QUARTERLY OF APPLIED MATHEMATICS

VOLUME XLVI DECEMBER · 1988 NUMBER 4

DECEMBER 1988, PAGES 593-603

STABILIZATION OF LINEAR SYSTEMS BY TIME-DELAY FEEDBACK CONTROLS II*

By

JIONGMIN YONG[1]

The University of Texas at Austin

Abstract. The notion of uniform r-stabilizability for linear autonomous systems is introduced. Some abstract characterizations and easy-to-check necessary conditions and sufficient conditions are given.

1. Introduction. We consider the following system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1.1}$$

where A is an $(n \times n)$-matrix, B is an $(n \times m)$-matrix, $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, and $t \in \mathbf{R}$. For given $r > 0$, we want to find a feedback control

$$u(t) = Kx(t - r), \tag{1.2}$$

which stabilizes (1.1), namely, the solution of

$$\dot{x}(t) = Ax(t) + BKx(t - r) \tag{1.3}$$

for any initial condition is asymptotically stable. This problem was initiated and discussed by the author in [11].

DEFINITION 1.1. Let $r > 0$ and system (1.1) (denoted by $[A, B]$) be given. System $[A, B]$ is said to be uniformly r-stabilizable if there exists a matrix K such that (1.3) (denoted by $[A, BK; r]$) is asymptotically stable. System $[A, B]$ is said to be uniformly r-stabilizable if there exists a matrix K such that $[A, BK; r']$ is asymptotically stable for all $r' \in [0, r]$.

The following result was proved in [11].

THEOREM 1.2. Suppose $[A, B]$ is completely controllable and

$$\sigma(A) \subseteq \mathbf{C}^- \cup \mathbf{C}^0, \tag{1.4}$$

where $\sigma(A)$ is the spectrum of $A, \mathbf{C}^- = \{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda < 0\}$ and $\mathbf{C}^0 = \{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda = 0\}$. Then, for any $r > 0$, $[A, B]$ is r-stabilizable. Suppose instead of (1.4), we have

$$\sigma(A) \subseteq \mathbf{C}^- \cup \mathbf{C}^0 \cup \{a\}, \tag{1.5}$$

*Received March 16, 1987.

[1]Current address: Department of Mathematics, Fudan University, Shanghai, China.

where $a > 0$, and the Jordan blocks of A corresponding to a are of order 1. Then, for any $r > 0$ with

$$ra < 1, \tag{1.6}$$

the system $[A, B]$ is r-stabilizable.

The purpose of this paper is to discuss the uniform r-stabilizability of $[A, B]$. We solve the problem for the case $\sigma(A) \subset \mathbf{C}^- \cup \{0\}$. For the general case, we give some abstract characterizations and some necessary conditions. Also, we study some of the two-dimensional cases. In some cases, we improve the result of Theorem 1.2 above.

2. Stabilization of polynomials–general case. For $n \geq 0$, we let

$$\mathscr{P}^n = \{z^n + p_{n-1}z^{n-1} + \cdots + p_0 \mid p_{n-1}, \ldots, p_0 \in \mathbf{R}\},$$
$$\mathscr{Q}^{n-1} = \{q_{n-1}z^{n-1} + \cdots + q_0 \mid q_{n-1}, \ldots, q_0 \in \mathbf{R}\}, \qquad \mathscr{Q}^{-1} \triangleq \{0\}.$$

Also, we denote

$$\mathbf{C}^- = \{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda < 0\},$$
$$\mathbf{C}^0 = \{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda = 0\},$$
$$\mathbf{C}^+ = \{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda > 0\}.$$

We introduce the following definitions.

Definition 2.1. An entire function $H(z)$ is said to be stable if

$$\mathscr{N}(H(z)) \equiv \{\lambda \in \mathbf{C} \mid H(\lambda) = 0\} \subset \mathbf{C}^-. \tag{2.1}$$

Definition 2.2. Let $r > 0$, $p(z) \in \mathscr{P}^n$.

(i) $p(z)$ is said to be r-stabilizable if there exists a $q(z) \in \mathscr{Q}^{n-1}$ such that $p(z) + q(z)e^{-rz}$ is stable.

(ii) $p(z)$ is said to be uniformly r-stabilizable if there exists a $q(z) \in \mathscr{Q}^{n-1}$ such that $p(z) + q(z)e^{-r'z}$ is stable for all $r' \in [0, r]$.

