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NONLINEAR CAPILLARY-GRAVITY WAVES IN MAGNETIC FLUIDS*

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Abstract. The nonlinear wave propagation of capillary-gravity waves on the surface of a ferrofluid of finite depth is investigated by employing the method of multiple scales. The stability analysis reveals the existence of different regions of instability. We show that the nonlinear modulational instability cannot be suppressed by the application of a strong magnetic field, however strong it may be. The influence of the magnetic field is not only quite signficant but its effect is also different for different regions of stability.

1. Introduction. The study of wave propagation in magnetic fluids has drawn considerable interest during the last few years. Zelazo and Melcher [1] (see also references [2]–[5]) examined theoretically as well as experimentally the plane wave propagation for two superposed magnetic fluids in the presence of a tangential magnetic field, and demonstrated that the magnetic field exerts a stabilizing influence on waves. In their investigation of the nonlinear evolution of wave packets on the surface of a magnetic fluid, Malik and Singh [6] showed that the wave train solution of constant amplitude is unstable against modulation if the product of the group velocity rate and the nonlinear interaction coefficient is negative. Furthermore, the magnetic field has a stabilizing influence on the modulational instability for small wavenumbers. However, in the studies mentioned above, the magnetic fluid taken was of infinite depth.

In hydrodynamics, the nonlinear wave propagation of capillary-gravity waves on the surface of an ideal fluid of finite depth has been extensively studied by various authors [7]–[11]. The gravity waves are always shown to be unstable in deep liquids and, when the fluid depth is finite, the stability characteristics vary quite significantly.

In this presentation, we look into the nonlinear wave propagation of capillary-gravity waves on the surface of a magnetic fluid of finite depth. We show that for the moderate values of the magnetic field and the wavenumber, gravity waves are stable or unstable against modulations according as the product of the wavenumber k and the fluid depth b is smaller or greater than a critical value. If the magnetic field is increased appreciably, then there is no cutoff value of kb above which the instability

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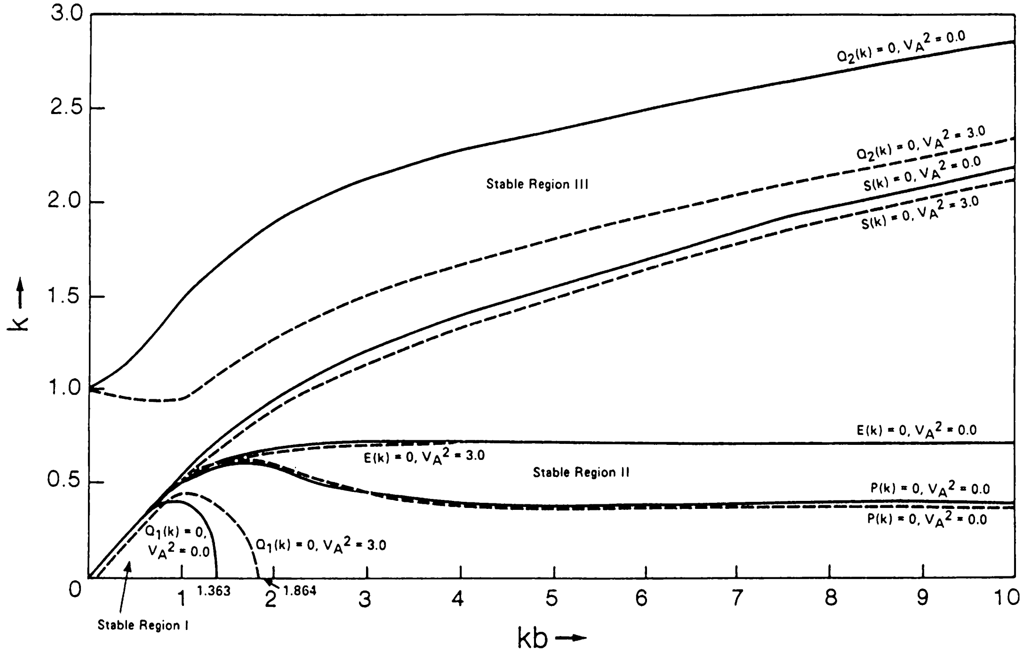

FIG. 1. Stability diagram showing different branches for $V_A^2 = 0.0$, 3.0 with $\mu = 1.5$.

can set in. These results are exhibited in Figs. 1–4. It is also interesting to observe that the higher values of the magnetic field generate two new regions of instability shown as regions R_1 and R_2 in Figs. 2 and 3.

In Secs. 2 and 3, we give the basic equations, review the linear theory, and derive the second-order solutions. The equation governing the evolution of the amplitude, obtained from the third-order solutions, is set out in Sec. 4. From this equation, we deduce the condition for modulational instability of nonresonant waves.

