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PLASTIC STRESSES INDUCED BY A RIGID RING EMBEDDED IN A THIN ANISOTROPIC PLATE UNDER UNIFORM TENSION*

BY

DAVID DURBAN[1]

Engineering Department, Cambridge University, England

Introduction. The presence of a rigid fastening ring (Fig. 1) in a thin infinite sheet, subjected to remote uniform tension σ_∞, is expected to affect the equibiaxial stress field in the vicinity of the rigid boundary. Here we present an analytical solution to this problem for pure power-hardening plastic materials, with transverse plastic anisotropy, modeled by

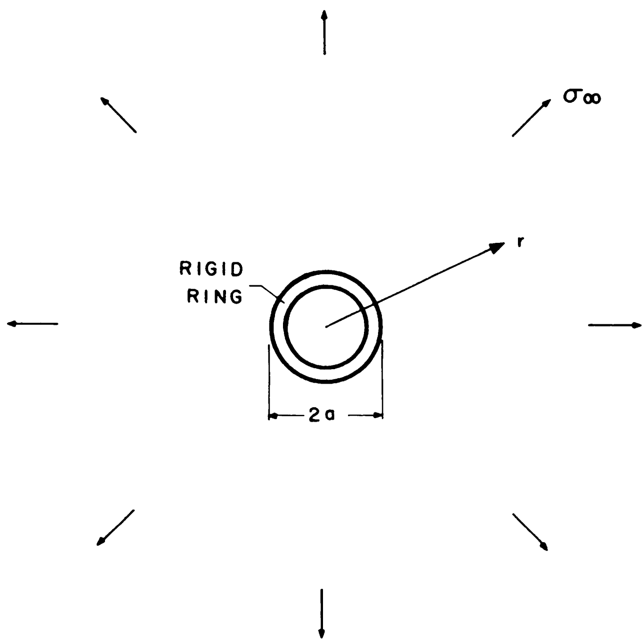

FIG. 1. A rigid ring is embedded in an infinite sheet subjected to remote uniform tension σ_∞. The radial coordinate is denoted by r and the rigid boundary is at $r = a$.

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[1]On sabbatical leave (until October 1987). Permanent address: Technion-Israel Institute of Technology, Haifa, Israel.

a theory proposed by Hill [1]. The effective stress σ_e is defined as

$$2(1 + R)\sigma_e^m = (1 + 2R)|\sigma_1 - \sigma_2|^m + |\sigma_1 + \sigma_2|^m, \tag{1}$$

where (σ_1, σ_2) are the in-plane principal stresses, and parameters (m, R) characterize the normal plastic anisotropy of the sheet. For the problem considered here we may identify the principal stresses with the polar components $(\sigma_r, \sigma_\theta)$. The dependence of the yield locus (1) on parameters (m, R) is illustrated in Fig. 2 for the relevant quadrant where both stress components are positive. Note that the standard Mises and Tresca loci are obtained with $(m = 2, R = 1)$ and $(m = 1, R = 0)$, respectively.

The stress concentration problem, for a rigid circular inclusion, has been solved by Yang [2] using an earlier anisotropic theory proposed by Hill in [3]. That theory is just a particular case of (1) when $m = 2$. A complete elasto/plastic solution to the problem, for

