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SPATIAL DECAY ESTIMATES FOR REACTION DIFFUSION SYSTEMS*

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1. Introduction. We shall be concerned with establishing estimates for spatial decay of solutions to systems of reaction diffusion equations. We consider both stationary and transient solutions of reaction diffusion systems of the form

$$\frac{\partial u_k}{\partial t} = \nabla \cdot (\alpha_k(u_k)\nabla u_k) + F_k(u) \quad \text{on } \Omega \times (0,\infty), \quad \text{for } k = 1 \text{ to } m \tag{1.1}$$

where Ω is a domain in R^3 with boundary $\partial\Omega$. Within the context of elasticity or fluid mechanics, such results are frequently known as Saint Venant's principles. Horgan and Knowles [7] provide an extensive and thoroughgoing survey of this area and the interested reader is referred thereto for background and a detailed bibliography.

In the work at hand we combine scalar results due to Horgan and Wheeler [9] and Horgan, Payne, and Wheeler [8] with techniques which have been recently developed for parabolic and elliptic systems. Speaking in the roughest possible terms, these methods employ functionals which control the growth of the reaction vector field $F = (F_i)$. These functionals were abstracted and applied to parabolic systems by Morgan [11]. They have been applied to elliptic systems by Fitzgibbon and Morgan [3], [4], and Fitzgibbon, Morgan, and Waggoner [5], [6].

Our motivation for employing these methods in the present setting is born in the following. Consider the reaction mechanism

$$\dot{v}_k = F_k(v), \qquad t > 0,\ k = 1 \text{ to } m. \tag{1.2}$$

If z is a steady state for (1.2) (i.e., $F(z) = (0, \dots, 0)^T$) and there exists a neighborhood U of z in R^m and a function $H: U \to R_+$ such that

$$H(w) = 0 \quad \text{if and only if } w = z \tag{1.3}$$

$$\partial H(w)F(w) < 0 \quad \text{for all } w \in U - \{z\} \tag{1.4}$$

then we can easily show that z is asymptotically stable. Such a function H is commonly referred to as a Lyapunov function for (1.2) relative to the steady state z.

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(1.3) states that H is uniquely minimized at z and (1.4) asserts that the vector field F 'points inwards' on level sets of H. That is, H provides a mechanism for controlling the growth of the reaction vector field F. Although its purpose in the setting of (1.2) is to provide decay in 'time', and we are primarily concerned in this note with spatial decay, we still find the Lyapunov functional to be an effective tool.

2. Preliminaries and review of scalar results. We confine our attention to the case where Ω is a cylinder with generators parallel to the x_3-axis and typical cross section at $x_3 = z$ denoted by S_z. We further suppose that $\partial\Omega$ contains a plane portion S_0 in the $x_3 = 0$ plane, and that Ω lies to one side of S_0, say in $x_3 > 0$ (see Fig. 1). It should be readily apparent that the results given below hold for a much larger class of domains and can be extended to arbitrary space dimensions.

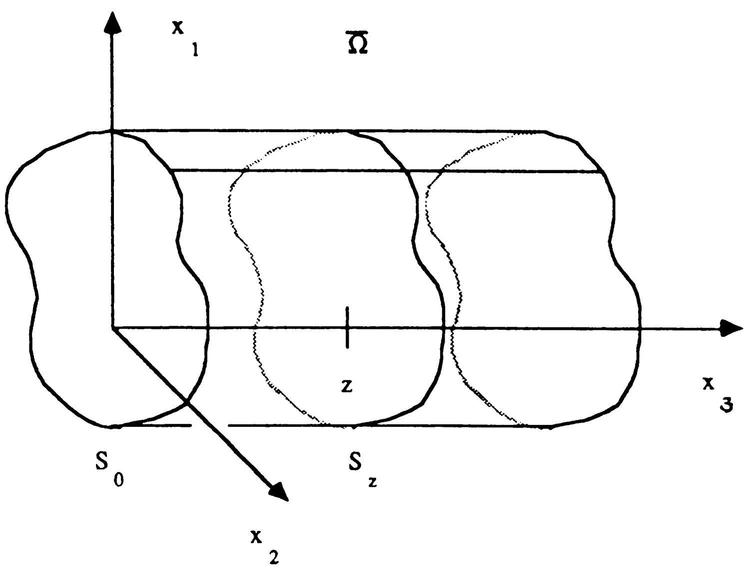

FIG. 1

Results for scalar equations are well known in this setting. For example, if Ω is bounded and we assume that $u \in C^2(\overline{\Omega})$ and that u satisfies Laplace's equation

$$\begin{aligned} \Delta u &= 0 && \text{on } \Omega \\ u &= 0 && \text{on } \partial\Omega - S_0 \\ u &= g(x_1, x_2) && \text{on } S_0 \end{aligned} \tag{2.1}$$

then we can apply a straightforward adaptation of the time dependent theory of Horgan and Wheeler [9] to obtain

$$|u(x)| \leq K \exp(-\sqrt{\lambda_1} x_3)\varphi(x_1, x_2) \quad \text{for } x \in \Omega. \tag{2.2}$$

