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UNIQUENESS RESULT FOR AN UNKNOWN COEFFICIENT IN A NONLINEAR DIFFUSION PROBLEM*

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Abstract. In this article we develop a number of monotonicity estimates of the solution to certain nonlinear diffusion equations. These estimates are then applied to prove a uniqueness result for an unknown diffusion coefficient from overspecified data measured on the boundary. Techniques presented here are based on arguments using the maximum principle.

1. Introduction. For real positive constants T, μ, ν, and $0 < r < 1$, we define

$$\begin{aligned} A &= \{a \text{ in } C^2(\mathbf{R}),\ \nu \le a(s) \le \mu,\ \text{for each } s \text{ in } \mathbf{R}\}, \\ D_1 &= \{h \text{ in } C^{1+r}[0,T],\ h(0) = 0 \text{ and } h(t) > 0,\ 0 < t \le T\}, \\ D_{2,1} &= \{h \text{ in } C^{2+r}[0,T],\ h(0) = 0 \text{ and } h' \text{ in } D_1\}. \end{aligned}$$

Let g in $D_{2,1}$ be given and let $u = u(x,t)$ be the solution $C^{2,1}$ of the nonlinear problem:

$$\begin{array}{lll} \partial_t u = a(u)\partial_{xx}u, & \text{in } Q_T = [0,1]\times[0,T], & \text{for } T > 0, \\ u(x,0) = 0, & & 0 \le x \le 1, \\ -\partial_x u(0,t) = g(t), & & 0 \le t \le T, \\ \partial_x u(1,t) = 0, & & 0 \le t \le T. \end{array} \tag{1.1}$$

Set

$$M = \max_{Q_T} u_\mu(x,t),$$

where $u_\mu = u_\mu(x,t)$ denotes the solution of the problem (1.1) in case that a is equal to μ.

DEFINITION. We say that (a_1, a_2) in A^2 satisfies the property (D) if one of the following conditions is realized:

1) $a_1(0) \ne a_2(0)$;

2) there exist s_0 and s_1 satisfying $0 \le s_0 < s_1 \le M$ such that $a_1(s) = a_2(s)$, $0 \le s \le s_0$, and $a_1(s) \ne a_2(s)$, $s_0 < s \le s_1$;

3) $a_1 = a_2$ in $[0, M]$.

The principal goal of this paper is to prove the following uniqueness theorem.

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THEOREM 1.1. Let a_1, a_2 in A satisfy the property (D), and let $u_i = u_i(x,t)$ denote the solution of the problem (1.1) in case that $a = a_i$, $i = 1, 2$. If $u_1(0,t) = u_2(0,t)$, $0 \leq t \leq T$, then

$$M_1 = M_2 = N \quad \text{and} \quad a_1(s) = a_2(s),\ 0 \leq s \leq N,$$

where $M_i = \max_{Q_T} u_i(x,t)$, $i = 1, 2$.

The conditions we assume for the unknown coefficient appear to be weaker than those found in [3] and the results more powerful.

The identifiability problem for the diffusion coefficient in the nonlinear parabolic equation with Dirichlet boundary conditions is treated by the author in [1,2] and Duchateau in [3].

This paper is organized as follows. Section 2 is devoted to establishing some monotonicity estimates for the solution of the problem (1.1). In Section 3 we prove Theorem 1.1.

2. Monotonicity estimates. To prove Theorem 1.1 we need some properties regarding the solution $u = u(x,t)$ of problem (1.1) and its derivative $\partial_{xx}u$. We obtain those properties by using the following proposition.

PROPOSITION 2.1. Let $z = z(x,t)$ in $C^{2,1}(Q_T)$ be the solution of the problem

$$\begin{aligned} &\alpha(x,t)\partial_{xx}z + \beta(x,t)z - \partial_t z = 0, && \text{in } Q_T, \\ &z(x,0) = 0, && 0 \leq x \leq 1, \\ &-\partial_x z(0,t) = p(t), && 0 \leq t \leq T, \\ &\partial_x z(1,t) = 0, && 0 \leq t \leq T. \end{aligned}$$

For α in $C^1(Q_T)$, β in $C^{r,r/2}(Q_T)$, $\alpha_0 \leq \alpha(x,t)$, where α_0 is some positive constant, and p in D_1, we have then

$$z(x,t) > 0, \qquad 0 \leq x \leq 1, \quad 0 < t \leq T.$$

This proposition follows from the strong maximum principle.

PROPOSITION 2.2. Let a in A. If $u = u(x,t)$ is function class $C^{2,1}$ and satisfies (1.1), then

$$0 \leq u_\nu(x,t) \leq u(x,t) \leq u_\mu(x,t) \quad \text{in } Q_T. \tag{2.1}$$

u_ν denotes the solution of the problem (1.1) in case that a is equal to ν. In addition $0 \leq u(x,t) \leq M$ in Q_T.

The proof is similar to the proof of Proposition 2.1 [1] and is omitted.

The problem of existence and uniqueness for the problem (1.1) is developed in [2,4].