2. Formulation and linear analysis. We consider the finite amplitude two-dimensional capillary-gravity Stokes wave propagation on the interface $z = 0$ which separates the two magnetic fluids. The fluid with density ρ, depth b, and permeability μ_1 occupies the region $z < 0$, whereas the medium $z > 0$ is occupied by the fluid of magnetic permeability μ_2 and negligible density. This configuration is subjected to an applied magnetic $\mathbf{H}(1, 0, 0)$ along the fluid interface. The fluids are assumed to be inviscid, incompressible, and initially quiescent. The motion is assumed to be irrotational under gravity $\mathbf{g}(0, 0, -1)$.

The basic equation governing the perturbed velocity potential ϕ ($\mathbf{V} = \nabla\phi$) is

$$\nabla^2\phi = 0, \qquad -b < z < \eta(x, t), \tag{1}$$

where $z = \eta(x, t)$ is the elevation of the free surface. The perturbation produces an additional magnetic field, say $\mathbf{h}$, which we assume to be generated from a potential $\psi(x, z)$ ($\mathbf{h} = -\nabla\psi$), satisfying the equations

$$\nabla^2\psi^{(1)} = 0, \qquad -b < z < \eta(x, t), \tag{2}$$

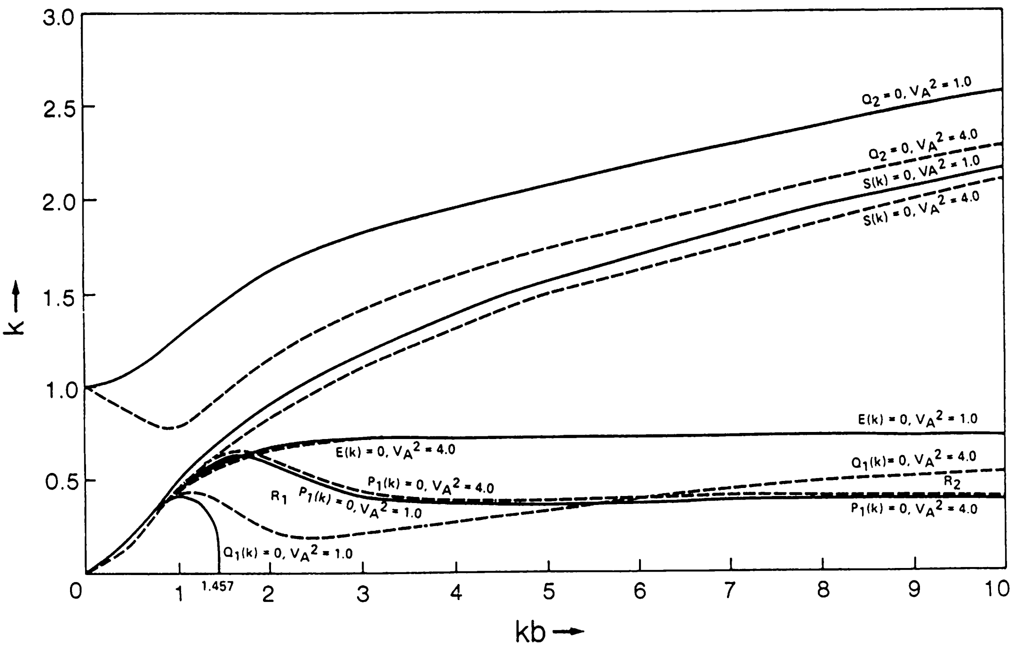

FIG. 2. Stability diagram showing various regions of stability for $V_A^2 = 1.0$, 4.0 with $\mu = 1.5$. R_1, R_2 are new regions of instability.