Here λ_1 is the smallest clamped membrane eigenvalue for the two dimensional problem

$$\Delta\varphi + \lambda\varphi = 0 \quad \text{on } S_0 \tag{2.3a}$$

$$\varphi = 0 \quad \text{on } \partial S_0 \tag{2.3b}$$

with φ a corresponding nonnegative eigenfunction, and we impose the further restriction that

$$K = \sup_{(x_1,x_2)\in S_0} \left[\frac{|g(x_1,x_2)|}{\varphi(x_1,x_2)}\right] < \infty. \tag{2.4}$$

Similarly, in the transient case, we can state well-known results for solutions of

$$\frac{\partial u}{\partial t} = \Delta u \qquad \text{on } \Omega \times [0,\infty) \tag{2.5a}$$

$$u = 0 \qquad \text{on } (\partial\Omega - S_0) \times [0,\infty) \tag{2.5b}$$

$$u = g(x_1,x_2,t) \qquad \text{on } S_0 \times [0,\infty) \tag{2.5c}$$

$$u(x,0) = 0 \qquad \text{on } \Omega. \tag{2.5d}$$

Here we assume $g(x_1,x_2,0) = 0$ and that

$$k(t) = \sup_{(x_1,x_2)\in S_0} \left[\frac{|g(x_1,x_2,t)|}{\varphi(x_1,x_2)}\right] < \infty \tag{2.6}$$

defines a bounded function on $[0,\infty)$. Horgan and Wheeler [9] then show that the solution u of (2.5a–d) satisfies

$$|u(x,t)| \le \exp(-\sqrt{\lambda_1}x_3)\varphi(x_1,x_2)K(t) \tag{2.7}$$

on $\overline{\Omega}X[0,\infty)$ for all continuously differentiable functions K such that

$$K(t) \ge k(t), \qquad K'(t) \ge 0, \qquad t \in [0,\infty). \tag{2.8}$$

We remark that a result of Serrin [12] demonstrates how a suitably modified φ may be constructed to alleviate the technical difficulties presented by (2.4) and (2.6).

We point out that the spatial decay rates given above are pointwise and equivalent for the stationary and transient cases. This situation changes somewhat in the case where Ω is a semi-infinite cylinder (see Fig. 2). In this case if u satisfies (2.5a–d) and we set

$$E(z,t) = \int_0^t \int_{S_z} |\nabla u(x_1,x_2,z,s)|^2 dx_1 dx_2 ds + \frac{1}{2}\int_{S_z} u(x_1,x_2,z,t) dx_1 dx_2 \tag{2.9}$$

for all $z, t \ge 0$, then Knowles [10] shows that

$$E(z,t) \le E(0,t)\exp(-2\sqrt{\lambda_1}z) \qquad z,t \ge 0. \tag{2.10}$$

In fact, cf. [10], this decay rate is identical with the decay rate associated with the steady state case. However, physical considerations lead one to believe that faster decay estimates should be possible in the transient case. Moreover, Horgan, Payne, and Wheeler [8] show that if

$$G(z,t) = \frac{1}{2\sqrt{\pi}} \int_0^t z\tau^{-3/2} \exp\left(\frac{-z^2}{4\tau} + \lambda_1\tau\right) d\tau \tag{2.11}$$