Proposition 2.3. Suppose $p(z) \in \mathscr{P}^n$.

(i) If $p(z)$ is stable, then for any $r > 0$, $p(z)$ is uniformly r-stabilizable.

(ii) If $p(z)$ is uniformly r-stabilizable, then $p(z)$ is r'-stabilizable for all $r' \in [0, r]$.

Proof. (i) Take $q(z) \equiv 0$. (ii) By definition. ■

From the above, we see that the only interesting case is that

$$\mathscr{N}(p(z)) \cap (\mathbf{C}^0 \cup \mathbf{C}^+) \neq \varnothing. \tag{2.2}$$

Our first result is related to Lemma 3.1 of [11].

Lemma 2.4. Suppose $p(z) \in \mathscr{P}^n$, $q(z) \in \mathscr{Q}^{n-1}$, $n \geq 0$, and

$$H(z; r') = e^{r'z}p(z) + q(z), \qquad r' \in [0, r].$$

Suppose $H(z; r')$ is stable for all $r' \in [0, r]$. Then there exists a $\hat{q} \in \mathbf{R}$ such that

$$H_1(z; r') \equiv zH(z; r') + \hat{q}$$

is stable for all $r' \in [0, r]$.

Proof. First we write

$$H(iy; r') = F(y; r') + G(y; r')i,$$

where $F(y; r')$ and $G(y; r')$ are real-valued functions. We claim that

$$\inf_{0 \le r' \le r} \{|y_0 G(y_0; r')| \, | F(y_0; r') = 0\} > 0. \tag{2.3}$$

Suppose it is not the case. Then there exist $\{r_m\} \subset [0, r]$, $y_m \in \mathbf{R}$ such that

$$\begin{cases} F(y_m; r_m) = 0, \\ \lim_{m \to \infty} |y_m G(y_m; r_m)| = 0. \end{cases}$$

Since one has

$$\lim_{y \to +\infty} |H(iy; r')| = +\infty, \quad \text{uniformly in } r' \in [0, r],$$

we can assume that $y_m \to y_0$, also $r_m \to r_0 \in [0, r]$. Then, by continuity, one has

$$F(y_0; r_0) = 0; \qquad y_0 G(y_0; r_0) = 0. \tag{2.4}$$

By the stability of $H(z; r_0)$, we have

$$F(0; r_0) = H(0; r_0) > 0.$$

Thus, $y_0 \neq 0$. Then we obtain from (2.4) that

$$H(iy_0; r_0) = F(y_0; r_0) + G(y_0; r_0)i = 0,$$

which contradicts the stability of $H(z; r_0)$. Thus, (2.3) holds. Then, as in [11], by taking $\hat{q}$ satisfying

$$0 < \hat{q} \le \tfrac{1}{2} \inf_{0 \le r' \le r} \{|y_0 G(y_0; r')| \, |F(y_0; r') = 0\}, \tag{2.5}$$

we see that $H_1(z; r') \equiv zH(z; r') + \hat{q}$ is stable for all $r' \in (0, r]$. We need to show that $zH_1(z; 0) + \hat{q}$ is also stable. In fact, if it is not stable, then the only possibility is that for some $\hat{y}_0 \in \mathbf{R}$,

$$\begin{aligned} 0 &= H_1(i\hat{y}_0; 0) + \hat{q} \\ &= \hat{q} + i\hat{y}_0(F(\hat{y}_0; 0) + G(\hat{y}_0; 0)i) \\ &= \hat{q} - \hat{y}_0 G(\hat{y}_0; 0) + i\hat{y}_0 F(\hat{y}_0; 0) \end{aligned}$$

since $\hat{q} > 0$, $\hat{y}_0 \neq 0$. Hence, $F(\hat{y}_0; 0) = 0$. But, then,

$$\begin{aligned} 0 < |\hat{y}_0 G(\hat{y}_0; 0)| = \hat{q} &\le \tfrac{1}{2} \inf_{0 \le r' \le r} \{|y_0 G(y_0; r')| \, |F(y_0; r') = 0\} \\ &\le \tfrac{1}{2} |\hat{y}_0 G(\hat{y}_0; 0)|, \end{aligned}$$

which is impossible. Hence, $zH_1(z; r') + \hat{q}$ is stable for all $r' \in [0, r]$. ∎

We should note that the above proof is based on a theorem of Pontryagin [9]. (See also [1], [4], [6], [11].) The original version of that result for our case was stated only for r being a positive integer. But it is easily seen that it is true for r being any positive number (see [6]). Also, we should note that in the above proof, we have to check for $r' = 0$ separately.