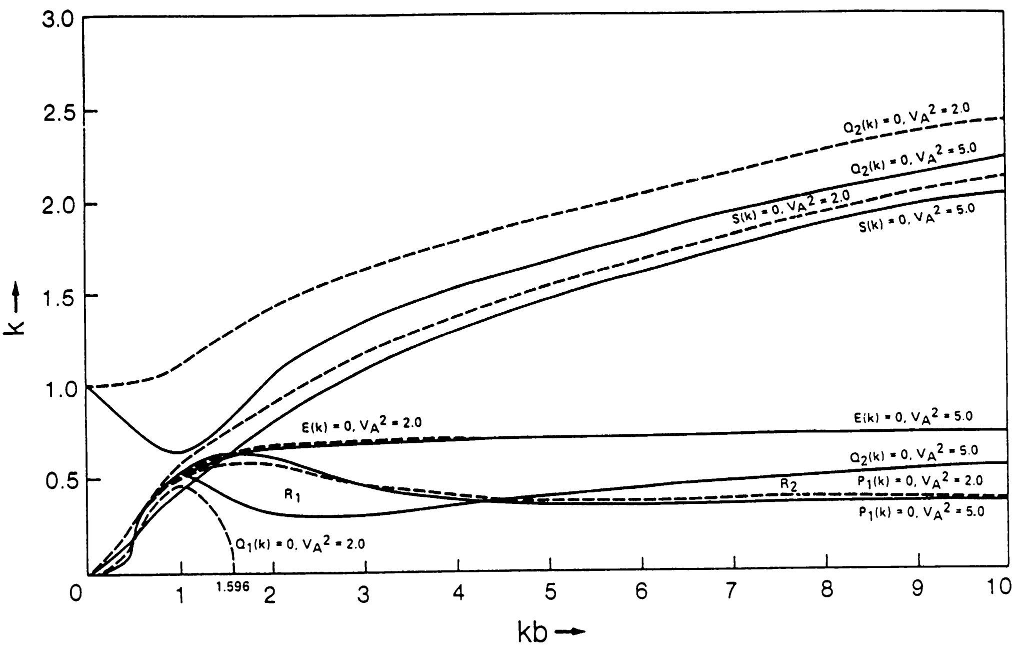

FIG. 3. Stability diagram showing regions of stability for $V_A^2 = 2.0$, 5.0 with $\mu = 1.5$. Here R_1, R_2 are new regions of instability.

and

$$\nabla^2\psi^{(2)} = 0, \qquad \eta(x,t) < z < \infty. \tag{3}$$

The velocity and magnetic potentials satisfy the conditions

$$|\nabla\phi| \to 0, \quad |\nabla\psi^{(1)}| \to 0 \qquad \text{as} \quad z \to -b, \tag{4}$$

$$|\nabla\psi^{(2)}| \to 0 \qquad \text{as} \quad z \to \infty. \tag{5}$$

The boundary conditions at the free interface $z = \eta(x,t)$ are

$$\frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial z} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} = 0, \tag{6}$$

$$\mu H_n^{(1)} = H_n^{(2)}, \tag{7}$$

$$H_T^{(1)} = H_T^{(2)}, \tag{8}$$

and

$$\frac{\partial\phi}{\partial t} + g\eta + \frac{1}{2}(\nabla\phi)^2 - \frac{T}{\rho}\frac{\partial^2\eta}{\partial x^2}\left(1 + \left(\frac{\partial\eta}{\partial x}\right)^2\right)^{-3/2} = \frac{(\mu-1)\mu_2}{8\pi\rho}\left[(H^{(1)})^2 + (\mu-1)(H_n^{(1)})^2\right], \tag{9}$$

where $\mu = \mu_1/\mu_2$ and where H_n and H_T represent the normal and the tangential components of the magnetic field, respectively [see Appendix]. We have confined ourselves to the magnetic fluids which exhibit linear magnetization properties.

To investigate the nonlinear interactions of small but finite amplitude waves, we apply the method of multiple scales. To that end, we introduce

$$x_n = \varepsilon^n x, \;\; t_n = \varepsilon^n t, \qquad n = 0, 1, 2, \tag{10}$$

and

$$\Phi(x,z,t) = \sum_{n=1}^{3} \varepsilon^n \Phi_n(z; x_0, x_1, x_2; t_0, t_1, t_2) + O(\varepsilon^4), \tag{11}$$

where Φ can be any of the physical quantities ϕ, ψ, η, and where ε represents a small parameter characterizing the steepness ratio of the wave. The boundary conditions (6)–(9) are prescribed at the interface $z = \eta(x,t)$. We, therefore, express all the physical quantities involved in terms of Maclaurin's series about $z = 0$. On using (10) and (11) into (1)–(9), and equating the coefficients of equal powers in ε, we obtain the linear as well as the successive higher-order equations. The hierarchy of the equations for each order can be obtained with the knowledge of the previous orders. The solution of the first-order problem in the form of progressive waves with respect to the lower scales x_0, t_0 is obtained as

$$\eta_1 = A\exp(i\theta) + \text{cc}, \tag{12}$$

$$\phi_1 = -i\omega k^{-1}\frac{\cosh k(z+b)}{\sinh(kb)}(A\exp(i\theta) - \overline{A}\exp(i\theta)) + \beta_1, \tag{13}$$

$$\psi_1^{(1)} = iBH\frac{\sinh k(z+b)}{\sinh(kb)}A\exp(i\theta) + \text{cc}, \tag{14}$$

$$\psi_1^{(2)} = iBH\exp(-kz)A\exp(i\theta) + \text{cc}, \tag{15}$$

where

$$B = \frac{(1-\mu)}{\mu\sigma(b)+1}, \tag{16}$$

$$\theta = kx_0 - \omega t_0, \tag{17}$$

with

$$\sigma(b) = \coth(kb). \tag{18}$$

The amplitude A is a function of x_1, x_2, t_1, t_2. The constant β_1 is assumed to be real and independent of the lower scales. The constant β_1, which allows us to eliminate secularities, is determined in Sec. 4. Lardner [12] has proved that the value of β_1 is zero for the propagation of capillary-gravity waves in deep water.