PROPOSITION 2.3. Let $u = u(x,t)$ denote the solution of problem (1.1) corresponding to a in A. Then

$$\partial_{xx}u(x,t) > 0, \qquad 0 \leq x \leq 1, \quad 0 < t \leq T.$$

Proof. Let

$$v = \partial_t u, \quad \text{in } Q_T,$$
$$L = (a \circ u)(x,t)\partial_{xx} - \partial_t,$$

and

$$h(x,t) = (a' \circ u)(x,t)u(x,t) \quad \text{in } Q_T.$$

v is then the solution of the following problem:

$$\begin{aligned} &L[v] + hv = 0, && \text{in } Q_T,\\ &v(x,0) = 0, && 0 \le x \le 1,\\ &-\partial_x v(0,t) = g'(t), && 0 \le t \le T,\\ &\partial_x v(1,t) = 0, && 0 \le t \le T. \end{aligned}$$

By Proposition 2.1 applied to v, we have

$$v(x,t) > 0, \qquad 0 \le x \le 1, \quad 0 < t \le T.$$

Thus,

$$\partial_{xx}u(x,t) = \partial_t u(x,t)[(a \circ u)(x,t)]^{-1} = v(x,t)[(a \circ u)(x,t)]^{-1}$$

in Q_T, and the proof is complete.

PROPOSITION 2.4. The solution $u = u(x,t)$ of problem (1.1) for a in A satisfies $u(x,t) \le u(0,t)$ in Q_T.

Proof. From Proposition 2.3 the function $h = u(0,\cdot)$ has a positive first derivative. For $z(x,t) = u(x,t) - h(t)$ in Q_T, we have then

$$\begin{aligned} &(a \cdot u)(x,t)\partial_{xx}z - \partial_t z = h'(t) \ge 0, && \text{in } Q_T,\\ &z(x,0) = 0, && 0 \le x \le 1,\\ &z(0,t) = 0, && 0 \le t \le T,\\ &\partial_x z(1,t) = 0, && 0 \le t \le T. \end{aligned}$$

The application of the maximum principle to z yields to $z \le 0$ in Q_T, and the proposition is proved.

3. Proof of Theorem 1.1. The equality $M_1 = M_2$ follows directly from Proposition 2.4 and $u_1(0,\cdot) = u_2(0,\cdot)$.

Now, we prove that

$$a_1 \ne a_2 \text{ in } [0,N] \text{ implies that } u_1(0,\cdot) \ne u_2(0,\cdot) \text{ in }]0,T].$$

a_1, a_2 satisfies property (D). Then, without loss of generality, we suppose that

i) $a_1(0) > a_2(0)$, or
ii) there exist two constants s_0 and s_1, $0 \le s_0 < s_1 \le N$, such that $a_1(s) = a_2(s)$ in $[0,s_0]$ and $a_1(s) > a_2(s)$ in $]s_0,s_1]$.

From the mean value theorem, remarking that $u_2(0,T) = N$ and $u_2(0,\cdot)$ is a strictly increasing function, we obtain:
For each s_0 and s_1, $0 \le s_0 < s_1 \le N$, there exist t_1 and T_1 such that

$$s_0 + \varepsilon \le u_2(0,t) \le s_1, \qquad t_1 \le t \le T_1,$$

with $s_0 + \varepsilon < s_1$.

But, $u(x,t)$ is twice continuously differentiable on x. Then

$$u_2(x,t) \ge u_2(0,t) - \varepsilon/2, \qquad 0 \le x \le x_\varepsilon, \quad t_1 \le t \le T_1,$$

with $x_\varepsilon = \varepsilon(2\max_{Q_T} |\partial_x u(x,t)|)^{-1}$. Hence,

$$s_0 + \varepsilon/2 \le u_2(x,t) \le s_1, \qquad 0 \le s \le x_\varepsilon, \quad t_1 \le t \le T_1,$$

and F, defined by

$$F = [a_1(u_2(x,t)) - a_2(u_2(x,t))]\partial_{xx} u_2(x,t)\exp(\lambda t) \quad \text{in } Q_{T_1}, \tag{3.1}$$

satisfies

$$F(x,t) \ge 0 \quad \text{in } Q_{T_1},$$

and

$$F(x,t) > 0 \quad \text{in } [0,x_\varepsilon] \times [t_1, T_1].$$

Let $w = (u_1 - u_2)\exp(\lambda t)$ in Q_{T_1}. Then w is the solution of the following linear parabolic problem:

$$\begin{aligned} &L[w] + (h(x,t) + \lambda)w = -F(x,t) && \text{in } Q_{T_1}, \\ &w(x,0) = 0, && 0 \le x \le 1, \\ &\partial_x w(0,t) = 0, && 0 \le t \le T_1, \\ &\partial_x w(1,t) = 0, && 0 \le t \le T_1, \end{aligned}$$

where

$$L = (a_1 \circ u_1)\partial_{xx} - \partial_t,$$

F is defined by (3.1), and h is some bounded function of the form

$$h(x,t) = (a_1 \circ c)(x,t)\partial_{xx} u_2(x,t) \quad \text{in } Q_{T_1}.$$

If we choose λ satisfying $\lambda > \max_{Q_T} h(x,t)$, we obtain, by application of the strong maximum principle to w, that there exists at least one t^* in $]0,T]$ such that

$$u_1(0,t^*) - u_2(0,t^*) = \exp(-\lambda t^*) w(0,t^*) > 0,$$

and the proof is complete.

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