THEOREM 2.5. Let $p(z) = z^n$. Then, for any $r > 0$, $p(z)$ is uniformly r-stabilizable.

Proof. By induction. For $n = 0$, it is trivially true. Suppose the conclusion is true for $n = k$, i.e., there exists a $q(z) \in \mathscr{Q}^{k-1}$ such that $z^k + q(z)e^{-r'z}$ is stable for all $r' \in [0, r]$. This is equivalent to $H(z; r') \equiv z^k e^{r'z} + q(z)$ being stable for all $r' \in [0, r]$. Then, by Lemma 2.4, there exists a $\hat{q} \in \mathbf{R}$, such that

$$zH(z; r') + \hat{q} = z(z^k e^{r'z} + q(z)) + \hat{q} = z^{k+1}e^{r'z} + (zq(z) + \hat{q})$$

is stable for all $r' \in [0, r]$. We are done. ■

THEOREM 2.6. Let $p(z) \in \mathscr{P}^n$ with

$$\mathscr{N}(p(z)) \subset \mathbf{C}^- \cup \{0\}. \tag{2.6}$$

Then, for any $r > 0$, $p(z)$ is uniformly r-stabilizable.

Proof. Let $p(z) = z^m p_0(z)$ with

$$\mathscr{N}(p_0(z)) \subset \mathbf{C}^-.$$

Then, by Theorem 2.5, there exists a $q_0(z) \in \mathscr{Q}^{m-1}$ such that

$$\mathscr{N}(e^{r'z}z^m + q_0(z)) \subset \mathbf{C}^-, \qquad \forall r' \in [0, r].$$

Then, we let $q(z) = q_0(z)p_0(z) \in \mathscr{Q}^{n-1}$, and we have

$$\begin{aligned}\mathscr{N}(e^{r'z}p(z) + q(z)) &= \mathscr{N}(p_0(z)(z^m e^{r'z} + q_0(z))) \\ &= \mathscr{N}(p_0(z)) \cup \mathscr{N}(z^m e^{r'z} + q_0(z)) \subset \mathbf{C}^-, \qquad \forall r' \in [0, r]. \quad ■\end{aligned}$$

In [11], we proved that if

$$\mathscr{N}(p(z)) \subset \mathbf{C}^- \cup \mathbf{C}^0, \tag{2.7}$$

then for any $r > 0$, $p(z)$ is r-stabilizable (not necessarily uniform). The following result shows that (2.7) is not enough for $p(z)$ to be uniformly r-stabilizable for any $r > 0$.

THEOREM 2.7. Suppose $p(z) \in \mathscr{P}^n$ with $\mathscr{N}(p(z)) \cap \mathbf{C}^0 \neq \varnothing$ and

$$a \equiv \max\{|\lambda| \,|\, \lambda \in \mathscr{N}(p(z)) \cap \mathbf{C}^0\} > 0. \tag{2.8}$$

Suppose $p(z)$ is uniformly r-stabilizable. Then

$$ra < 2\pi. \tag{2.9}$$

Before proving this theorem, let us make some observations which will lead to some abstract characterizations of the uniform r-stabilizability.

Let $p(z) \in \mathscr{P}^n$, $q(z) \in \mathscr{Q}^{n-1}$, $r > 0$. We assume (2.2) holds. Then, by [5], we can see that

$$Y(p, q) \equiv \{y \geq 0 \,|\, |p(iy)| = |q(iy)|\} \neq \varnothing. \tag{2.10}$$

It is clear that $Y(p, q)$ is symmetric with respect to the origin, i.e., $y \in Y(p, q)$ iff $-y \in Y(p, q)$. We define

$$\varphi(y) = \arg(p(iy)), \quad \text{for } p(iy) \neq 0,$$
$$\psi(y) = \arg(q(iy)), \quad \text{for } q(iy) \neq 0.$$

For definiteness, we let $\varphi(y) = \varphi(y + 0)$, $\psi(y) = \psi(y + 0)$, for y at which $p(iy) = 0$ or $q(iy) = 0$.