For the solution given by Eqs. (12)–(18) to be nontrivial, the frequency ω and the wavenumber k must satisfy the dispersion relation

$$D(\omega, k) = -\frac{\omega^2}{k}\sigma(b) + \frac{T}{\rho}k^2 + gk + \frac{(\mu-1)^2 k V_A^2}{(\mu\sigma(b)+1)} = 0, \tag{19}$$

where

$$V_A^2 = \frac{\mu_2 H^2}{4\pi\rho}. \tag{20}$$

The dispersion relation (19) was initially obtained, as well as experimentally confirmed, by Zelazo and Melcher [1] (see also Rosensweig [2]). However, to derive the equation for the evolution of travelling waves, we need to proceed to the second-order and higher-order problems.

3. Second-order problems. Following the procedure developed in [6], the nonsecularity condition for the second-order perturbation η_2 turns out to be

$$\frac{\partial A}{\partial t_1} + V_g \frac{\partial A}{\partial x_1} = 0, \tag{21}$$

where $V_g = d\omega/dk$ is the group velocity of the wave train. Equation (21) implies that A is constant along the characteristic $x_1 - V_g t_1 =$ constant. The uniformly valid solutions for the second-order problem now take the form

$$\begin{aligned}\phi_2 = & -\frac{\omega}{k}(z+b)\frac{\sinh k(z+b)}{\sinh(kb)}\frac{\partial A}{\partial x_1}\exp(i\theta) + \text{cc} \\ & + ip_2\frac{\cosh 2k(z+b)}{\sinh(2kb)}A^2\exp(2i\theta) + \text{cc} + \beta_2,\end{aligned} \tag{22}$$

$$\begin{aligned}\psi_2^{(1)} = & -\frac{i}{k}(z+b)BH\frac{\sinh k(z+b)}{\sinh(kb)}\frac{\partial A}{\partial x_1}\exp(i\theta) \\ & + iq_2 H\frac{\sinh 2k(z+b)}{\sinh(2kb)}A^2\exp(2i\theta) + \text{cc},\end{aligned} \tag{23}$$

$$\begin{aligned}\psi_2^{(2)} = & \frac{i}{k}(1+kz)BH\exp(-kz)\frac{\partial A}{\partial x_1}\exp(i\theta) \\ & + iq_3 H\exp(-2kz)A^2\exp(2i\theta) + \text{cc},\end{aligned} \tag{24}$$

$$\begin{aligned}\eta_2 = & \Lambda\exp(2i\theta) - i\left[\frac{1}{k}(1+kb\sigma(b)\frac{\partial A}{\partial x_1} + \frac{1}{\omega}\frac{\partial A}{\partial t_1}\right]\exp(i\theta) \\ & + \text{cc} + \xi_2(x_1, x_2; t_1, t_2),\end{aligned} \tag{25}$$

where

$$p_2 = -(\omega/k)(k\sigma(b) - \Lambda), \tag{26}$$

$$q_2 = \frac{(1-\mu)[\Lambda - k(\mu + \sigma(b))(1 + \mu\sigma(b))^{-1}]}{\mu\sigma(2b) + 1} \tag{27}$$

$$q_3 = q_2 + kB_1(\sigma(b) + 1)(1 - \mu), \tag{28}$$

$$\Lambda = \frac{1}{D(2\omega, 2k)}\left[\frac{\omega^2}{2}(3 - \sigma^2(b)) - 2\omega^2\sigma(2b)\sigma(b) + k^2(\mu - 1)(V_A^2/2)\{-4B(\mu + \sigma(b)) \times (\mu\sigma(2b) + 1)^{-1} + B^2(1 - \mu\sigma^2(b)) + (1 - \mu) + 2B(\sigma(b)(2 - \mu))\}\right], \tag{29}$$

$$\xi_2 = -\frac{1}{g}\left(\frac{\partial\beta_1}{\partial t_1} + 2q_1|A|^2\right), \tag{30}$$

$$q_1 = \frac{1}{2}[\omega^2(\sigma^2(b) - 1) + (\mu - 1)k^2V_A^2\{(\mu\sigma^2(b) + 1)B^2 + (\mu - 1) + 2B\mu\sigma(b)\}], \tag{31}$$

$$B_1 = [1 + \mu\sigma(b)]^{-1},$$

$$D(2\omega, 2k) = -\frac{2\omega^2}{k}\sigma(2b) + 4\frac{Tk^2}{\rho} + g + \frac{2kV_A^2(\mu - 1)^2}{\mu\sigma(2b) + 1}. \tag{32}$$

Here, we assume that $D(2\omega, 2k) \neq 0$. The case when $D(2\omega, 2k) = 0$ corresponds to the case of second harmonic resonance which can be dealt with along the same lines as outlined by Kant and Malik [13]. The value of the arbitrary constant β_2 in Eq. (22) can be determined from the higher-order equations. The quantity ξ_2 in Eq. (30) represents the induced mean motion or the zero frequency correction to show modulation of the fundamental mode.

