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# ELASTO/PLASTIC STRAIN CONCENTRATION AT A CIRCULAR HOLE EMBEDDED IN AN ANISOTROPIC SHEET*

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**Introduction.** The influence of plastic orthotropy on the stress concentration factor at a circular hole, embedded in a thin sheet, has been discussed in a number of papers [1]-[6]. Much less research has apparently been devoted to the related problem of the strain concentration factor. It should be noted in this context that present day fatigue theories [7] give equal weight to both stress and strain concentration factors.

Budiansky [4] has presented results for the strain concentration factor in rigid/power-hardening materials with various theories of plastic anisotropy. That analysis, however, is of asymptotic value only since the neglect of elastic strains produces a constant (load independent) strain concentration factor. In reality one would expect a strain concentration factor that increases with the remote load beyond initial yielding.

This paper investigates the strain concentration problem for elastic/perfectly-plastic solids using Hill's [8] anisotropic yield criterion along with the associated flow rule. We show that within the framework of small strains it is possible to reduce the system of governing equations to a single differential equation of the first order. The solution of this equation can be expressed by quadratures, and in terms of simple functions when the orthotropy parameter $m$ takes the values of 1 or 2. For the latter cases we derive simple algebraic expressions for the strain concentration factors. The solution covers the entire elasto/plastic range up to complete yielding of the sheet.

Graphical and numerical examples indicate that plastic orthotropy, of the type considered here, has strong influence on the magnitude of strain concentration at the hole. It appears that the sensitivity to parameter $R$ increases as parameter $m$ decreases. These results are of particular interest due to the fact that the stress concentration factor is independent of both parameters. We also show that the present solution agrees with two other available solutions for elastic/hardening materials at the limit of zero hardening.

**Analysis.** A circular hole, of radius $a$, is located at the center of an infinite sheet which is subjected to remote uniform tension $\sigma_\infty$. The standard elastic solution for

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the pre-yielding stress field is simply

$$\sigma_r = \left(1 - \frac{1}{\rho^2}\right)\sigma_\infty, \qquad \sigma_\theta = \left(1 + \frac{1}{\rho^2}\right)\sigma_\infty, \tag{1}$$

where $\rho$ denotes the nondimensional (with respect to $a$) radial coordinate. The stress and strain concentration factors at the hole follow from (1), with the usual notation, as

$$k = \frac{\sigma_\theta(\rho = 1)}{\sigma_\infty} = 2 \tag{2}$$

and

$$k_\varepsilon = \frac{\varepsilon_\theta(\rho = 1)}{\varepsilon_z(\rho = \infty)} = \frac{1}{\nu} \tag{3}$$

where $\nu$ is Poisson's ratio.

Hill's [8] yield condition for plane stress fields with transverse material orthotropy states that

$$2(1 + R)Y^m = (1 + 2R)(\sigma_\theta - \sigma_r)^m + (\sigma_\theta + \sigma_r)^m \tag{4}$$

where $Y$ is the yield stress and $(m, R)$ are material parameters. As it stands, criterion (4) holds in the region where both $(\sigma_\theta - \sigma_r)$ and $(\sigma_\theta + \sigma_r)$ are nonnegative. This will include the elastic field (1) and also the plastic stresses to be determined subsequently.

With the stress components (1) we find that yielding starts at the hole ($\rho = 1$) when the load reaches the level

$$\sigma_\infty = \frac{Y}{2} \tag{5}$$

regardless of parameters $(m, R)$. Increasing the remote tension beyond initial yielding (5) will create a plastic zone bounded by the hole's boundary ($\rho = 1$) and the elasto/plastic interface ($\rho = \rho_i$). It is conceivable that the extent of the plastic zone will increase with $\sigma_\infty$ up to complete plastification of the sheet. This will happen when $\sigma_r = \sigma_\theta = \sigma_\infty$ at infinity satisfies Eq. (4), which gives the limit load

$$\sigma_\infty = (2 + 2R)^{1/m}\frac{Y}{2}. \tag{6}$$

The present elastic/perfectly-plastic solution is therefore valid only for loads below the limit (6). It is worth noting here that for both the von-Mises criterion ($m = 2, R = 1$) and the Tresca criterion ($m = 1, R = 0$) the plastic process is terminated (6) at the same tension $\sigma_\infty = Y$.

