

LEAST SQUARES APPROXIMATION OF LYAPUNOV EXPONENTS*

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Abstract. Discrete least squares approximations are shown to converge for the Lyapunov exponents of dynamical systems. Numerical examples demonstrate the approximation's utility.

Introduction. The limit explicit in the definition of Lyapunov exponents, L.E., (3), (4), has been found through the evaluation of $\ln \lambda_n$ and $q(t)$ and formation of the ratio $\ln \lambda_n/q(t)$ for increasing t , [2], [4], and [6]. It is shown, in the following, that this limit may be expressed as $\lim_{t \rightarrow \infty} a_2$, where a_2 is a coefficient in the least squares approximation of $\ln \lambda_n$. The definition of L.E. used here includes the customary definition as a special case with $q(t) \equiv t$. The L.E., β , for dynamical systems in which $\ln \lambda_n \sim \beta \ln t$, $t \rightarrow \infty$, [8], (12), is found through a suitable choice of approximating functions. $\sim \equiv$ asymptotic equivalence. A numerical study indicates the utility of the derived algorithms.

Lyapunov exponents. Consider a continuous dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

where \mathbf{x} are spatial curvilinear coordinates in an n -dimensional Euclidean phase space, E_n . Under general conditions the solution of (1), [1],

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{2}$$

is one-to-one, continuous together with its inverse and continuously differentiable with respect to \mathbf{X} and t . \mathbf{X} are material coordinates.

A subspace, V_m , of dimension $m < n$ may be defined by the parametric equation $\mathbf{X} = \mathbf{X}(\mathbf{u})$ in material coordinates and, after the deformation (2), by $\mathbf{x} = \mathbf{x}(\mathbf{u})$ in spatial coordinates. \mathbf{u} has m components. Let $da_{(m)}$ and $dA_{(m)}$ be spatial and material area elements of V_m respectively. The m -dimensional L.E. associated with a trajectory originating at \mathbf{X} is defined as, [2], [3],

$$\chi(\mathbf{X}, V_m) \equiv \limsup_{t \rightarrow \infty} (\ln(da_m/dA_m))/q(t) \tag{3}$$

where $q(t) \equiv t$. The one-dimensional L.E. for the arc, ds , with tangent, \mathbf{n} , associated with a trajectory having initial conditions respectively of dS , \mathbf{N} , \mathbf{X} is defined as, [2],

$$\chi(\mathbf{X}, \mathbf{n}) \equiv \limsup_{t \rightarrow \infty} (\ln \lambda_n)/q(t) \tag{4}$$

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where $\lambda_n \equiv ds/dS$, $(ds)^2 = g_{ij}dx^i dx^j$, $(dS)^2 = G_{IJ}dX^I dX^J$ and $q(t) \equiv t$. In the following, only exact L.E. are considered and therefore sup will be omitted, [2].

Since $D(dA_m)/Dt = 0$ and $D(dA^{i_1 \dots i_m})/Dt = 0$, (3) may be expressed as, [4],

$$\chi(\mathbf{X}, V_m) = \lim_{t \rightarrow \infty} (1/q) \int_{t_0}^t (1/m!) \frac{1}{da_{(m)}} \frac{D}{Dt} (da^{i_1 \dots i_m}) \frac{1}{da_{(m)}} da_{i_1 \dots i_m} dt \tag{5}$$

where $da^{i_1 \dots i_m}$ is the spatial area tensor and

$$\frac{1}{da_{(m)}} \frac{D}{Dt} da^{i_1 \dots i_m} = v^{i_1, r} \frac{1}{da_{(m)}} da^{r i_2 \dots i_m} + \dots + v^{i_m, r} \frac{1}{da_{(m)}} da^{i_1 i_2 \dots i_{m-1} r}. \tag{6}$$

Denote da_m/dA_m or λ_n by $h(t)$. If $q(t) = t$ and $\ln h(t) \sim \alpha t$ then the limit in (3) or (4) is α . If $\ln h(t) \sim \beta \ln t$, $t \rightarrow \infty$, then the limits in (3) or (4) vanish, [5]. In this case, if $q(t) = \ln t$, then the limit in (3) or (4) is β . Subsequently, the coefficient, a_2 , in a least squares approximation of $f(t) \equiv \ln h(t)$ is shown to converge to the limit α or β .