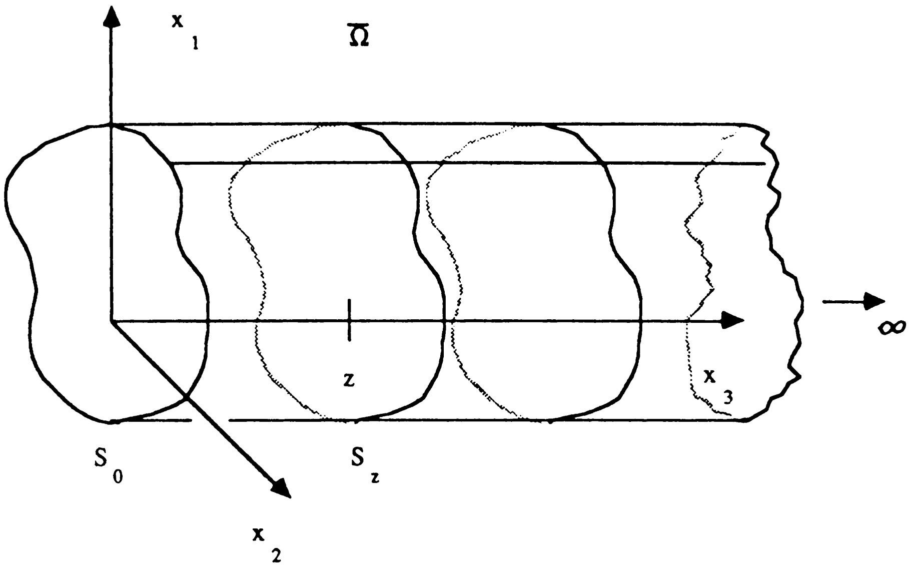

FIG. 2.

then

$$\left(\int_{S_z}(u(x_1,x_2,z,t))^2dx_1dx_2\right)^{1/2} \le \left[\max_{[0,t]}\left(\int_{S_0}g^2dA\right)^{1/2}\right]G(z,t) \quad \text{for } z,t\ge 0$$

$$\le \left[\max_{[0,t]}\left(\int_{S_0}g^2dA\right)^{1/2}\right]\frac{2z(t/\pi)^{1/2}\exp(-\lambda_1 t)}{z^2-4\lambda_1 t^2}\exp\left(\frac{-z^2}{4t}\right) \quad \text{for } z,t\ge 0. \tag{2.12}$$

These estimates show that for fixed t the spatial decay is ultimately controlled by the factor $\exp(-z^2/4t)$ rather than $\exp(-2\sqrt{\lambda_1}z)$.

3. Decay estimates for stationary solutions of systems. Throughout this section Ω denotes a bounded domain of the form depicted by Fig. 1. We consider the following quasilinear system on Ω

$$-\nabla\cdot(\alpha_k(u_k)\nabla u_k) = F_k(u) \quad \text{on } \Omega,\ k=1 \text{ to } m \tag{3.1a}$$

$$u_k = z_k \quad \text{on } \partial\Omega - S_0,\ k=1 \text{ to } m \tag{3.1b}$$

$$u_k = f_k \quad \text{on } S_0,\ k=1 \text{ to } m \tag{3.1c}$$

where $F_k\colon R^m \to R$, $f_k\colon S_0 \to R$, and $\alpha_k\colon R\to R_+$ are continuous for all $k=1$ to m. In addition we suppose there exists $a>0$ such that

$$\alpha_k(v)\ge a \quad \text{for all } v\in R,\ k=1 \text{ to } m, \tag{3.2}$$

and that $z=(z_k)\in R^m$ satisfies

$$F_k(z)=0 \quad \text{for all } k=1 \text{ to } m \tag{3.3}$$

(i.e., z is a steady state for the reaction mechanism described by (1.2)). Further, we assume that (3.1a–c) admits a classical solution $u=(u_k)\in C^2(\Omega,R^m)\cap C(\overline{\Omega},R^m)$.

We now postulate the existence of a Lyapunov structure similar to the one described for the reaction mechanism (1.2). To be precise, we assume there exists $M \subseteq R^m$ which is the closure of a convex region, such that $u(x) \in M$ for all $x \in \overline{\Omega}$. We also assume there exists a function $H \in C^2(M, R_+)$ of the form

$$H(v) = \sum_{i=1}^{m} h_i(v_i) \tag{3.4}$$

such that the following are true

$$\text{for each } v \in M, \qquad H(v) = 0 \quad \text{if and only if } v = z \tag{H1}$$

$$\text{the Hessian } \partial^2 H(v) \text{ is nonnegative definite for all } v \in M \tag{H2}$$

$$\partial H(v) F(v) \leq 0 \quad \text{for all } v \in M. \tag{H3}$$

We remark that (H1) and (H3) are essentially restatements of (1.3) and (1.4), whereas the separability assumption (3.4) and the convexity assumption (H2) allow us to manage the diffusion terms in (3.1a). At the risk of belaboring the obvious, we point out that the multiplication of (H3) is that of the $1 \times m$ row vector $\partial H(v)$ by the $m \times 1$ column vector $F(v)$.