Once a plastic zone has been activated the circumferential stress at the hole remains constant and equal to $Y$. The stress concentration factor is then given trivially by

$$k = \frac{Y}{\sigma_\infty} \tag{7}$$

for any $(m, R)$ and with $\sigma_\infty$ varying from initial yield (5) to complete plastification (6).

Turning now to the more interesting problem, of determining the strain concentration factor, we begin by establishing the relation between the elastic/plastic interface location and the remote tension stress. This can be done quite easily since the stress analysis is here independent of kinematics. Within the elastic zone $\rho_i \leq \rho \leq \infty$ we have the known relations

$$\sigma_r = \sigma_\infty - \frac{A}{\rho^2}, \qquad \sigma_\theta = \sigma_\infty + \frac{A}{\rho^2} \tag{8}$$

where $A$ is a constant. Within the plastic zone $1 \leq \rho \leq \rho_i$ we introduce the parametric representation [3]

$$\sigma_r = S_r(\alpha)Y, \qquad \sigma_\theta = S_\theta(\alpha)Y \tag{9}$$

where

$$S_r(\alpha) = \frac{1}{2}(2+2R)^{1/m}[(\cos\alpha)^{2/m} - (1+2R)^{-1/m}(\sin\alpha)^{2/m}] \tag{10}$$

$$S_\theta(\alpha) = \frac{1}{2}(2+2R)^{1/m}[(\cos\alpha)^{2/m} + (1+2R)^{-1/m}(\sin\alpha)^{2/m}]. \tag{11}$$

This representation satisfies identically the anisotropic yield condition (4). The stress free boundary condition at the hole implies, via (10), that

$$\alpha(\rho = 1) = \alpha_a = \arctan\sqrt{1+2R} \tag{12}$$

Inserting relations (10)–(11) in the equations of radial equilibrium

$$\rho\frac{d\sigma_r}{d\rho} + \sigma_r - \sigma_\theta = 0 \tag{13}$$

results in, after one integration, the $\rho(\alpha)$ relation

$$\rho = \exp\int_\alpha^{\alpha_a} f(\alpha)\,d\alpha \tag{14}$$

where

$$f(\alpha) = \frac{1}{m}[(1+2R)^{1/m}(\cot\alpha)^{(2-m)/m} + \cot\alpha]. \tag{15}$$

For $m = 1$ integral (14) gives the closed form expression

$$\rho = \left(\frac{\sin\alpha_a}{\sin\alpha}\right)^{2+2R} \tag{16}$$

while for $m = 2$ we get

$$\rho = \sqrt{\frac{\sin\alpha_a}{\sin\alpha}}\exp\left[\frac{\sqrt{1+2R}}{2}(\alpha_a - \alpha)\right]. \tag{17}$$

Relation (17) agrees with Budiansky's result [2, eq. (22)] for a pure power law material at the limit of $n \to \infty$.

Stress continuity requirements at the elastic/plastic interface are satisfied, from (8) and (9), by the equations

$$\sigma_\infty - \frac{A}{\rho_i^2} = S_r(\alpha_i)Y, \qquad \sigma_\infty + \frac{A}{\rho_i^2} = S_\theta(\alpha_i)Y \tag{18}$$

where $\alpha_i$ is the value of parameter $\alpha$ at the interface. The solution of (18) follows, with the aid of (10)–(11), as

$$\cos\alpha_i = \frac{(2\sigma_\infty/Y)^{m/2}}{\sqrt{2+2R}} \tag{19}$$

$$A = \left(\frac{\rho_i^2}{2}\right)\left(\frac{2+2R}{1+2R}\right)^{1/m}(\sin\alpha_i)^{2/m}Y. \tag{20}$$

