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ASYMPTOTIC ANALYSIS OF SURFACE WAVES DUE TO OSCILLATORY WAVE MAKER*

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Abstract. The initial value problem of surface waves generated by a harmonically oscillating vertical wave-maker immersed in an infinite incompressible fluid of finite constant depth is presented. The resulting motion is investigated using the generalized function method, and an asymptotic analysis for large times and distances is given for the free surface elevation.

1. Introduction. The classical problem of forced two-dimensional wave motion with outgoing surface waves at infinity generated by a harmonically oscillating vertical wave-maker immersed in water was solved by Havelock [1]. Rhodes-Robinson [2] reinvestigated the same problem, making allowance for the presence of surface tension. Pramanik [3] considered the initial value problem of waves generated by a moving oscillatory surface pressure against a vertical cliff and gave a uniform asymptotic analysis for the unsteady case. Debnath and Basu [4] treated the same problem taking into account the effect of surface tension.

In this paper we consider the transient development of two-dimensional linearized gravity waves generated by a harmonically oscillating vertical wave-maker immersed in a homogeneous incompressible inviscid fluid. With the help of an initial-value formulation and generalized function method developed by Debnath and Rosenblat [5], the integral representation of the surface elevation is obtained through an application of the Laplace and the generalized cosine Fourier transforms of the equations of motion. These integrals are then analyzed asymptotically for large time and distance. The transient waves are determined by the stationary phase method combined with the contour integration method.

2. Formulation. We are concerned with the transient development of two-dimensional surface waves produced by a harmonically oscillating wave-maker in a nonviscous incompressible fluid, neglecting any effect due to surface tension at the free surface of the fluid. If the motion is generated originally from rest by the oscillations of the wave-maker, it will be irrotational throughout all time and we may describe the motion in terms of a velocity potential $\phi(x, y; t)$. Take the origin O at the mean level of the free surface and the axis Oy pointing vertically downward along the

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wave-maker. Thus the region of the fluid is of semi-infinite horizontal extent. Let the fluid be bounded at some fixed finite constant depth h. The unsteady motions are generated in the fluid by the continuous oscillations of the wave-maker; let it oscillate horizontally with velocity $U(y;t)$ given by

$$U(y;t) = u(y)e^{i\omega t}H(t), \tag{2.1}$$

where $u(y)$ is an arbitrary function of y, ω is the frequency, and $H(t)$ is the unit step function.

The velocity potential satisfies an initial boundary value problem in which

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{2.2}$$

in the fluid region, $0 \le x < \infty$, $0 \le y \le h$, $t > 0$, with the bottom condition

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on } y = h,\ t > 0. \tag{2.3}$$

The linearized dynamic and kinematic conditions are

$$\left.\begin{array}{l} \frac{\partial\phi}{\partial t} = g\eta \\ \frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t} \end{array}\right\} \quad \text{on } y = 0,\ t > 0, \tag{2.4}$$

where $\eta = \eta(x;t)$ is the elevation of the free surface above its mean level and g is the acceleration due to gravity.

At the wave-maker,

$$\frac{\partial\phi}{\partial x} = U(y;t) \quad \text{on } x = 0,\ t > 0, \tag{2.5}$$

and the initial conditions are

$$\phi = \eta = 0, \quad \text{when } t = 0. \tag{2.6}$$

Also we suppose that ϕ, η are defined in the generalized sense of Lighthill [6].

3. Solution of the problem. We introduce the Fourier cosine transform with respect to x and the Laplace transform with respect to t as

$$\overline{F}_c(k, y; s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx\, dx \int_0^\infty e^{-st} F(x, y; t)\, dt, \tag{3.1}$$

where the subscript c and the bar in the transformed function refer to the cosine Fourier and Laplace transforms, respectively.