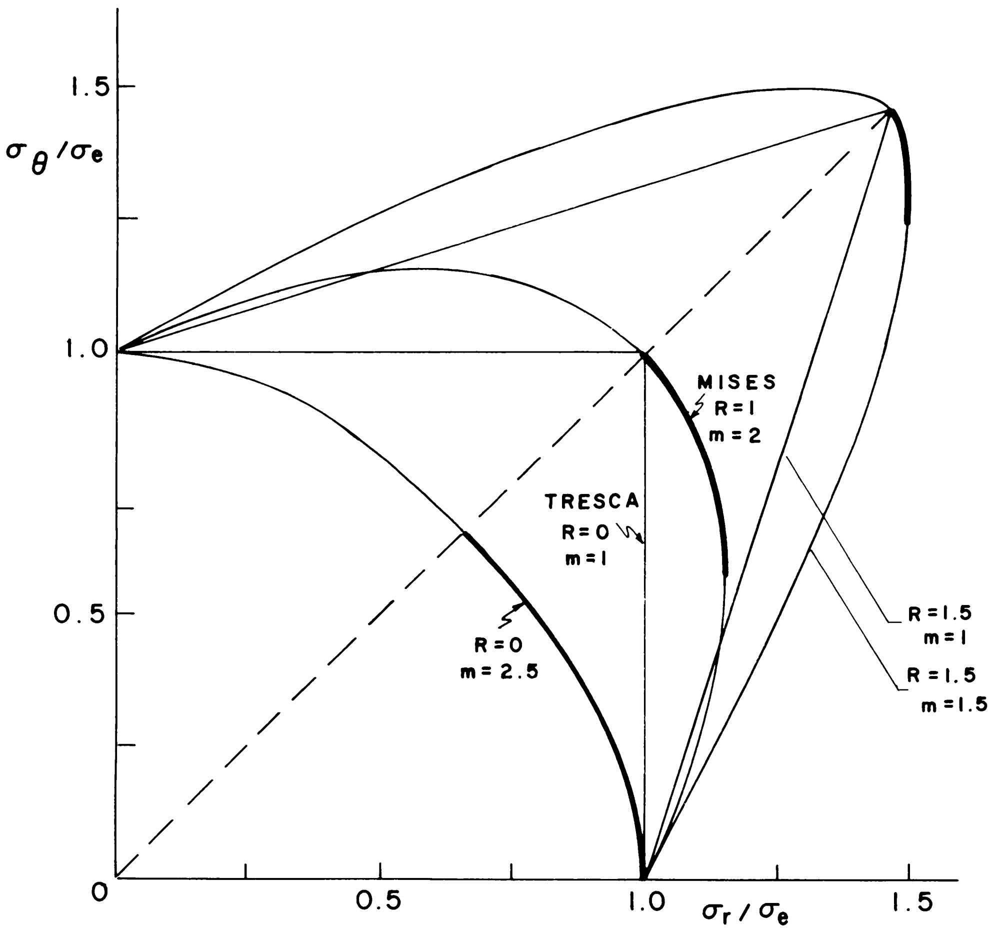

FIG. 2. Variation of the yield locus with parameters (m, R) when both stress components are positive. The Mises material is described by $(m = 2, R = 1)$, and the Tresca material is described by $(m = 1, R = 0)$. The heavy lines show the operative parts of the yield locus. With $m = 1$ only the point $\sigma_r = \sigma_\theta = \sigma_\infty = (1 + R)\sigma_e$ is operative.

a restricted version of (1) with $m = 1$, and accounting for linear strain hardening, has been given recently in [4].

The analogous problem of the stress concentration at a circular hole has been treated in several papers [5–8]. The analysis by Budiansky [7] contains a general investigation with different families of constitutive relations accounting for plastic anisotropy.

The solution in this paper is within the usual framework of axially-symmetric, plane-stress, small-strain plasticity. Elastic strains are neglected and changes in the thickness at the rigid boundary are permitted. Contact is made with the earlier result given in [2] for $m = 2$.

Analysis. Anticipating that $\sigma_r > \sigma_\theta > 0$ throughout the entire field, we rewrite definition (1) in the form

$$2(1 + R)\sigma_e^m = (1 + 2R)(\sigma_r - \sigma_\theta)^m + (\sigma_r + \sigma_\theta)^m. \tag{2}$$

Relation (2) is identically satisfied by the parametric representation

$$\sigma_r - \sigma_\theta = J(\sin\alpha)^{2/m}\sigma_e, \tag{3a}$$

$$\sigma_r + \sigma_\theta = H(\cos\alpha)^{2/m}\sigma_e, \tag{3b}$$

where

$$H = (2 + 2R)^{1/m}, \qquad J = \left(\frac{2 + 2R}{1 + 2R}\right)^{1/m}. \tag{4}$$

The stress components are therefore given by

$$\sigma_r = S_r\sigma_e, \qquad \sigma_\theta = S_\theta\sigma_e, \tag{5}$$

with

$$S_r = \tfrac{1}{2}\left[H(\cos\alpha)^{2/m} + J(\sin\alpha)^{2/m}\right], \tag{6a}$$

$$S_\theta = \tfrac{1}{2}\left[H(\cos\alpha)^{2/m} - J(\sin\alpha)^{2/m}\right]. \tag{6b}$$