These two relations complete the stress analysis since the two stress components $(\sigma_r, \sigma_\theta)$ are now determined over the entire elastoplastic zone at any load level. The location of the elastic/plastic interface is obtained from (14) as

$$\rho_i = \exp\int_{\alpha_i}^{\alpha_a} f(\alpha)\,d\alpha \tag{21}$$

with $\alpha_i$ given by (19). For the particular values of $m = 1, 2$ we find from (16)–(17) the explicit relations

$$\rho_i = \left(\frac{1+2R}{2+2R-2\sigma_\infty/Y}\right)^{1+R} \qquad m = 1 \tag{22}$$

$$\rho_i = \left[\frac{1+2R}{2+2R-(2\sigma_\infty/Y)^2}\right]^{1/4}\exp\left[\frac{\sqrt{1+2R}}{2}\left(\arctan\sqrt{1+2R} - \arccos\frac{2\sigma_\infty/Y}{\sqrt{2+2R}}\right)\right] \qquad m = 2. \tag{23}$$

A further specification for the Tresca and Mises solids reads

$$\rho_i = \left(2 - \frac{2\sigma_\infty}{Y}\right)^{-1} \text{(Tresca: } m = 1,\ R = 0) \tag{24}$$

$$\rho_i = \left\{\frac{4}{3}\left[1 - \left(\frac{\sigma_\infty}{Y}\right)^2\right]\right\}^{-1/4}\exp\left[\frac{\sqrt{3}}{2}\left(\frac{\pi}{3} - \arccos\frac{\sigma_\infty}{Y}\right)\right] \quad \text{(Mises: } m = 2,\ R = 1). \tag{25}$$

As expected, $\rho_i$ becomes very large when $\sigma_\infty$ approaches the limit load (6).

The next part of the analysis employs the constitutive relation of the flow theory associated with the yield condition (4). Thus, within the usual framework of small strain plasticity in conjunction with the normality rule, we have the three equations

$$\dot\varepsilon_r = \frac{1}{E}(\dot\sigma_r - \nu\dot\sigma_\theta) + \dot\lambda[(S_\theta + S_r)^{m-1} - (1+2R)(S_\theta - S_r)^{m-1}] \tag{26}$$

$$\dot\varepsilon_\theta = \frac{1}{E}(\dot\sigma_\theta - \nu\dot\sigma_r) + \dot\lambda[(S_\theta + S_r)^{m-1} + (1+2R)(S_\theta - S_r)^{m-1}] \tag{27}$$

$$\dot\varepsilon_z = -\frac{\nu}{E}(\dot\sigma_r + \dot\sigma_\theta) - 2\dot\lambda(S_\theta + S_r)^{m-1} \tag{28}$$

where the superposed dot denotes differentiation with respect to a time-like parameter and $\lambda$ is left undetermined. Now, the stresses in the plastic zone $1 \le \rho \le \rho_i$ remain time independent at each radius after initial yielding. This is evident from the $\rho(\alpha)$ relation (14) which does not involve the load $\sigma_\infty$. The stresses (9) depend therefore only on the radius $\rho$, so that the stress rates in (26)–(27) vanish identically and we are left with the strain rates

$$\dot{\varepsilon}_r = \dot{\lambda}[(S_\theta + S_r)^{m-1} - (1+2R)(S_\theta - S_r)^{m-1}] \tag{29}$$

$$\dot{\varepsilon}_\theta = \dot{\lambda}[(S_\theta + S_r)^{m-1} + (1+2R)(S_\theta - S_r)^{m-1}] \tag{30}$$

$$\dot{\varepsilon}_z = -2\dot{\lambda}(S_\theta + S_r)^{m-1}. \tag{31}$$

Eliminating $\dot{\lambda}$ between (29)–(30) gives

$$\dot{\varepsilon}_r - g(\alpha)\dot{\varepsilon}_\theta = 0 \tag{32}$$

where, with the aid of (10)–(11),

$$g(\alpha) = \frac{1 - (1+2R)^{1/m}(\tan\alpha)^{2(m-1)/m}}{1 + (1+2R)^{1/m}(\tan\alpha)^{2(m-1)/m}}. \tag{33}$$