Typically $M = R_+^m$ or R^m although neither need be the case. Basically, we are requiring the existence of a separable, nonnegative convex functional H such that the reaction vector field F does not point out of the bounded convex regions determined by the level sets of H. We shall not take up the questions of existence but remark that existence results have been established with the aid of such functionals and the interested reader is referred to recent papers by Fitzgibbon and Morgan [3], [4], and Fitzgibbon, Morgan, and Waggoner [6].

In order to obtain decay estimates for solutions of (3.1a–c), we define a nonlinear functional V on M via the formula

$$V(v) = \sum_{k=1}^{m} \int_{z_k}^{v_k} \alpha_k(s) \, dh_k(s). \tag{3.5}$$

One easily applies (H1) and (3.2) to obtain

$$V(v) \geq aH(v) \geq 0 \quad \text{for all } v \in M \tag{3.6}$$

Furthermore, it follows from (H1), (H2), and (H3) that

$$-\Delta[V(u)] \leq 0 \qquad \text{on } \Omega \tag{3.7a}$$

$$V(u) = 0 \qquad \text{on } \partial\Omega - S_0 \tag{3.7b}$$

$$V(u) = V(f) \quad \text{on } S_0. \tag{3.7c}$$

Consequently, if we suppose

$$k = \sup_{S_0} \frac{V(f(x))}{\varphi(x)} < \infty \tag{3.8}$$

then we can apply the comparison principle for elliptic equations (2.2) and (3.6) to obtain

$$aH(u(x)) \leq V(u(x)) \leq k \exp(-\sqrt{\lambda_1} x_3) \varphi(x_1, x_2) \quad \text{for all } x \in \overline{\Omega}. \tag{3.9}$$

We note that if

$$\sup_{S_0} \frac{h_k(f_k(x))}{\varphi(x)} < \infty \quad \text{for all } k = 1 \text{ to } m \tag{3.10}$$

then (3.8) is satisfied.

In Sec. 5 we provide specific examples of systems and demonstrate how the above structure may be used to provide explicit spatial decay estimates.

4. Decay estimates for transient solutions of systems. Throughout this section Ω denotes a domain depicted by either Figs. 1 or 2. We consider weakly coupled parabolic systems of the form

$$\frac{\partial u_k}{\partial t} = \nabla \cdot (\alpha_k(u_k)\nabla u_k) + F_k(u) \quad \text{on } \Omega \times [0,\infty),\ k = 1 \text{ to } m \tag{4.1a}$$

$$u_k = z_k \quad \text{on } (\partial\Omega - S_0) \times [0,\infty),\ k = 1 \text{ to } m \tag{4.1b}$$

$$u_k = f_k \quad \text{on } S_0 \times [0,\infty),\ k = 1 \text{ to } m \tag{4.1c}$$

subject to the initial conditions

$$u_k = z_k \quad \text{on } \Omega \times \{0\},\ k = 1 \text{ to } m. \tag{4.1d}$$

Here F_k, z_k, and α_k are given as in Sec. 3 and $f_k : S_0 \times [0,\infty) \to R$ is continuous and satisfies $f_k(x,0) = z_k$ for all $x \in S_0$, $k = 1$ to m. We again postulate the existence of M and H as in Sec. 3, with the obvious modification that $u(x,t) \in M$ for all $x \in \overline{\Omega}$ and $t \geq 0$. Criteria concerning the existence of forward invariant regions for parabolic systems are given by Bates [1], Cheuh, Conley, and Smoller [2], and Weinberger [14].

Continuing as in Sec. 3, we define V via (3.5) and then define a function W by

$$W(x,t) = \int_0^t V(u(x,s))ds \tag{4.2}$$

for all $x \in \overline{\Omega}$ and $t \geq 0$. We then multiply both sides of (4.1a) by $h_k'(u_k)$, integrate in time, and apply (H2), (H3) to obtain

$$H(u) \leq \Delta W \quad \text{on } \Omega \times (0,\infty). \tag{4.3}$$

Hence, from the definition of H, for each fixed $t > 0$ we have

$$-\Delta W(x,t) \leq 0 \quad \text{for all } x \in \Omega \tag{4.4a}$$

$$W(x,t) = 0 \quad \text{for all } x \in \partial\Omega - S_0 \tag{4.4b}$$

$$W(x,t) = \int_0^t V(f(x,s))ds \quad \text{for all } x \in S_0. \tag{4.4c}$$

Consequently, if

$$K_1(t) = \sup_{S_0} \frac{\int_0^t V(f(x,s))ds}{\varphi(x)} < \infty \tag{4.5}$$

and Ω is a bounded domain depicted by Fig. 1 then the comparison principle for elliptic equations (2.2) and (3.6) imply

$$a\int_0^t H(u(x,s))ds \leq W(x,t) \leq K_1(t)\exp(-\sqrt{\lambda_1}x_3)\varphi(x_1,x_2) \tag{4.6}$$

for all $x \in \overline{\Omega}$.