For the class of materials considered here it is possible to integrate the constitutive relations of the flow theory associated with (1). This leads to the deformation theory-type constitutive relations

$$\varepsilon_r = \Gamma_r\varepsilon, \qquad \varepsilon_\theta = \Gamma_\theta\varepsilon \tag{7}$$

where $(\varepsilon_r, \varepsilon_\theta)$ are the usual strain components,

$$\Gamma_r = \frac{1}{H}(\cos\alpha)^{2(m-1)/m} + \frac{1}{J}(\sin\alpha)^{2(m-1)/m}, \tag{8a}$$

$$\Gamma_\theta = \frac{1}{H}(\cos\alpha)^{2(m-1)/m} - \frac{1}{J}(\sin\alpha)^{2(m-1)/m}, \tag{8b}$$

and ε is the effective plastic strain determined by the uniaxial characteristic

$$\varepsilon = (\sigma_e/\sigma_0)^n, \tag{9}$$

where σ_0 and n are material constants.

With u denoting the radial displacement, we have the kinematical relations

$$\varepsilon_r = du/dr, \qquad \varepsilon_\theta = u/r, \tag{10}$$

which may be conjoined to form the compatibility equation

$$r\,d\varepsilon_\theta/dr + \varepsilon_\theta - \varepsilon_r = 0. \tag{11}$$

Now, we combine the equation of equilibrium

$$r\,d\sigma_r/dr + \sigma_r - \sigma_\theta = 0 \tag{12}$$

with (11) and eliminate the radial coordinate, thus obtaining the differential relation

$$(\sigma_r - \sigma_\theta)\,d\varepsilon_\theta + (\varepsilon_r - \varepsilon_\theta)\,d\sigma_r = 0. \tag{13}$$

Substituting in (13) the stresses from (5) and the strains from (7) gives, with the aid of (9),

$$d\sigma_e/\sigma_e = f(\alpha)\,d\alpha, \tag{14}$$

where

$$f(\alpha) = -\frac{(S_r - S_\theta)\Gamma_\theta' + (\Gamma_r - \Gamma_\theta)S_r'}{(S_r - S_\theta)n\Gamma_\theta + (\Gamma_r - \Gamma_\theta)S_r} \tag{15}$$

and the prime denotes differentiation with respect to α. A further substitution of (6) and (8) results in

$$f(\alpha) = \left(\frac{2}{m}\right)\frac{(m-1)\tan\alpha + (m-2)I(\tan\alpha)^{(m-2)/m} + I^2(\tan\alpha)^{(3m-4)/m}}{n - (n-1)I(\tan\alpha)^{(2m-2)/m} + I^2(\tan\alpha)^{(2m-4)/m}}, \tag{16}$$

where

$$I = H/J = (1 + 2R)^{1/m}. \tag{17}$$

The integral of Eq. (14) provides the solution of the problem; once the dependence of σ_e on α has been determined, the corresponding expressions for the stresses and the strains follow from (5) and (7). The spatial profiles of these quantities can be found by transforming, from parameter α to the radial coordinate r, through the relation

$$\frac{dr}{r} = -\frac{S_r' + S_r f(\alpha)}{S_r - S_\theta}\,d\alpha, \tag{18}$$

which is obtained from (12) with the aid of (5) and (14).

In order to solve (14) we need to know the boundary data. At infinity, where $\sigma_r = \sigma_\theta = \sigma_\infty$, we have from (3a) that $\alpha = 0$, while from (1) we find that $\sigma_e = (2/H)\sigma_\infty$. At the inner boundary, where $u = 0$, the circumferential strain ε_θ has to vanish or, from (8b),

$$\tan\alpha_a = I^{-m/(2m-2)}. \tag{19}$$

Thus, parameter α varies from zero at infinity to the value given by (19) at $r = a$. The stress points on the yield locus will therefore move along the corresponding heavy parts shown in Fig. 2. Note, however, that with $m = 1$ parameter α is identically equal to zero over the entire field, and the only operative part of the yield locus is the corner point $\sigma_r = \sigma_\theta = \sigma_\infty = (1 + R)\sigma_e$. The solution for this special case is different from the one given by (14), as will be described shortly.