Equation (32) can be integrated (at a given radius parameter $\alpha$ is time independent) along the straining path, from the initial yield strains $(\varepsilon_r^*, \varepsilon_\theta^*)$ up to the current strains. This gives

$$\varepsilon_r - \varepsilon_r^* - g(\alpha)(\varepsilon_\theta - \varepsilon_\theta^*) = 0. \tag{34}$$

The yield strains, however, depend on the stresses (at a given radius) through the usual Hookean relations

$$\varepsilon_r^* = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) = \frac{Y}{E}[S_r(\alpha) - \nu S_\theta(\alpha)] \tag{35}$$

$$\varepsilon_\theta^* = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) = \frac{Y}{E}[S_\theta(\alpha) - \nu S_r(\alpha)] \tag{36}$$

where we have used representation (9) along with the observation that the stresses remain constant (at a given radius) after initial yielding. Substituting the strains (35)–(36) in (34) we get the equation

$$\varepsilon_r - g(\alpha)\varepsilon_\theta = h(\alpha) \tag{37}$$

where

$$h(\alpha) = \frac{Y}{E}[(S_r - \nu S_\theta) - g(\alpha)(S_\theta - \nu S_r)]. \tag{38}$$

Finally, we combine the compatibility condition

$$\varepsilon_r = \rho\frac{d\varepsilon_\theta}{d\rho} + \varepsilon_\theta \tag{39}$$

and the $\rho(\alpha)$ relation (14) with (37) to obtain the governing differential equation

$$\begin{aligned}\frac{d\varepsilon_\theta}{d\alpha} &- \left(\frac{2}{m}\right)(1+2R)^{1/m}(\tan\alpha)^{(m-2)/m}\varepsilon_\theta \\ &= \left(\frac{Y}{E}\right)\left(\frac{1}{m}\right)\left(\frac{2+2R}{1+2R}\right)^{1/m}(\sin\alpha)^{2/m}(\cot\alpha) \\ &\quad \cdot [(1+\nu) - (1-\nu)(1+2R)^{2/m}(\tan\alpha)^{2(m-2)/m}].\end{aligned} \tag{40}$$

Equation (40) is supplemented by the two boundary conditions

$$\alpha = \alpha_a \colon \varepsilon_\theta = \left(\frac{2\nu\sigma_\infty}{E}\right) k_\varepsilon \tag{41}$$

$$\alpha = \alpha_i \colon \varepsilon_\theta = \frac{Y}{E}[S_\theta(\alpha) - \nu S_r(\alpha)] \tag{42}$$

with $\alpha_i$ given by (19). The solution of system (40)–(42) provides the dependence of the strain concentration factor on the remote tension.

**Numerical examples and discussion.** Equation (40) is of first order and can be solved by quadratures. For the special cases of $m = 1, 2$ it is possible to express the solution in terms of simple functions. With $m = 1$ Eq. (40) becomes

$$\begin{aligned}\frac{d\varepsilon_\theta}{d\alpha} &- 2(1+2R)(\cot\alpha)\varepsilon_\theta \\ &= \left(\frac{Y}{E}\right)\left(\frac{2+2R}{1+2R}\right)(\sin\alpha\cos\alpha)[(1+\nu) - (1-\nu)(1+2R)^2(\cot\alpha)^2]\end{aligned} \tag{43}$$

with the solution

$$\begin{aligned}\varepsilon_\theta = &\left(\frac{Y}{E}\right)\left(\frac{2+2R}{1+2R}\right)\left[\left(\frac{1-\nu}{2}\right)(1+2R) - \frac{(1+\nu)+(1-\nu)(1+2R)^2}{4R}(\sin\alpha)^2\right] \\ &+ C(\sin\alpha)^{2(1+2R)}\end{aligned} \tag{44}$$

where $C$ is an integration constant. Compliance with boundary conditions (41)–(42) gives the strain concentration factor

$$k_\varepsilon = \left(\frac{Y}{4\nu R\sigma_\infty}\right)\left[\frac{(1+2R)^{1+2R}}{(2+2R-2\sigma_\infty/Y)^{2R}} - 1\right]. \tag{45}$$