LEMMA 2.8. $p(z)+q(z)e^{-r'z}$ is stable for all $r'\in[0,r]$ iff $p(z)+q(z)$ is stable and

$$\psi(y)+\pi-r'y-\varphi(y)\not\equiv 0\ (\mathrm{mod}\,2\pi),\qquad \forall y\in Y(p,q),\qquad \forall r'\in[0,r]. \tag{2.11}$$

Proof. Let

$$\delta(p,q;r')\stackrel{\Delta}{=}\max\mathrm{Re}\mathscr{N}(p(z)+q(z)e^{-r'z}). \tag{2.12}$$

Then, we see that $p(z)+q(z)e^{-r'z}$ is stable for all $r'\in[0,r]$ iff

$$\delta(p,q;r')<0,\qquad \forall r'\in[0,r]. \tag{2.13}$$

By [2], we know that $\delta(p,q;\cdot)$ is continuous on $[0,\infty)$. Thus, (2.12) is equivalent to the following: $p(z)+q(z)$ is stable and

$$\mathscr{N}(p(z)+q(z)e^{-r'z})\cap\mathbf{C}^0=\varnothing,\qquad \forall r'\in[0,r]. \tag{2.14}$$

The above means that

$$p(iy)\neq -q(iy)e^{-r'y},\qquad \forall y\in\mathbf{R}.$$

Then, the conclusion of our lemma follows. ■

We note that

$$\lim_{y\to\infty}\frac{|q(iy)|}{|p(iy)|}=0. \tag{2.15}$$

Thus, one has

$$\hat{y}=\max Y(p,q)\equiv\hat{y}(p,q)<\infty. \tag{2.16}$$

The following corollary is obvious (see (2.11)).

COROLLARY 2.9. If $p(z)+q(z)e^{-r'z}$ is stable for all $r'\in[0,r]$, then

$$r\hat{y}(p,q)<2\pi. \tag{2.17}$$

Proof of Theorem 2.7. By (2.8), we see that

$$ai\in\mathscr{N}(p(z)).$$

Thus, by (2.15), we see that

$$Y(p,q)\cap(a,\infty)\neq\varnothing.$$

Hence, $a<\hat{y}(p,q)$ and (2.9) follows from (2.17). ■

Next, we let

$$\mathscr{Q}_0^{n-1}(p)=\{q\in\mathscr{Q}^{n-1}\,|\,\delta(p,q;0)<0\}, \tag{2.18}$$

where $\delta(p,q;r')$ is defined in (2.12). Then, for any $q\in\mathscr{Q}_0^{n-1}(p)$, $0\notin Y(p,q)$ and

$$\psi(y)+\pi-\varphi(y)\not\equiv 0\,(\mathrm{mod}\,2\pi),\qquad \forall y\in Y(p,q).$$

Thus,

$$0<\theta(p,q)\equiv\min_{y\in Y(p,q)}\left\{\frac{1}{y}\left(\psi(y)+\pi-\varphi(y)-2\pi\left[\frac{\psi(y)+\pi-\varphi(y)}{2\pi}\right]\right)\right\}, \tag{2.19}$$

where $[x]$ is the greatest integer $\leq x$.

Lemma 2.10. Let $p(z) \in \mathscr{P}^n$ and let (2.2) hold. Let $q(z) \in \mathscr{Q}^{n-1}$. Then, $p(z) + q(z)e^{-r'z}$ is stable for all $r' \in [0, r]$ iff

$$\begin{cases} q(z) \in \mathscr{Q}_0^{n-1}(p), \\ r < \theta(p, q). \end{cases} \tag{2.20}$$

Proof. Sufficiency. If (2.20) holds, then

$$0 < ry < \psi(y) + \pi - \varphi(y) - 2\pi \left[\frac{\psi(y) + \pi - \varphi(y)}{2\pi}\right] < 2\pi, \qquad \forall y \in Y(p, q).$$

Hence (2.11) holds and Lemma 2.8 applies.

Necessity. If $p(z) + q(z)e^{-r'z}$ is stable for all $r' \in [0, r]$, then it is necessary that $q(z) \in \mathscr{Q}_0^{n-1}(p)$. Now, if $r \geq \theta(p, q)$, then there exists a $y_1 \in Y(p, q)$, $r_1 \in [0, r]$, such that

$$r_1 = \frac{1}{y_1}\left(\psi(y_1) + \pi - \varphi(y_1) - 2\pi\left[\frac{\psi(y_1) + \pi - \varphi(y_1)}{2\pi}\right]\right).$$

Thus,

$$\psi(y_1) + \pi - r_1 y_1 - \varphi(y_1) \equiv 0 \ (\mathrm{mod}\, 2\pi).$$

This implies $iy_1 \in \mathscr{N}(p(z) + q(z)e^{-r_1 z})$ and then $p(z) + q(z)e^{-r_1 z}$ is not stable. Hence (2.20) must be true. ■

Then, we have the following characterization of the uniform r-stabilizability.