4. Third-order solutions. In order to obtain the amplitude modulation for the travelling wave, we formulate the third-order problem. The expression for ξ_2 is obtained by considering the constant terms in Eq. (6) up to the third order. We get the following nonsecularity condition for η_3:

$$\frac{\partial}{\partial t_1}\xi_2 + b\frac{\partial^2\beta_1}{\partial x_1^2} + 2\omega\sigma(b)\frac{\partial|A|^2}{\partial x_1} = 0. \tag{33}$$

We assume the dependence of the quantities ξ_2, β_1, and A on x_1 and t_1 only. Substitution from (21) and (30) into (33) yields

$$\frac{\partial\beta_1}{\partial x_1} = (V_g^2 - gb)^{-1}(2q_1V_g + 2\omega g\sigma(b))|A|^2 + F(x_2, t_2), \tag{34}$$

where $F(x_2, t_2)$ can be determined by imposing suitable initial or boundary conditions. In addition, we require that there be no steady flow at infinity, implying thereby that

$$\frac{\partial\beta_1}{\partial x_1} \to 0, \qquad A = A_0, \qquad \text{as} \quad x_1 \to \infty. \tag{35}$$

This condition furnishes

$$F(x_2, t_2) = -(V_g^2 - gb)^{-1}(2\omega\sigma(b) + 2q_1V_g). \tag{36}$$

Equation (34) is valid only if we assume that $V_g^2 \neq gb$. When $V_g^2 = gb$, there are resonance interactions between the short capillary waves and the long gravity waves such that the group velocity of the short wave coincides with the phase velocity of the gravity wave. The analysis given in this paper is not valid in the vicinity of such a resonance phenomenon, and the resonant solution may be obtained by following the procedure outlined by Benney [14, 15]. However, we omit the function $F(x_2, t_2)$ since it can be eliminated by a simple transformation which has no effect on the character of the solution. The quantity ξ_2 in (33) representing the induced motion shall be furnished by substituting (34) and (36) into (30). It is interesting to remark here that if the fluid is of infinite depth, then ξ_2 vanishes.

On introducing (22)–(32) into the third-order perturbation problem, we obtain the nonsecularity condition:

$$i\frac{\partial A}{\partial \tau} + P\frac{\partial^2 A}{\partial \xi^2} = Q|A|^2 A, \tag{37}$$

$$x_2 - V_g t_2 = \xi, \qquad t_2 = \tau, \tag{38}$$

$$P = \frac{1}{2}\frac{dV_g}{dk}, \tag{39}$$

and the nonlinear interaction coefficient

$$\begin{aligned} Q = {} & \frac{-k}{2\omega\sigma(b)}[\omega^2 2(\Lambda - k\sigma(b))(1 - \sigma(b)\sigma(2b)) + \Lambda + \xi_2 - \tfrac{5}{2}k\sigma(b)] + \tfrac{3}{2}k^4 T \\ & - \omega^2\sigma(b)(\Lambda(2\sigma(2b) + \sigma(b)) + \xi_2\sigma(b) + (k/2)(3 - 4\sigma(b)\sigma(2b))) \\ & + \frac{k^2 V_A^2}{2}(\mu - 1)\Big[4q_2((\mu\sigma(b)(\sigma(2b) + 1)B + (\mu + 1)\sigma(2b)) \\ & + 2B\Lambda((2\mu - 1)\sigma(b) - 2(1 + \mu)) + 2B\xi_2\sigma(b) \\ & + kB\sigma(b)(4B(\mu + 1) + 3) + 4k(\mu - 1)B + \frac{2(l_1 - l_2)\sigma(b)}{(\mu\sigma(b) + 1)}\Big] \\ & - 2\omega\sigma(b)[(\sigma^2(b) - 1)\omega^2 V_g + 2\omega\sigma(b)](V_g^2 - b)^{-1}, \end{aligned} \tag{40}$$

with

$$l_1 = (\Lambda + \xi_2)(\mu - 1)^2 B_1 + 2(q_3 - \mu q_2) - \tfrac{3}{2}kB_1(1 - \mu), \tag{41}$$