At the limit of $R \to 0$ we find from (45) the Tresca-solid solution

$$k_\varepsilon = \left(\frac{Y}{2\nu\sigma_\infty}\right)\left\{1 - \ln\left[2\left(1 - \frac{\sigma_\infty}{Y}\right)\right]\right\}. \tag{46}$$

Several examples of the variation of $k_\varepsilon$ with the remote tension are shown in Fig. 1. The strain concentration factor becomes unbounded as the limit load (6) is approached, but the present small strain model is valid only within a limited range of $k_\varepsilon$.

With $m = 2$ Eq. (40) becomes

$$\frac{d\varepsilon_\theta}{d\alpha} - \sqrt{1+2R}\,\varepsilon_\theta = \left(\frac{Y}{E}\right)\sqrt{\frac{2+2R}{1+2R}}[\nu(1+R) - R]\cos\alpha \tag{47}$$

with the solution

$$\varepsilon_\theta = \left(\frac{Y}{E}\right)\sqrt{\frac{2+2R}{1+2R}}\left(\frac{\nu(1+R)-R}{2+2R}\right)(\sin\alpha - \sqrt{1+2R}\cos\alpha) + C\exp(\sqrt{1+2R}\,\alpha). \tag{48}$$

The strain concentration factor follows as

$$k_\varepsilon = \left(\frac{Y}{2\nu\sigma_\infty}\right)\frac{\exp[\sqrt{1+2R}(\alpha_a - \alpha_i)]}{\sqrt{2+2R}}(\cos\alpha_i + \sqrt{1+2R}\sin\alpha_i) \tag{49}$$

where $\alpha_i$ is given by (19) with $m = 2$. Sample calculations of $k_\varepsilon$ from (49) are depicted in Fig. 2. Here, in contrast to the case $m = 1$, the strain concentration factor attains a finite value at the limit load $\sigma_\infty/Y = \sqrt{2+2R}/2$. This is given by

$$k_\varepsilon = \frac{\exp[\sqrt{1+2R}\arctan\sqrt{1+2R}]}{\nu(2+2R)} \tag{50}$$

with the particular value of $k_\varepsilon = [\exp(\pi/\sqrt{3})]/4\nu$ for the Mises material ($R = 1$).

TABLE 1. Strain concentration factors for $m = 1.5$.

| $R$ \ $\sigma_\infty/Y$ | 0.5 | 0.6 | 0.7 | 0.8 | 1.0 | 1.2 | 1.4 | 1.6 |
|---|---|---|---|---|---|---|---|---|
| 0 | 3.00 | 3.06 | 3.25 | | | | | |
| 1 | 3.00 | 3.06 | 3.21 | 3.46 | 4.41 | 6.75 | | |
| 2 | 3.00 | 3.05 | 3.20 | 3.45 | 4.23 | 5.70 | 8.53 | 17.3 |