Least squares approximation. Given an approximating function, $G(t) = a_1 + a_2 q(t)$, where a_1 and a_2 are constants. For a least squares approximation of $f(t)$, a_2 is determined by

$$a_2 = \left(N \sum f_i q_i - \left(\sum f_i \right) \left(\sum q_i \right) \right) / A, \tag{7}$$

where $f_i \equiv f(t_i)$, $A \equiv N \sum q_i^2 - (\sum q_i)^2$ and $\sum \equiv$ summation on $i = 1, N$ discrete values of t , [7]. It follows that if $t_i = \Delta \cdot i$, $q(t) \equiv t$, and $\lim_{t \rightarrow \infty} (f(t)/t) = \alpha$ then $\lim_{N \rightarrow \infty} a_2 = \alpha$. This may be shown by setting $t_i = \Delta \cdot i$ in (7) and noting that $\sum i = (N^2 + N)/2$ and $\sum i^2 = N^3/3 + N^2/2 + N/6$. (7) becomes

$$a_2 = (12/\Delta) \left(\sum (i f_i) / (N(N^2 - 1)) - \sum f_i / (2N(N - 1)) \right). \tag{8}$$

By assumption $\lim_{t \rightarrow \infty} (f(t)/t) = \alpha$. Then $\lim_{i \rightarrow \infty} Z_i = 0$ where $Z_i = (f_i/t_i) - \alpha$. It follows that given any $\epsilon > 0$ then $\exists N_\epsilon \ni |Z_i| < \epsilon$ for all $i > N_\epsilon$. Substituting $f_i = (Z_i + \alpha)i \cdot \Delta$ into the second and third quotients in (8) and taking the limit as $N \rightarrow \infty$ gives $\Delta\alpha/3$ and $\Delta\alpha/4$ respectively. Substituting these expressions into (8) yields the result.

Consider the case in which $q(t) = \ln t$, $h(t) = Ct^\beta(1 + g(t))$, $C > 0$ is a constant, $g(t)$ is bounded, $g(t) > -1$ and $\lim_{t \rightarrow \infty} g(t) = 0$. Setting $t_i = \Delta \cdot i$ and $f(t) = \beta \ln t + \ln C(1 + g(t))$ in (7) gives

$$a_2 = \beta + \frac{1}{A} \left[N \sum (\ln i \ln C(1 + g_i)) - \sum \ln i \sum \ln C(1 + g_i) \right] \tag{9}$$

where $A = N \sum (\ln i)^2 - (\ln N!)^2$. Note that $\ln N! = (N + \frac{1}{2}) \ln(N+1) - N - 1 + f_2(N+1)$, $|f_2(N+1)| < 1$, $\ln(1 + g) \leq g$ for $0 \leq g < 1$, $|\ln(1 - g)| \leq 3g/2$ for $0 \leq g \leq 0.58$, $\ln(N+1) = \ln N + 2(\frac{1}{2N+1} + \frac{1}{3}(\frac{1}{2N+1})^3 + \dots)$, [7], $\sum \ln i = \ln N!$ and $\sum \ln^2 i = N(\ln^2 N - 2 \ln N + 2) + \ln^2 N(1 + o(1))$ as $N \rightarrow \infty$. The last relationship follows from the Euler-Maclaurin summation formula, [12] and [13]. It follows that $A \sim N^2$ and the second term on the right hand side of (9) vanishes as $N \rightarrow \infty$. Then $\lim_{N \rightarrow \infty} a_2 = \beta$.