$$l_2 = 2(q_2\sigma(2b) + q_3) + (\xi_2 - \Lambda)B(1 + \sigma(b)). \tag{42}$$

5. Discussion. Equation (39) describes the modulation of a one-dimensional weakly nonlinear dispersive wave which is governed by a nonlinear Schrödinger equation. It is well known [11] that the solutions of this equation are stable if and only if $PQ > 0$. All the physical quantities are normalized with respect to characteristic length $(T/(\rho g))^{1/2}$ and time $(T/(\rho g^3))^{1/2}$. Figures 1–3 are stability diagrams for capillary gravity waves showing the nondimensional wavenumber k versus kb for different values of the square of the normalized Alfven's velocity. The result of Kawahara [10] for the ideal hydrodynamic capillary-gravity waves can be obtained by setting $V_A^2 = 0$ in the nonlinear Schrödinger equation (37). If in Eq. (37) we take the limit when $kb \to \infty$, we recover the results obtained earlier by Malik and Singh [6] for capillary-gravity waves in ferrofluids of infinite depth. We should remark here that for ideal fluids in the absence of surface tension, the gravity waves are stable if $kb < 1.363$ for the shallow depth [11], and unstable otherwise. We show that the introduction of surface tension effect gives rise to many branches in the stability diagram (see Figs. 1–3). The different branches sketched in Figs. 1–3 for $\mu = 1.5$ correspond to the group velocity rate $P(k) = 0$ and the nonlinear interaction coefficient $Q(k) = 0$, respectively. The branches $E(k) \equiv D(2\omega, 2k) = 0$ and $S(k) = V_g^2 - gh = 0$ correspond to the second harmonic resonance and the capillary-gravity resonance, respectively, for the different values of V_A^2. The coefficient $Q(k)$ given by Eq. (40) becomes infinite in the vicinity of these curves. The analysis given in this paper then does not apply and the precise location of these branches in the stability diagrams cannot be assumed to be given accurately. It is to be noted that $Q(k)$ changes sign across these transition curves.

There are two other curves, $Q_1(k) = 0$, $Q_2(k) = 0$, across which $Q(k)$ changes sign. The branches corresponding to $S(k) = 0$ and $Q_2(k) = 0$ disappear from the stability diagram if $kb \to \infty$, i.e., the depth is infinite. Figure 1 shows the different branches for $V_A^2 = 0.0$ and $V_A^2 = 3.0$. There exist three stable regions: one bounded by the curves $Q_1(k) = 0$ and the kb-axis; second when $P(k) = 0$ and $E(k) = 0$; third when $S(k) = 0$ and $Q_2(k) = 0$. It is interesting to observe that introduction of the magnetic field increases the stability range of the region I, implying the stable influence of the magnetic field for small values of k. In the stable region II, the increase in intensity H of the magnetic field or increase in the permeability μ does not have a significant effect. The region III shrinks due to the application of the magnetic field, thereby indicating the destabilizing effect of the magnetic field. The regions II and III show similar behaviors if either the magnetic field or the magnetic permeability is increased still further (see Figs. 2 and 3). However, the transition branch $Q_1(k) = 0$ (see Fig. 4) depicts a very interesting behavior. This branch meets the kb-axis for the values of V_A^2 less than 3.5 yielding critical values of kb above which the instability shall set in. This critical value increases with increase of V_A^2 as long as the value of V_A^2 is less than 3.5. When $V_A^2 > 3.5$, this curve does not cross the kb-axis. Physically, when $V_A^2 = 0$, the value of kb is equal to 1.363 for Stokes waves. With the increase in V_A^2 up to the value 3.5, the value of kb increases with the increase of the magnetic field strength showing the stabilizing influence of the magnetic field. When V_A^2 is greater than 3.5, we get two more bands of instability shown in Figs. 2 and 3 as regions R_1 and R_2, respectively. When $V_A^2 = 4.0$, or 4.5, or 5.0, the regions R_1 and R_2 shrink to zero at $kb = 5.491$, 4.768, and 4.340, respectively. Figure 5 gives graphs: $Q_1(k) = 0$, $Q_2(k) = 0$, $E(k) = 0$, $S(k) = 0$, and $P(k) = 0$ for $kb = 2.0$ and for $kb = 5.0$.

We conclude that while the nonlinear modulational instability cannot be completely suppressed by increasing the magnetic field or using the ferrofluids of higher permeabilities, the presence of a magnetic field does greatly affect the range of stable wavenumbers.