Theorem 2.11. Let $p(z) \in \mathscr{P}^n$ and (2.2) hold. Then $p(z)$ is uniformly r-stabilizable iff

$$r < \sup_{q \in \mathscr{Q}_0^{n-1}(p)} \theta(p, q) \triangleq \theta(p). \tag{2.21}$$

Next, let us look at another characterization of the uniform r-stability of $p(z)$. Let $p(z) \in \mathscr{P}^n$, $q(z) \in \mathscr{Q}^{n-1}$ and let (2.2) hold. Then, by [2] and [5], we see that

$$R(p, q) = \min\{r \geq 0 \mid \delta(p, q; r) \geq 0\} \tag{2.22}$$

is well-defined. We let

$$R(p) = \sup_{q \in \mathscr{Q}^{n-1}} r(p, q). \tag{2.23}$$

Then we have the following:

Theorem 2.12. Let $p(z) \in \mathscr{P}^n$ and let (2.2) hold. Then $p(z)$ is uniformly r-stabilizable iff

$$r < R(p). \tag{2.24}$$

The proof is obvious.

By Theorems 2.11 and 2.12, we see that

$$\theta(p) = R(p), \tag{2.25}$$

provided $p(z) \in \mathscr{P}^n$ with (2.2).

3. Stabilization of polynomials of degree two. In this section, we will discuss some interesting cases for polynomials of degree two.

From [11], we know that $p(z) = z^2 - az$, with $a > 0$, is r-stabilizable if $ra < 1$. Unexpectedly, we have the following:

THEOREM 3.1. Let $p(z) = z^2 - az$, $a > 0$. Then for any $r > 0$, $p(z)$ is uniformly r-stabilizable.

Proof. Let $q_0 > 0$, $q_1 > a > 0$ be undetermined. Let

$$q(z) = q_1 z + q_0.$$

Then, it is clear that $q(z) \in \mathscr{Q}_0^1(p)$ (see (2.18)), namely $p(z) + q(z)$ is stable. Now, we consider the following equation:

$$|p(iy)| = |q(iy)|. \tag{3.1}$$

It is equivalent to

$$y^4 + (a^2 - q_1^2)y^2 - q_0^2 = 0.$$

Then, notice $q_0 > 0$ and $q_1 > a$, we see that $Y(p,q)$ (see (2.10)) only contains the following element:

$$y_0 = \sqrt{\frac{q_1^2 - a^2 + \sqrt{(q_1^2 - a^2)^2 + 4q_0^2}}{2}}.$$

It is clear that as $q_0 \to 0$, $q_1 \to a$, we have $y_0 \to 0$. Thus, for any $r > 0$, one can find $q_0 > 0$ and $q_1 > a$, such that

$$ry_0 < \tan^{-1} \frac{a}{y_0}. \tag{3.2}$$

We claim that for such a choice of q_0 and q_1, $p(z)+q(z)e^{-r'z}$ is stable for all $r' \in [0, r]$. In fact, let us define, for $y > 0$,

$$\varphi(y) \triangleq \arg(p(iy)) = \arg(-y^2 - ayi) = -\pi - \tan^{-1} \frac{a}{y} \in \left(-\pi, -\frac{\pi}{2} \right),$$
$$\psi(y) \triangleq \arg(q(iy)) = \arg(q_0 + q_1 yi) = \tan^{-1} \frac{q_1 y}{q_0}.$$

Then

$$\psi(y) + \pi - r'y - \varphi(y) = \tan^{-1} \frac{q_1 y}{q_0} + 2\pi - r'y + \tan^{-1} \frac{a}{y}.$$

Noting (3.2), we have, for all $r' \in [0, r]$, that

$$\begin{aligned} 0 &< \tan^{-1} \frac{q_1 y_0}{q_0} - ry_0 + \tan^{-1} \frac{a}{y_0} \le \tan^{-1} \frac{q_1 y_0}{q_0} - r'y_0 + \tan^{-1} \frac{a}{y_0} \\ &\le \tan^{-1} \frac{q_1 y_0}{q_0} + \tan^{-1} \frac{a}{y_0} < \pi. \end{aligned}$$

Thus,

$$\psi(y_0) + \pi - r'y_0 - \varphi(y_0) \not\equiv 0 \pmod{2\pi}, \qquad \forall r' \in [0, r].$$

Then, our conclusion follows from Lemma 2.8. ■

By Theorem 2.7, we see that if $p(z) = z^2 + a^2$ is uniformly r-stabilizable, then $ra < 2\pi$. Our next result gives a sufficient condition for $z^2 + a^2$ being uniformly r-stabilizable.