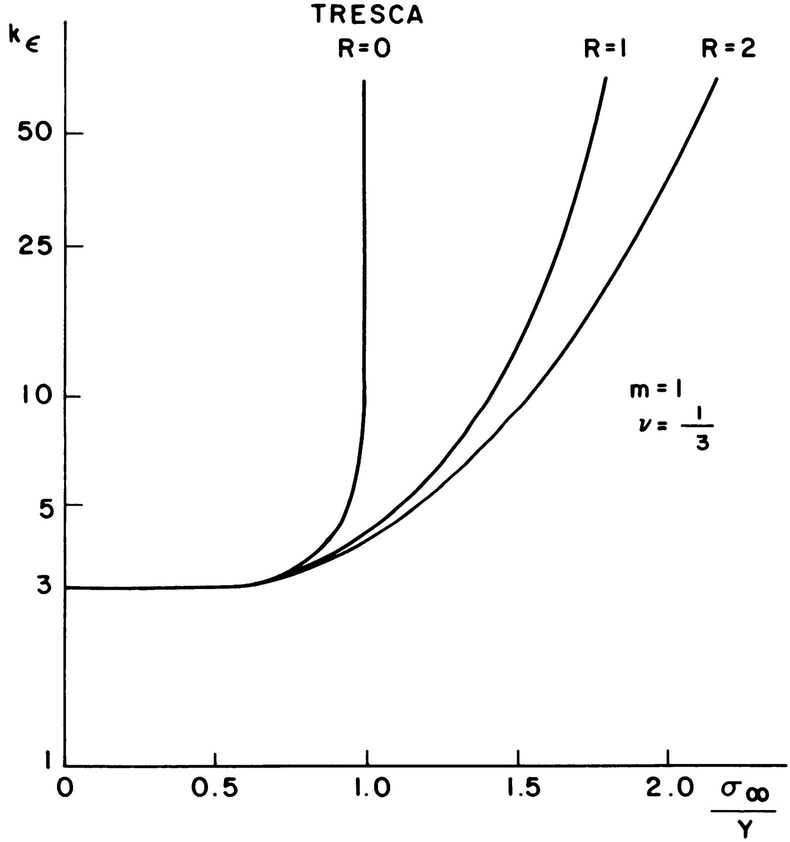


FIG. 1. Variation of the strain concentration factor with the remote tension stress for $m = 1$. The Tresca material is given by $R = 0$.

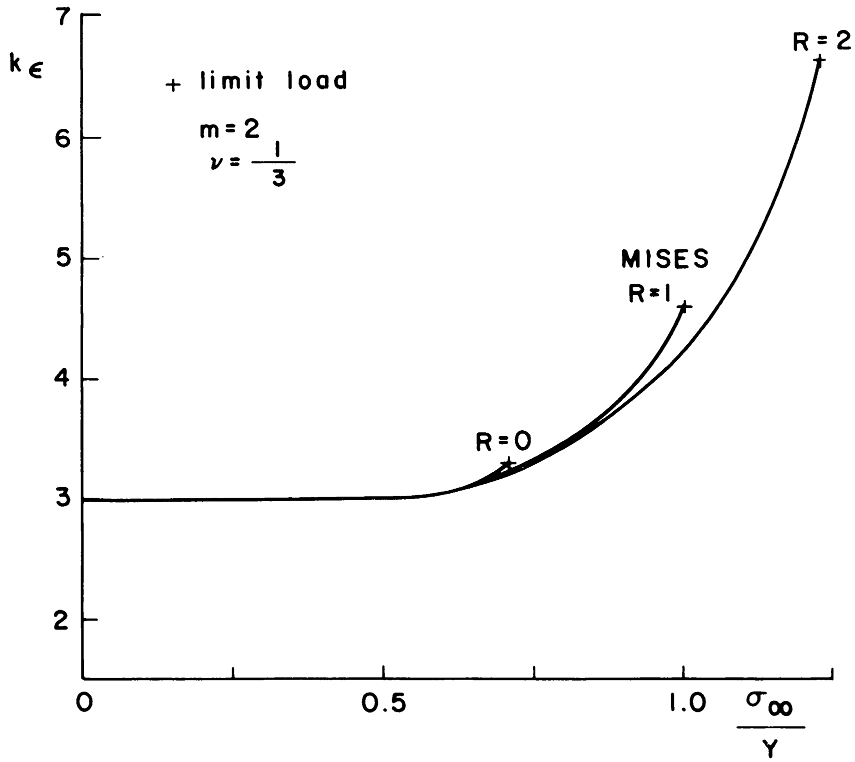


FIG. 2. Variation of the strain concentration factor with the remote tension stress for $m = 2$. The Mises material is given by $R = 1$.