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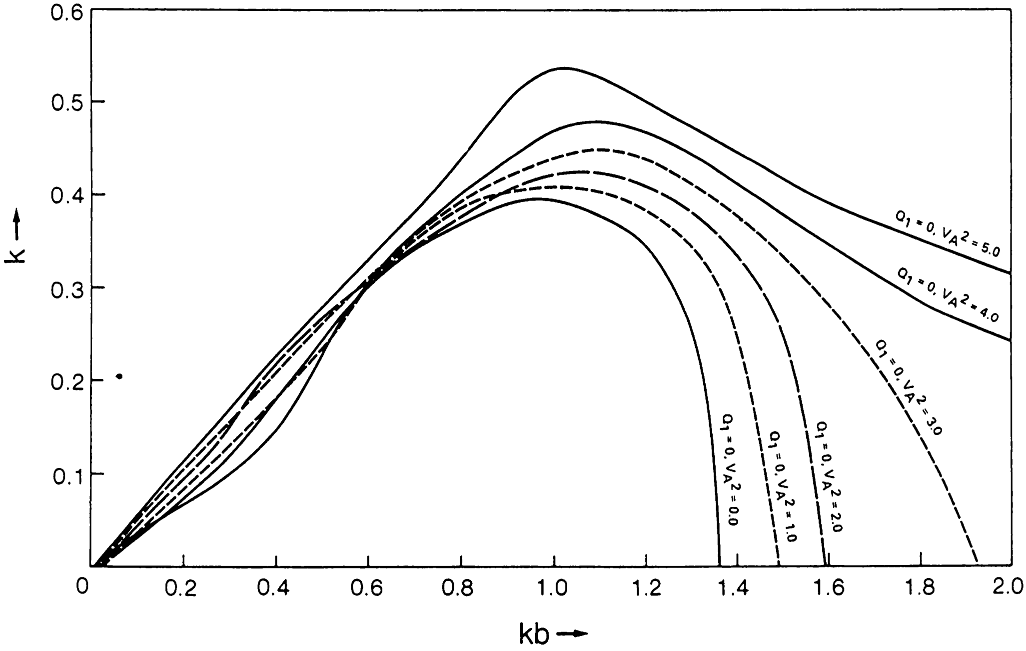

FIG. 4. The variation of nonlinear interaction coefficient $Q_1(k)$ against k and kb for $V_A^2 = 0.0$, 1.0, 2.0, 3.0, 4.0, and 5.0 with $\mu = 1.5$. The branch $Q_1(k)$ does not meet the b-axis for $V_A^2 = 4.0$ and 5.0.

Appendix. Derivation of the magnetic stress terms in Eq. (9). The dynamical condition that the normal stress should be continuous across the perturbed interface is

$$\left[\rho\frac{\partial\phi}{\partial t} + \frac{\rho}{2}(\nabla\phi)^2 + g\eta\rho - T\left(\frac{\partial^2\eta}{\partial x^2}\right)\left(1+\left(\frac{\partial\eta}{\partial x}\right)^2\right)^{-3/2}\right]n_i + T_{ij}^{(1)}n_j = T_{ij}^{(2)}n_j, \quad \text{(A.1)}$$

where $T_{ij}^{(l)}(l = 1, 2)$ represents the magnetic stress tensor given by (see Rosensweig [2], p. 112):

$$\mathbf{T}_{ij}^{(l)} = -\frac{\delta_{ij}}{4\pi}\int_0^H\left[\mu - \rho\left(\frac{\partial\mu}{\partial\rho}\right)_{H,T_0}\right]\mathbf{H}\,d\mathbf{H} + \frac{\mu\mathbf{HH}}{4\pi}, \qquad (l = 1, 2). \qquad \text{(A.2)}$$

Here, $\mu = \mu(H, T_0, \rho)$, where T_0 is the temperature and $\mathbf{n}$ is the outward drawn normal. We assume that the media is linear. In incompressible flow problems, the second term on the right-hand side of (A.2) can be absorbed into pressure or dropped with no effect on the fluid flow dynamics (see Rosensweig [2], p. 115). Finally,