THEOREM 3.2. Let $p(z) = z^2 + a^2$, $a > 0$. Suppose $r > 0$ satisfying

$$ra < \pi/\sqrt{2}. \tag{3.3}$$

Then, $p(z)$ is uniformly r-stabilizable.

Proof. Let $q(z) = q_1 z + q_0$, $q_1 > 0$, $q_0 \in (-a^2, 0)$ be undetermined. Then, we see that $p(z) + q(z)$ is stable and in the present case, equation (3.1) is equivalent to

$$y^4 - (q_1^2 + 2a^2)y^2 + a^4 - q_0^2 = 0.$$

Thus, it has exactly two solutions on $[0, \infty)$, namely

$$y_\pm = \sqrt{\frac{q_1^2 + 2a^2 \pm \sqrt{(q_1^2 + 2a^2)^2 - 4(a^4 - q_0^2)}}{2}}, \qquad y_+ \in (a, \infty), \quad y_- \in (0, a). \tag{3.4}$$

Next, we let, for $y > 0$,

$$\varphi(y) \triangleq \arg(p(iy)) = \arg(a^2 - y^2) = \begin{cases} 0, & y \in [0, a), \\ \pi, & y \in (a, \infty). \end{cases}$$

$$\psi(y) \triangleq \arg(q(iy)) = \arg(q_0 + q_1 yi) = \pi - \tan^{-1} \frac{q_1 y}{|q_0|} \in \left(\frac{\pi}{2}, \pi\right].$$

Then, from (3.4), we see that as $q_0 \to -a^2$, $q_1 \to 0$, one has $y_+ \to \sqrt{2}a$, $y_- \to 0$. Thus, for $r < \pi/(\sqrt{2}a)$, there exist $q_1 > 0$ and $0 > q_0 > -a^2$, such that

$$r < \frac{1}{y_+}\left(\pi - \tan^{-1} \frac{q_1 y_+}{|q_0|}\right). \tag{3.5}$$

Then, under this choice of q_0, q_1, we have for all $r' \in [0, r]$,

$$\begin{aligned} \pi &> \psi(y_+) + \pi - r' y_+ - \varphi(y_+) \equiv \pi - \tan^{-1} \frac{q_1 y_+}{|q_0|} - r' y_+ \\ &\geq \pi - \tan^{-1} \frac{q_1 y_+}{|q_0|} - r y_+ > 0, \\ 2\pi &> \psi(y_-) + \pi - r' y_- - \varphi(y_-) \equiv 2\pi - \tan^{-1} \frac{q_1 y_-}{|q_0|} - r' y_- \\ &\geq 2\pi - \tan^{-1} \frac{q_1 y_-}{|q_0|} - r y_- \\ &> 2\pi - \tan^{-1} \frac{q_1 y_+}{|q_0|} - r y_+ > 0. \end{aligned}$$

Hence, we have

$$\psi(y_\pm) + \pi - r' y_\pm - \varphi(y_\pm) \not\equiv 0 \pmod{2\pi}, \qquad \forall r' \in [0, r].$$

Then, the theorem follows from Lemma 2.8. ■

THEOREM 3.3. Let $p(z) = z^2 - p_1 z + p_0$, $p_1 > 0$, $p_0 > 0$. Suppose $p(z)$ is uniformly r-stabilizable. Then

$$r\sqrt{p_0} < 2\pi. \tag{3.6}$$

Proof. Let $q(z) = q_1 z + q_0$ be such that $p(z) + q(z)e^{-r'z}$ is stable for all $r' \in [0, r]$. Then, it is necessary that $q_1 > p_1 > 0$ and $q_0 > -p_0$. In the present case, equation (3.1) is equivalent to

$$y^4 + (p_1^2 - 2p_0 - q_1^2)y^2 + p_0^2 - q_0^2 = 0.$$

Thus the largest solution of (3.1) on $(0, \infty)$ is

$$\begin{aligned} y_0(q_0, q_1) &\equiv \sqrt{\frac{q_1^2 - p_1^2 + 2p_0 + \sqrt{(q_1^2 - p_1^2)^2 + 4p_0(q_1^2 - p_1^2) + 4q_0^2}}{2}} \\ &> y_0(q_0, p_1) = \sqrt{p_0 + |q_0|} \geq \sqrt{p_0}. \end{aligned}$$