Figures 1–2 display the importance of plastic orthotropy in evaluating strain concentration factors. It appears that the sensitivity to parameter $R$ increases as parameter $m$ decreases. It is also seen that the values of $k_\varepsilon$ are much higher with $m = 1$ than with $m = 2$. These findings are supported by results of numerical integration of Eq. (40), for intermediate values of $m$, as given in Tables 1–2. It is interesting to note that while the strain concentration factor is considerably dependent on plastic orthotropy, the stress concentration factor (7) is unaffected by parameters $(m, R)$.

TABLE 2. Strain concentration factors for $R = 1$.

| $m$ \ $\sigma_\infty/Y$ | 0.5 | 0.6 | 0.7 | 0.8 | 1.0 | 1.2 | 1.4 | 1.6 |
|---|---|---|---|---|---|---|---|---|
| 1.0 | 3.00 | 3.06 | 3.21 | 3.46 | 4.31 | 5.97 | 9.51 | 19.3 |
| 1.25 | 3.00 | 3.06 | 3.21 | 3.46 | 4.34 | 6.25 | 11.7 | |
| 1.50 | 3.00 | 3.06 | 3.21 | 3.46 | 4.41 | 6.75 | | |
| 1.75 | 3.00 | 3.06 | 3.21 | 3.48 | 4.45 | | | |
| 2.0 | 3.00 | 3.06 | 3.22 | 3.48 | 4.49 | | | |

The present study is restricted to perfectly-plastic behaviour beyond initial yield. An analogous investigation for hardening solids would be mainly numerical with the exception of two notable cases; with $m = 2$ and for a power-law hardening material Budiansky discovered [2] an exact solution for the stress concentration factor. Within the loading range bounded by (5)–(6) the expression for $k$ reads (eq. (20) in [2])

$$k = \left(\frac{Y}{\sigma_\infty}\right)\left[\frac{(n+1+2R)+(n-1)\sqrt{1+2R}\sqrt{\frac{1+R}{2}\left(\frac{Y}{\sigma_\infty}\right)^2-1}}{n(1+R)(Y/\sigma_\infty)}\right]^{(n+1+2R)/(n^2+1+2R)} \cdot \exp\left[\frac{(n-1)\sqrt{1+2R}}{n^2+1+2R}(\alpha_a-\alpha_i)\right] \tag{51}$$

where $n$ is the hardening parameter and $\alpha_i$ is given by (19) with $m = 2$. It is now straightforward to show that the strain concentration factor associated with (51) is simply

$$k_\varepsilon = \frac{k}{2\nu}\left(k\frac{\sigma_\infty}{Y}\right)^{n-1}. \tag{52}$$

At the limit load $\sigma_\infty/Y = \sqrt{2+2R}/2$ we recover from (51)–(52), with $n \to \infty$, the same formula as (50).

The second known solution for elastic/hardening solids has been given recently in [6] for the case of $m = 1$ and with a linear-hardening characteristic. The stress concentration factor is determined in [6] by the solution of a transcendental equation, but at the limit load we have the simple relation

$$k = 1 + \sqrt{(1-\eta)\left(\frac{R}{1+R}\right)^2 + \eta} \tag{53}$$

where $\eta = E_T/E$ and $E_T$ denotes the constant tangent modulus. It is easily verified that the strain concentration factor associated with (53) is

$$k_\varepsilon = \frac{1 + \sqrt{(1-\eta)\left(\frac{R}{1+R}\right)^2 + \eta} - \frac{1-\eta}{1+R}}{2\nu\eta}. \tag{54}$$

The solution for the perfectly-plastic model is approached by (54), as $\eta \to 0$, with the asymptotic behaviour

$$k_\varepsilon \sim \frac{R}{\nu(1+R)\eta} \qquad R \neq 0 \tag{55}$$

$$k_\varepsilon \sim \frac{1}{2\nu\sqrt{\eta}} \qquad R = 0 \text{ (Tresca)}. \tag{56}$$

The limit-load strain concentration factor (54) becomes therefore unbounded as the tangent modulus approaches zero, in agreement with solution (45).

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