag{70}$$

expressed in terms of the quantities defined in (68), becomes

$$e_{\alpha\beta} = \eta_{\alpha\beta} + \tfrac{1}{2}\Omega_{3\alpha}\Omega_{3\beta},$$
$$e_{\alpha 3} = \eta_{\alpha 3},$$
$$e_{33} = \eta_{33} + \tfrac{1}{2}\Omega_{\omega 3}\Omega^{\omega}_{\cdot 3} \tag{71}$$

Equation $(71)_2^4$ shows that in the framework of this approximate theory the transverse shearing strains are described by linear strain measures.

Employment in (11) of (71) and by paralleling further the developments in Sections 8.1–8.9, (see also [50–52]), all these allow to derive the field equations of the refined theory of multilayered shells characterized by small displacement gradients, small rotation about the normal (Ω_{12}) and moderately small rotations $\Omega_{3\alpha}$. Such an approximate theory may be developed both for the first-order transverse shear-deformation and the high-order shell theories as well.[5]

---

[4] See also [53].

[5] Added in proof: The theory of elastic plates was considered, along these lines, in the paper "Higher-order Moderate Rotation Theories for Elastic Anisotropic Plates" by L. Librescu and R. Schmidt, to appear in the volume "Finite Rotations in Structural Mechanics" (ed.) W. Pietraszkiewicz, Springer-Verlag, (under press), 1986.

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