In the case where Ω is the semi-infinite cylinder depicted in Fig. 2 we can obtain the following results. First we suppose there exists $b > 0$ such that

$$\alpha_k(v) \le b \quad \text{for all } v \in R,\ k = 1 \text{ to } m. \tag{4.7}$$

Then from (3.5) we obtain

$$V(v) \le bH(v) \quad \text{for all } v \in M. \tag{4.8}$$

Thus, from (4.2), (4.3), and (4.8) we obtain

$$W_t \le b\Delta W \qquad \text{on } \Omega \times (0, \infty) \tag{4.9a}$$

$$W = 0 \qquad \text{on } (\partial\Omega - S_0) \times [0, \infty) \tag{4.9b}$$

$$W = \int_0^t V(f(x,s))ds \quad \text{on } S_0 \times [0, \infty) \tag{4.9c}$$

$$W = 0 \qquad \text{on } \Omega \times \{0\}. \tag{4.9d}$$

Therefore, from the comparison principle for parabolic equations, W satisfies decay estimates of the form (2.10) and (2.12). In the case of (2.12), the estimate takes the form

$$\left(\int_{S_z} (W(x_1, x_2, z, t))^2 dx_1 dx_2\right)^{1/2} \le \left[\max_{[0,bt]} \int_{S_0} (g(x_1, \tau))^2 dx\right]^{1/2} G(z, bt) \tag{4.10}$$

for all $z, t \le 0$, where

$$g(x, \tau) = \int_0^\tau V(f(x,s))ds.$$

We remark that if we assume

$$\alpha_k \equiv 1 \quad \text{for all } k = 1 \text{ to } m \tag{4.11}$$

then much better estimates can be obtained. One simply applies (H1), (H2), and (H3) to obtain

$$[H(u)]_t \le \Delta[H(u)] \quad \text{on } \Omega \times (0, \infty) \tag{4.12a}$$

$$H(u) = 0 \qquad \text{on } (\partial\Omega - S_0) \times [0, \infty) \tag{4.12b}$$

$$H(u) = H(f) \qquad \text{on } S_0 \times [0, \infty) \tag{4.12c}$$

$$H(u) = 0 \qquad \text{on } \Omega \times \{0\}. \tag{4.12d}$$

Hence, if Ω is a bounded domain as depicted by Fig. 1 and

$$K_2(t) = \sup_{S_0} \frac{H(f(x,t))}{\varphi(x)} < \infty \tag{4.13}$$

then from the comparison principle for parabolic equations and (2.7) we obtain

$$H(u(x,t)) \le \exp(-\sqrt{\lambda_1}x_3)\varphi(x_1, x_2)K(t) \tag{4.14}$$

for all $x \in \overline{\Omega}$, $t \ge 0$ and continuously differentiable functions K such that

$$K(t) \ge K_2(t) \quad \text{and} \quad K'(t) \ge 0 \quad \text{for all } t \ge 0. \tag{4.15}$$

Again, estimates of the form (2.10) and (2.12) can be obtained when Ω is depicted by Fig. 2. For example, (2.12) implies

$$\left(\int_{S_z} (H(u(x_1, x_2, z, t)))^2 dx_1 dx_2\right)^{1/2} \leq \left[\max_{[0,t]} \int_{S_0} (H(f(x, \tau)))^2 dx\right]^{1/2} G(z, t) \quad (4.16)$$

for all $z, t \geq 0$. Further estimation can be given as in (2.12).