A convenient measure of the stress field near the rigid ring is given by the stress concentration factor, defined as

$$k = \sigma_e(r = a)/\sigma_\infty. \tag{20}$$

Thus, integration of (14) from infinity to the inner boundary gives

$$k = \left(\frac{2}{H}\right)\exp\int_0^{\alpha_a} f(\alpha)\,d\alpha. \tag{21}$$

Illustrative numerical results obtained from (21) are shown in Fig. 3. Also shown are the reference curves $k = 2/H$, which give the ratio σ_e/σ_∞ for a uniform sheet under equibiaxial tension. Note that this is also the asymptotic value of (21) as n becomes very large. It may be concluded from Fig. 3 that the n-sensitivity of the local stress field near the ring increases with parameter m but decreases with parameter R.

When $m = 2$, definition (1) and its associated constitutive relations are reduced to the earlier anisotropic theory given by Hill [3]. The integral in (21) can then be expressed in a closed form, as observed already in [2], namely

$$k = \sqrt{\frac{2}{1+R}}\left[\frac{n+1+2R}{\sqrt{2(1+R)(1+2R)}}\right]^{(n+1+2R)/(n^2+1+2R)}$$
$$\cdot \exp\left[-\frac{(n-1)\sqrt{1+2R}}{n^2+1+2R}\arctan\frac{1}{\sqrt{1+2R}}\right]. \tag{22}$$

A further specification for the Mises material, with $R = 1$, reads

$$k = \left(\frac{n+3}{2\sqrt{3}}\right)^{(n+3)/(n^2+3)}\exp\left[-\frac{\pi(n-1)}{2\sqrt{3}\,(n^2+3)}\right]. \tag{23}$$

Analogous expressions for the stress concentration at a circular hole (free boundary) are given in [7], [6], and [5].

For materials with $m = 1$ the entire field is in an equibiaxial state of stress with the obvious result

$$k = 1/(1+R). \tag{24}$$

The analogous expression for the hole problem [7] is

$$k = \left(\frac{1}{1+R}\right)\left(\frac{n}{1+2R}\right)^{(1+2R)/(n-1-2R)}. \tag{25}$$

When $R = 0$ we obtain from (24) and (25) the stress concentration factors for the standard Tresca material.

The constitutive relations for the rigid-ring problem when $m = 1$ are not given by the usual normality rule since the whole field is at the corner regime of the yield locus. Instead we use the plastic work-equivalence relation, which leads here to the relation

$$\varepsilon_r + \varepsilon_\theta = \frac{1}{1+R}\varepsilon. \tag{26}$$

Inserting (9) and (10) in (26) results in a differential equation for the radial displacement, with the solution

$$u = \frac{(\sigma_\infty/\sigma_0)^n}{2(1+R)^{n+1}}\left(r - \frac{a^2}{r}\right). \tag{27}$$