Computation of L.E. The Lorenz and Rossler equations together with an equation given by Davis, [8], are subsequently used as test cases:

Lorenz equations, [9],

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) & a &= 16.0, \\ \dot{x}_2 &= x_1(b - x_3) - x_2 & b &= 45.92, \\ \dot{x}_3 &= x_1x_2 - cx_3 & c &= 4.0; \end{aligned} \tag{10}$$

Rossler equations, [10],

$$\begin{aligned} \dot{x}_1 &= -(x_2 + x_3) & a &= 0.15, \\ \dot{x}_2 &= x_1 + ax_2 & b &= 0.20, \\ \dot{x}_3 &= b + x_3(x_1 - c) & c &= 10.0; \end{aligned} \tag{11}$$

Davis equations, [8],

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1.5x_2^2 - kx_1^4)/x_1 & k &= 2. \end{aligned} \tag{12}$$

For the given set of parameters (10), (11) possess chaotic solutions. The solution to (12) is given in [8] as $x_1 = ab((t + p)^2 + b^2)^{-1}$ with $a^2 = 2/k$. If dv is a volume element of the phase space then it may be shown that, [11], $\ln dv = \int v_{k,k} dt$ where $v_k \equiv \dot{x}_k$. For (12)

$$\ln dv = -3(\ln(((t + p)^2 + b^2)/(p^2 + b^2))) \tag{13}$$

and $\lim_{t \rightarrow \infty} (\ln dv / \ln t) = -6$. It follows that the sum of the L.E. of (12), $\chi_1 + \chi_2 = -6$, and -8.656 if \log_2 is used in (13) rather than \ln , [6].

A fourth order Runge-Kutta integrator was used to solve (10), (11), and (12) with $\Delta t = 0.01$. The L.E. were found, [2, 4, 6, 14], through the evolution of $\ln \lambda_n$, $q(t)$ and the ratio $\ln \lambda_n / q(t)$, subsequently referred to as Case 1, and the evolution of a_2 , with $\Delta = 0.1$, (8) or (9), referred to as Case 2. For (10) with $t = 100$ and $q(t) = t$, Case 1 gave L.E.=2.03, .0139, -32.348 while Case 2 gave 2.18, $-.0045$, -32.48 . The corresponding results for $t = 1000$ were 2.15, .002, -32.455 , and 2.16, $-.0008$, -32.457 respectively. Reference [6] reports values of 2.15, .003, and -32.45 . The initial convergence of Case 2 appears to have been more rapid than that of Case 1.

For the Rossler equation, (11), with $q(t) = t$, L.E. for Cases 1 and 2 were 0.126, $-.0003$, -14.2 , and 0.127, $-.00025$, -14.1 respectively. Reference [6] reports values of 0.13, 0.00, and -14.1 .

L.E. for Cases 1 and 2, (12), with $q(t) = \ln t$, $p = .1$, $b = .1$, and $t = 200$ were $\chi_1 = -3.633$, $\chi_2 = -8.191$, $\chi_1 + \chi_2 = -11.825$, and $\chi_1 = -2.916$, $\chi_2 = -5.692$, $\chi_1 + \chi_2 = -8.607$ respectively. The exact value of $\chi_1 + \chi_2 = -8.656$, (13), which is in good agreement with $\chi_1 + \chi_2 = -8.607$ from Case 2. $\chi_1 + \chi_2 = -11.825$ for Case 1 which differs considerably from the exact value.

Conclusions. Numerical tests of the least squares algorithms for the computation of L.E. demonstrate their convergence rates to be equal to and in some instances, (12), greater than those of previous methods. The poor convergence of Case 1 for (12) is attributable to the slow convergence to zero of the ratio of the term $3 \cdot \ln(p^2 + b^2)$, (13), and $\ln t$. The rate of convergence of the least squares algorithm, Case 2, is not significantly affected by this term.

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