Hence, (3.6) follows from (2.9). ■

THEOREM 3.4. Let $p(z) = z^2 - a^2$, $a > 0$. Suppose $p(z)$ is uniformly r-stabilizable. Then

$$ra < \frac{\pi}{2\sqrt{2}}. \tag{3.7}$$

Proof. Let $q(z) = q_1 z + q_0$ be such that $p(z) + q(z)e^{-r'z}$ is stable for all $r' \in [0, r]$. Then, it is necessary that $q_1 > 0$ and $q_0 > a^2$. In the present case, equation (3.1) is equivalent to

$$y^4 + (2a^2 - q_1^2)y^2 + a^4 - q_0^2 = 0.$$

Thus, the largest root of it on $(0, \infty)$ is

$$\begin{aligned} y_0 &= \sqrt{\frac{q_1^2 - 2a^2 + \sqrt{(q_1^2 - 2a^2)^2 + 4q_0^2 - 4a^4}}{2}} \\ &\geq \sqrt{\frac{q_1^2 - 2a^2 + |q_1^2 - 2a^2|}{2}} \\ &= \sqrt{\max(q_1^2, 2a^2)} \geq \sqrt{2}a. \end{aligned} \tag{3.8}$$

On the other hand,

$$\begin{aligned} \varphi(y) &\equiv \arg(p(iy)) = \arg(-y^2 - a^2) \equiv \pi, \\ \psi(y) &\equiv \arg(q(iy)) = \tan^{-1}\frac{q_1 y}{q_0}, \qquad y \geq 0. \end{aligned}$$

Thus, by Lemma 2.8, for all $r' \in [0, r]$,

$$\psi(y_0) + \pi - r'y_0 - \varphi(y_0) = \tan^{-1}\frac{q_1 y_0}{q_0} - r'y_0 \not\equiv 0 \pmod{2\pi}.$$

This implies that

$$ry_0 < \tan^{-1}\frac{q_1 y_0}{q_0} < \frac{\pi}{2}.$$

Then, (3.7) follows from (3.8) and the above. ■

It is easy to see that all the polynomials we have discussed in this section satisfy (2.2). It seems to us that easy-to-check sufficient conditions for the uniform r-stabilizability of general polynomials (even of degree 2) are very hard to get. We will continue the investigation in future publications.

4. Feedback controls with time-delay. In this section, we will apply the results obtained in previous sections to linear autonomous systems. For simplicity, we only discuss single-input systems.

THEOREM 4.1. Suppose $[A, b]$ is completely controllable.

(i) Suppose

$$\sigma(A) \subseteq \mathbf{C}^- \cup \{0\}. \tag{4.1}$$

Then, for any $r > 0$, $[A, b]$ is uniformly r-stabilizable.

(ii) Suppose

$$\{0, a\} \subseteq \sigma(A) \subseteq \mathbf{C}^- \cup \{0\} \cup \{a\}, \tag{4.2}$$

with $a > 0$ being a simple eigenvalue of A. Then, for any $r > 0$, $[A, b]$ is uniformly r-stabilizable.

(iii) Suppose

$$\{\pm ai\} \subseteq \sigma(A) \subseteq \mathbf{C}^- \cup \{0\} \cup \{\pm ai\}, \tag{4.3}$$

with $a > 0$, and $\pm ai$ are simple eigenvalues of A and $r > 0$,

$$ra < \pi/\sqrt{2}. \tag{4.4}$$

Then $[A, b]$ is uniformly r-stabilizable.

Proof. Since $[A, b]$ is completely controllable, by [10], we know that $[A, b]$ has the following canonical representation:

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -p_0 & -p_1 & \cdots & -p_{n-2} & -p_{n-1} \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{4.5}$$

Then, if $k = (-q_0, -q_1, \dots, -q_{n-1})^{\mathrm{T}} \in \mathbf{R}^n$, one has that, for $r > 0$,

$$\det(zI - A - bk^{\mathrm{T}}e^{-rz}) = z^n + p_{n-1}z^{n-1} + \cdots + p_0 + (q_{n-1}z^{n-1} + \cdots + q_0)e^{-rz}.$$

Thus, we see that the (uniform) r-stabilizability of $[A, b]$ is equivalent to the (uniform) r-stabilizability of $p(z) = \det(zI - A)$. Hence, the results of §§2, 3 are applicable.