Application of (3.1) to the system (2.2)–(2.6) gives

$$\frac{d^2}{dy^2}\bar{\phi}_c - k^2\bar{\phi}_c = \sqrt{\frac{2}{\pi}}\overline{U}, \tag{3.2}$$

$$\frac{d}{dy}\bar{\phi}_c = 0 \quad \text{on } y = h, s > 0, \tag{3.3}$$

$$\left.\begin{array}{r} \bar{\phi}_c = \frac{g}{s}\bar{\eta}_c \\ \frac{d}{dy}\bar{\phi}_c = s\bar{\eta}_c \end{array}\right\} \quad \text{on } y = 0,\ s > 0. \tag{3.4}$$

The solution of (3.2) is

$$\bar{\phi}_c(k, y; s) = \overline{A}_c(k; s)e^{ky} + \overline{B}_c(k; s)e^{-ky} + \sqrt{\frac{2}{\pi}} \int_0^y k^{-1} \sinh k(y - \xi)\overline{U}(\xi; s)\, d\xi, \quad (3.5)$$

where $\overline{A}_c(k; s)$ and $\overline{B}_c(k; s)$ are functions to be determined.

The transformed boundary conditions (3.4) are satisfied if

$$\overline{A}_c(k; s) = \frac{gk + s^2}{2ks}\bar{\eta}_c, \qquad \overline{B}_c(k; s) = \frac{gk - s^2}{2ks}\bar{\eta}_c. \quad (3.6)$$

From (3.3), (3.5), and (3.6) we get

$$\sqrt{\frac{\pi}{2}}\bar{\eta}_c = -\frac{s}{s^2 + \alpha^2} \int_0^h \frac{\cosh k(h - \xi)}{\cosh kh} \overline{U}(\xi, s)\, d\xi, \quad (3.7)$$

$$\begin{aligned}\sqrt{\frac{\pi}{2}}\bar{\phi}_c = &\int_0^y k^{-1} \sinh k(y - \xi)\overline{U}(\xi; s)\, d\xi \\ &- k^{-1} \sinh ky \int_0^h \frac{\cosh k(h - \xi)}{\cosh kh} \overline{U}(\xi; s)\, d\xi \\ &- \frac{g \cosh k(h - y)}{(s^2 + \alpha^2) \cosh kh} \int_0^h \frac{\cosh k(h - \xi)}{\cosh kh} \overline{U}(\xi; s)\, d\xi, \end{aligned} \quad (3.8)$$

where $\alpha^2 = gk \tanh kh$.

The inverse Laplace and cosine Fourier transforms together with the convolution theorem for the Laplace transform give

$$\eta(x; t) = -\frac{2}{\pi} \int_0^\infty \cos kx\, dk \int_0^h \frac{\cosh k(h - \xi)}{\cosh kh}\, d\xi \int_0^t U(\xi; \tau) \cos \alpha(t - \tau)\, d\tau,$$

$$\begin{aligned}\phi(x, y; t) = &\frac{2}{\pi} \int_0^\infty \cos kx\, dk \left[\int_0^y k^{-1} \sinh k(y - \xi) U(\xi, t)\, d\xi \right. \\ &\left. -k^{-1} \int_0^h \frac{\cosh k(h - \xi)}{\cosh kh} U(\xi, t)\, d\xi \right] \\ &- \frac{2g}{\pi} \int_0^\infty \cos kx\, dk \int_0^h \frac{\cosh k(h - y) \cosh k(h - \xi)}{\cosh^2 kh}\, d\xi \\ &\times \int_0^t \alpha^{-1} \sin \alpha(t - \tau) U(\xi, \tau)\, d\tau. \end{aligned}$$

Now using the particular form of $U(y, t)$ as given in (2.1), we have

$$\eta(x; t) = -\frac{2}{\pi} \int_0^\infty \beta(k) \cos kx\, dk \int_0^t e^{i\omega\tau} \cos \alpha(t - \tau)\, d\tau,$$

$$\begin{aligned}\phi(x, y; t) = &\frac{2}{\pi} e^{i\omega t} \int_0^\infty [\gamma(k, y) - \beta(k) k^{-1} \sinh ky] \cos kx\, dk \\ &- \frac{2g}{\pi} \int_0^\infty \frac{\beta(k)}{\cosh kh} \cosh k(h - y) \cos kx\, dk \int_0^t \alpha^{-1} e^{i\omega\tau} \sin \alpha(t - \tau)\, d\tau, \end{aligned}$$