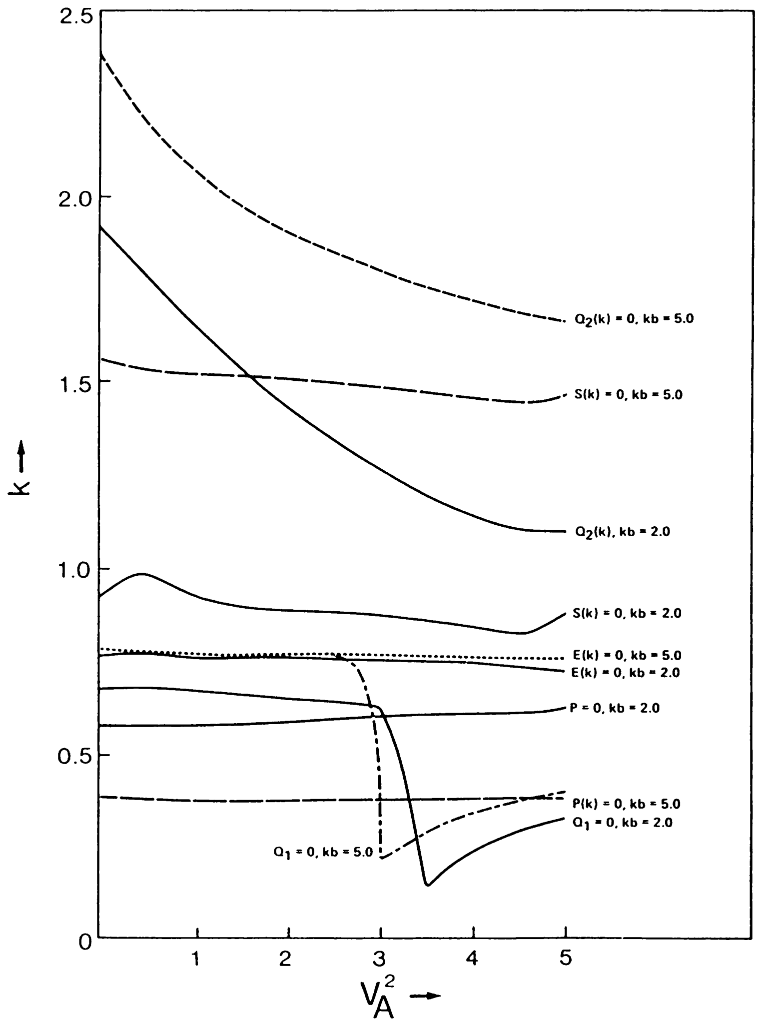

FIG. 5. The variation of different transition curves against k and V_A^2 for $kb = 2.0$ and 5.0 with $\mu = 1.5$.

substituting (A.2) into (A.1), we get

$$\rho\frac{\partial\phi}{\partial t} + \frac{\rho}{2}(\nabla\phi)^2 + g\eta\rho - T\left(\frac{\partial^2\eta}{\partial x^2}\right)\left(1 + \left(\frac{\partial\eta}{\partial x}\right)^2\right)^{-3/2} + \frac{\mu_1}{8\pi}(\mathbf{H}_n^{(1)^2} - \mathbf{H}_T^{(1)^2}) = \frac{\mu_2}{8\pi}(\mathbf{H}_n^{(2)^2} - \mathbf{H}_T^{(2)^2}). \qquad \text{(A.3)}$$

We now wish to express $\mathbf{H}^{(2)}$ in terms of $\mathbf{H}^{(1)}$ in Eq. (A.3). Using Eqs. (7)–(8) into (A.3) yields

$$\rho\frac{\partial\phi}{\partial t}+\frac{\rho}{2}(\nabla\phi)^2+g\eta\rho-T\left(\frac{\partial^2\eta}{\partial x^2}\right)\left(1+\left(\frac{\partial\eta}{\partial x}\right)^2\right)^{-3/2}$$
$$=\frac{(\mu_1-\mu_2)}{8\pi}\mathbf{H}_T^{(1)^2}+\frac{\mathbf{H}_n^{(1)^2}}{8\pi}\left(\frac{\mu_1^2}{\mu_2}-\mu_1\right)$$
$$=\frac{(\mu_1-\mu_2)}{8\pi}(\mathbf{H}_T^{(1)^2}+\mu\mathbf{H}_n^{(1)^2}), \tag{A.4}$$

where $\mu=\mu_1/\mu_2$.

Another equivalent form of (A.4) is

$$\rho\frac{\partial\phi}{\partial t}+\frac{\rho}{2}(\nabla\phi)^2+g\eta\rho-T\left(\frac{\partial^2\eta}{\partial x^2}\right)\left(1+\left(\frac{\partial\eta}{\partial x}\right)^2\right)^{-3/2}$$
$$=\mu_2\frac{(\mu-1)}{8\pi}(\mathbf{H}^{(1)^2}+(\mu-1)\mathbf{H}_n^{(1)^2}). \tag{A.5}$$

Let us consider a boundary between two ferrofluids. We shall confine to a special case by taking the magnetic permeability in media 2 to be equal to unity, i.e., $\mu_2=1$. Equation (A.5) then reduces to the form

$$\rho\frac{\partial\phi}{\partial t}+\frac{\rho}{2}(\nabla\phi)^2+g\eta\rho-T\left(\frac{\partial^2\eta}{\partial x^2}\right)\left(1+\left(\frac{\partial\eta}{\partial x}\right)^2\right)^{-3/2}$$
$$=\frac{(\mu-1)}{8\pi}(\mathbf{H}_T^{(1)^2}+\mu\mathbf{H}_n^{(1)^2}).$$
$$=\frac{(\mu-1)}{8\pi}(\mathbf{H}^{(1)^2}+(\mu-1)\mathbf{H}_n^{(1)^2}). \tag{A.6}$$

The equation (A.6) has been used by Shliomis [16], Malik and Singh [5]–[6], and Kant and Malik [4], [13].

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