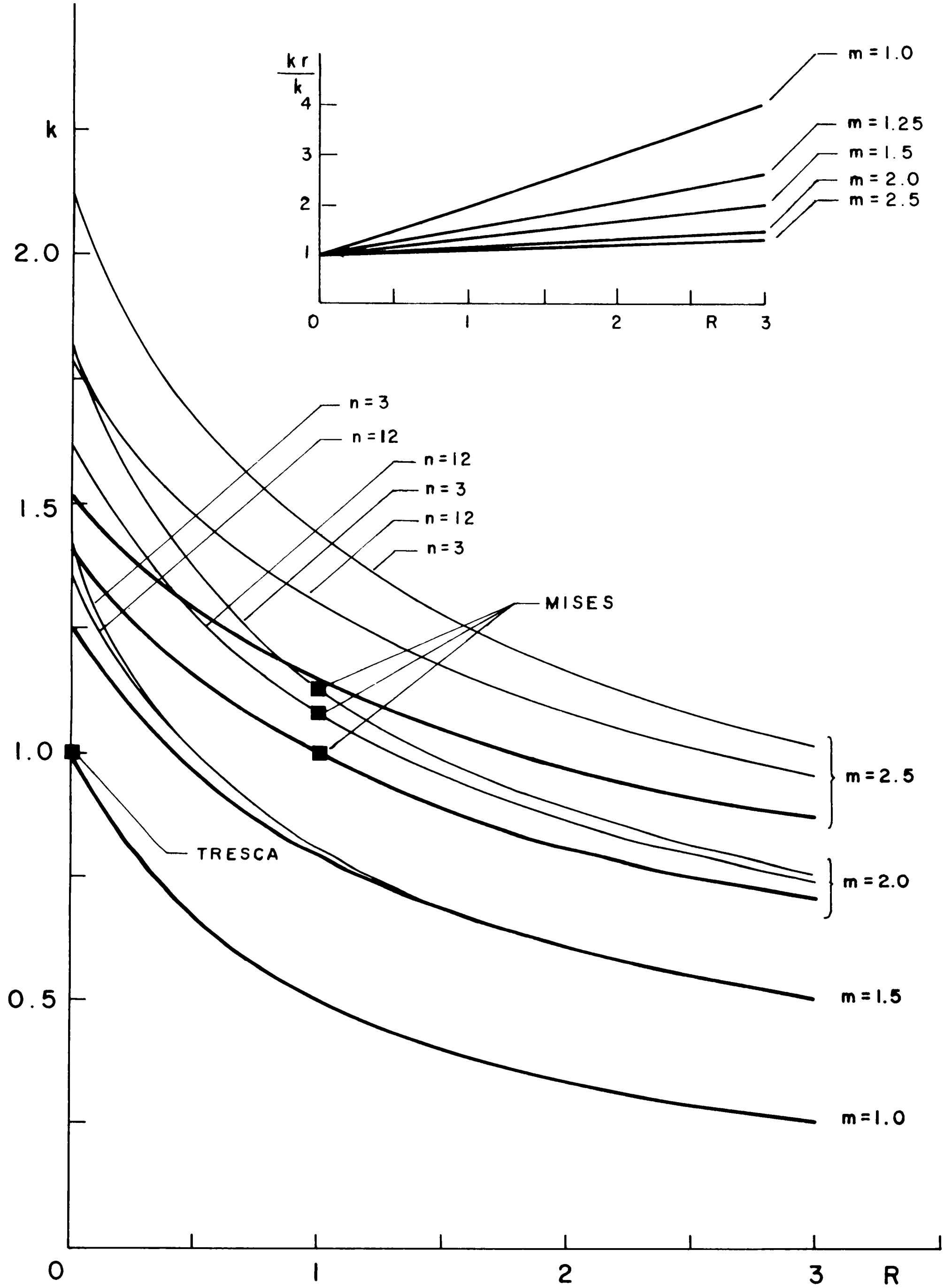

FIG. 3. Stress concentration factor for different values of the anisotropy parameters (m, R) and hardening parameter n. Note the results for the Mises material $(m = 2, R = 1)$ and the Tresca material $(m = 1, R = 0)$. The heavy lines show the reference level $k = 2/H$ for a uniform sheet. The insert shows k_r/k for different m and R.

It follows from (27) that the strain-rates ratio

$$\dot{\varepsilon}_\theta/\dot{\varepsilon}_r = (r^2 - a^2)/(r^2 + a^2) \tag{28}$$

remains bounded between 0 (at the hole) and 1 (at infinity).

Finally, we mention that the radial stress concentration factor defined as $k_r = \sigma_r(r = a)/\sigma_\infty$ is equal to $S_r(\alpha_a)k$ or, with the aid of (6a) and (19),

$$k_r = (J/2)(1 + I^{m/(m-1)})^{(m-1)/m}k. \tag{29}$$

Thus, the ratio k_r/k is independent of n. That ratio is always greater than one (Fig. 3), so that $k_r > k$. With $m = 1$ and $m = 2$ we get the simple relations

$$\left.\frac{k_r}{k}\right|_{m=1} = 1 + R, \qquad \left.\frac{k_r}{k}\right|_{m=2} = \frac{1 + R}{\sqrt{1 + 2R}}. \tag{30}$$

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