where

$$\beta(k) = \int_0^h u(\xi) \frac{\cosh k(h - \xi)}{\cosh kh}\, d\xi, \qquad \gamma(k, y) = \int_0^y k^{-1} \sinh k(y - \xi) u(\xi)\, d\xi,$$

i.e.,

$$\eta(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\beta(k)}{\alpha^2-\omega^2}[i\omega\cos\alpha t - \alpha\sin\alpha t - i\omega e^{i\omega t}]\cos kx\,dk, \tag{3.9}$$

$$\begin{aligned}\phi(x,y;t) = {} & \frac{2}{\pi}e^{i\omega t}\int_0^\infty [\gamma(k,y)-\beta(k)k^{-1}\sinh ky]\cos kx\,dk \\ & + \frac{2g}{\pi}\int_0^\infty \frac{\beta(k)\cosh k(h-y)}{\alpha(\alpha^2-\omega^2)\cosh kh}[\alpha\cos\alpha t + i\omega\sin\alpha t - \alpha e^{i\omega t}]\cos kx\,dk.\end{aligned} \tag{3.10}$$

4. Asymptotic behavior of $\eta(x;t)$ for large values of x and t. We are interested in the waves after a large time at a large distance. To investigate the principal feature of the wave motion, it suffices to work only with the free surface elevation function $\eta(x;t)$.

Write

$$\eta = I + J,$$

where

$$I = -\frac{2}{\pi}i\omega e^{i\omega t}\int_0^\infty \frac{\beta(k)}{\alpha^2-\omega^2}\cos kx\,dk, \tag{4.1}$$

$$J = \frac{2}{\pi}\int_0^\infty \frac{\beta(k)}{\alpha^2-\omega^2}(i\omega\cos\alpha t - \alpha\sin\alpha t)\cos kx\,dk. \tag{4.2}$$

The first integral represents the steady-state solution while the second represents the transient solution. It is convenient to rewrite (4.1) and (4.2) as follows:

$$I = \frac{i}{2\pi}e^{i\omega t}\sum_{n=1}^{2} I_n, \qquad J = \frac{i}{2\pi}\sum_{n=1}^{4} J_n,$$

where

$$I_1, I_2 = \mp\int_0^\infty \frac{\beta(k)}{\alpha\mp\omega}(e^{ikx}+e^{-ikx})\,dk,$$

$$J_1, J_2 = \int_0^\infty \frac{\beta(k)}{\alpha-\omega}e^{i(\omega t\pm kx)}\,dk,$$

$$J_3, J_4 = -\int_0^\infty \frac{\beta(k)}{\alpha+\omega}e^{-i(\omega t\mp kx)}\,dk.$$

We follow the method of Debnath and Rosenblat [5] to evaluate these wave integrals.

The main contribution to the asymptotic value of the above integrals for large t and x comes from the poles and stationary points of the integrands. It is noted that each of the integrals I_1, J_1, and J_2 contains one pole at $k = k_0$ where k_0 is the only real positive root of the equation

$$\sqrt{gk_0\tanh k_0 h} = \omega. \tag{4.3}$$

In addition, the integrals J_2 and J_3 contain one stationary point, at $k = k_1$, which is the root of the equation

$$\frac{d\alpha}{dk} = \frac{x}{t}. \tag{4.4}$$

It may be observed that the function $d\alpha/dk$ decreases monotonically from $\sqrt{gh}$ to 0 as k varies from 0 to ∞. Hence Eq. (4.4) has a real root k_1. On the other hand, the integrals I_2 and J_4 contain neither poles nor stationary points in the range of integration.