Thus, (i) follows from Theorem 2.6. To prove (ii), we notice that under our assumption, one has

$$p(z) \equiv \det(zI - A) = (z^2 - az)p_0(z),$$

where $\mathscr{N}(p_0(z)) \subseteq \mathbf{C}^- \cup \{0\}$. Then, for any $r > 0$, by Theorem 3.1, one has q_1, $q_0 \in \mathbf{R}$, such that

$$z^2 - az + (q_1 z + q_0)e^{-r'z} \equiv e^{-r'z}g(z; r')$$

is stable for all $r' \in [0, r]$. Then by the proof of Lemma 2.4 and Theorem 2.6, we get (ii). We can similarly prove (iii) by using Theorem 3.2 instead of using Theorem 3.1 in the proof of (ii). ■

THEOREM 4.2. Suppose $[A, b]$ is completely controllable. Let

$$a = \max\{|\lambda| \mid \lambda \in \sigma(A) \cap \mathbf{C}^0\}. \tag{4.6}$$

Suppose $[A, b]$ is uniformly r-stabilizable. Then

$$ra < 2\pi. \tag{4.7}$$

Proof. Using the representation (4.5), we can get our conclusion immediately by applying Theorem 2.7. ■

It is clear that we can state and prove some other necessary conditions for the uniform r-stabilizability of $[A, b]$ (corresponding to Theorems 3.3 and 3.4). Also, the corresponding results of the above for multi-input systems $[A, B]$ can also be stated and proved. We omit these details here. To close this section, we give the following

THEOREM 4.3. Suppose $[A, b]$ is completely controllable and

$$\{0, a\} \subseteq \sigma(A) \subseteq \mathbf{C}^- \cup \mathbf{C}^0 \cup \{a\}, \tag{4.8}$$

with $a > 0$ being a simple eigenvalue of A. Then, for any $r > 0$, $[A, b]$ is r-stabilizable.

This result improves Theorem 6.1 of [11] (stated as Theorem 1.2 in §1) in some sense.

Proof of Theorem 4.3. In the present case, we have, by using (4.5),

$$p(z) \equiv \det(zI - A) = (z^2 - az)p_0(z), \tag{4.9}$$

with $\mathcal{N}(p_0(z)) \subseteq \mathbf{C}^- \cup \mathbf{C}^0$. Then by Theorem 3.1, for any $r > 0$ there exist $q_1, q_0 \in \mathbf{R}$, such that $e^{rz}(z^2 - az) + q_1 z + q_0$ is stable. Then, using the technique of [11], we can get the r-stabilizability of $[A, b]$. ■

REFERENCES

[1] R. Bellman and K. L. Cooke, *Differential-Difference Equations*, Academic Press, New York, London, 1963

[2] R. Datko, *A procedure for determination of the exponential stability of certain differential-difference equations*, Quart. Appl. Math. **36**, 279–292 (1978)

[3] L. E. El'sgol'ts and S. B. Norkin, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*, Academic Press, New York, London, 1973

[4] J. K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, Berlin, 1977

[5] J. K. Hale, E. F. Infante and E. P. Tsen, *Stability in linear delay equations*, J. Math. Anal. Appl. **105**, 533–555 (1985)

[6] V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional Differential Equations*, Academic Press, London, 1986

[7] X.-J. Li, *On the absolute stability of systems with time lags*, Acta Math. Sinica **13**, 558–573 (1963); English transl. in Chinese Math. **4**, 609–626 (1963)

[8] J. M. Mahaffy, *A test for stability of linear differential delay equations*, Quart. Appl. Math. **40**, 193–202 (1982)

[9] L. S. Pontryagin, *On the zeros of some elementary transcendental functions*, Amer. Math. Soc. Transl., Ser. 2, **1**, 95–110 (1955)

[10] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 2nd Edition, Springer-Verlag, New York, 1979

[11] J. Yong, *Stabilization of linear systems by time-delay feedback controls*, Quart. Appl. Math. **45**, 377–388 (1987)