Now the contribution from the pole of the integral I_1 can be evaluated using the formula for the asymptotic development of the generalized Fourier transform developed by Lighthill [6]; that is, if $f(k)$ has a simple pole at $k = k_0$ in $a < k_0 < b$, then as $|x| \to \infty$,

$$\int_a^b f(k)e^{ikx}\,dk \sim i\pi \operatorname{sgn} x e^{ik_0x}(\text{residue of } f(k) \text{ at } k = k_0) + O\left(\frac{1}{|x|}\right). \tag{4.5}$$

Using this formula, it is easy to see that as $x \to \infty$

$$I \sim \frac{e^{i\omega t}\beta(k_0)}{2\alpha'(k_0)}(e^{ik_0x} - e^{-ik_0x}), \tag{4.6}$$

where $\alpha'(k_0)$ is the derivative of α at $k = k_0$.

The stationary phase method (Copson [7]) can be used to evaluate the transient component of J (that is, the contribution from the stationary points),

$$J_{\text{tr}} \sim \frac{i\beta(k_1)}{2\pi}\sqrt{\frac{2\pi}{t|\alpha''(k_1)|}}\left\{\frac{e^{i[t\alpha(k_1)-k_1x-\pi/4]}}{\alpha(k_1)-\omega} - \frac{e^{-i[t\alpha(k_1)-k_1x-\pi/4]}}{\alpha(k_1)+\omega}\right\} + O\left(\frac{1}{t}\right), \tag{4.7}$$

where J_{tr} denotes the transient part of J for large t.

Finally, we calculate the contribution to J from its polar singularity. This can easily be estimated by formula (4.5),

$$J_{\text{polar}} \sim -\frac{e^{i\omega t}\beta(k_0)}{2\alpha'(k_0)}(e^{ik_0x} + e^{-ik_0x}). \tag{4.8}$$

Write $\eta = \eta_{\text{st}} + \eta_{\text{tr}}$, where η_{st} is the steady-state solution and η_{tr} is the transient component. The first term in η is the polar contribution to I and J, which is given by

$$\eta_{\text{st}}(x, t) = \frac{-\beta(k_0)}{\alpha'(k_0)}e^{i(\omega t - k_0x)} + O\left(\frac{1}{x}\right) \tag{4.9}$$

and the transient solution η_{tr} is given by (4.7).

5. Asymptotic solution in the case of infinite depth. In the case of infinitely deep water, that is, when $h \to \infty$, the functions $\beta(k)$, $\alpha(k)$, the pole k_0, and the stationary point k_1 are all simpler in form and are given by

$$\beta(k) = \int_0^\infty u(\xi)e^{-k\xi}\,d\xi, \qquad \alpha(k) = \sqrt{gk},$$
$$k_0 = \omega^2/g, \qquad k_1 = gt^2/(4x^2).$$

Therefore, in this case the asymptotic solution for $\eta(x,t)$ can be obtained independently or from (4.7) and (4.9) by formally letting $h \to \infty$,

$$\eta_{\text{st}}(x;t) \sim \frac{-2\omega}{g}\beta(\omega^2/g)e^{i[\omega t - (\omega^2/g)x]}, \tag{5.1}$$

$$\eta_{\text{tr}}(x;t) \sim \frac{i}{2}\sqrt{\frac{g}{\pi}}\frac{t}{x^{3/2}}\beta\left(\frac{gt^2}{4x^2}\right)\left\{\frac{e^{i[gt^2/(4x)-\pi/4]}}{gt/(2x)-\omega} - \frac{e^{-i[gt^2/(4x)-\pi/4]}}{gt/(2x)+\omega}\right\}. \tag{5.2}$$

6. Conclusion. The above analysis reveals that the transient solution η_{tr} as given by (4.7) and (5.2) for liquids of constant finite depth and of infinite depth, respectively, decays rapidly to zero as time $t \to \infty$. Thus the ultimate steady state is established in the limit and is given by (4.9), (5.1). These solutions represent outgoing progressive waves propagating with phase velocity ω/k_0 and g/ω, respectively.

These results justify the use by previous authors such as Rhodes-Robinson [2] of the condition at infinity known as the Sommerfeld radiation condition when investigating steady-state harmonic surface wave problems. The application of this condition instead of the boundedness condition at infinity was necessary to render the solution unique.

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