5. Examples and further comments. We first consider a two-component quasilinear system

$$u_t = \nabla \cdot (\alpha_1(u)\nabla u) - \mu^2 g(u) \exp\left(\gamma - \frac{\gamma}{v}\right) \quad \text{on } \Omega \times (0, \infty) \quad (5.1a)$$

$$v_t = \nabla \cdot (\alpha_2(v)\nabla v) + \beta\mu^2 g(u) \exp\left(\gamma - \frac{\gamma}{v}\right) \quad \text{on } \Omega \times (0, \infty) \quad (5.1b)$$

where Ω is a bounded domain depicted by Fig. 1. We assume that

$$u = v = 0 \quad \text{on } (\partial\Omega - S_0) \times [0, \infty) \quad (5.1c)$$

and

$$u = f_1, \qquad v = f_2 \quad \text{on } S_0 \times [0, \infty). \quad (5.1d)$$

and initially we have

$$u(x, 0) = v(x, 0) = 0 \quad \text{on } \Omega. \quad (5.1e)$$

The coefficients of diffusivity are assumed to satisfy (3.2) and the function g is assumed to be continuous and to satisfy $g(s) \geq g(0) = 0$ for $s \geq 0$.

This system models an irreversible nonisothermic reaction, [6]. The dependent variables u and v represent concentration and temperature of the reactant. The constants β and γ are positive and denote the Prater temperature and the Arrhenius number, respectively. μ^2 denotes the Thiele number. Frequently $g(u) = u^p$ where $p \geq 0$ is the order of the reaction.

We let $M = R_+^2$ and define our reaction vector field on R_+^2 by

$$F_1(u, v) = -\mu^2 g(u) \exp\left(\gamma - \frac{\gamma}{v}\right) \quad (5.2)$$

$$F_2(u, v) = \beta\mu^2 g(u) \exp\left(\gamma - \frac{\gamma}{v}\right) \quad (5.3)$$

with the obvious extension to $v = 0$. It is known [5] that M is a forward invariant region for (5.1a–e). It is clear that $\beta F_1(u, v) + F_2(u, v) = 0$ for all $(u, v) \in M$. Thus, if we define $H(u, v) = \beta u + v$, then it is evident that H satisfies (H1), (H2), and (H3). Consequently, if (4.5) holds then we obtain the estimate (4.6). Furthermore, if f_1 and f_2 satisfy (3.8) or (3.10), then stationary solutions of (5.1a–d) satisfy the pointwise estimate

$$a(\beta u(x) + v(x)) \leq V(u(x), v(x)) \leq K \exp(-\sqrt{\lambda_1} x_3)\varphi(x_1, x_2) \quad (5.4)$$

for all $x \in \overline{\Omega}$, where K is obtained via (3.8). (See [5], [6] for existence results.)

In our final example we consider the following semilinear parabolic system:

$$a_t = d_1\Delta a - ab^2 + N_1 b^3 \quad \text{on } \Omega\times(0,\infty) \tag{5.5a}$$
$$b_t = d_2\Delta b + ab^2 - N_1 b^3 - \sigma(b - N_2 c) \quad \text{on } \Omega\times(0,\infty) \tag{5.5b}$$
$$c_t = d_3\Delta c + \sigma(b - N_2 c) \quad \text{on } \Omega\times(0,\infty). \tag{5.5c}$$

This system of partial differential equations arises from the Schlögl model of chemical reactions due to Gray–Scott [13]. The constants d_1, d_2, d_3, N_1, N_2, and σ are assumed positive. Here $\Omega \subseteq R^3$ is a bounded region satisfying the description given in Fig. 1. We impose the boundary conditions

$$a = b = c = 0 \quad \text{on } (\partial\Omega - S_0)\times[0,\infty) \tag{5.5d}$$

and

$$a = f_1, \qquad b = f_2, \qquad c = f_3 \quad \text{on } S_0\times[0,\infty). \tag{5.5e}$$

Initially we have

$$a(x,0) = b(x,0) = c(x,0) = 0 \quad \text{on } \Omega \tag{5.5f}$$

where $f_1, f_2, f_3 \geq 0$ on $S_0 \times [0,\infty)$.

We set $F(a,b,c) = (-ab^2 + N_1b^3, ab^2 - N_1b^3 - \sigma(b - N_2c), \sigma(b - N_2c))^T$ on R^3_+. It is known [5] that $M = R^3_+$ is a forward invariant region for (5.5a–f). Moreover, it is readily verified that $H(a,b,c) = a + b + c$ satisfies (H1), (H2), and (H3). Thus, we may obtain spatial decay estimates of the form (4.6). In addition, if $d_1 = d_2 = d_3 = 1$ then we obtain decay estimates of the form (4.14).

On unbounded domains of the form depicted in Fig. 2 we may obtain L_2 decay estimates of the form (4.16).

We remark that the assumption of diffusion constants equal to one in the discussion following (4.11) was made for convenience. Estimates similar to (4.14) and (4.16) can be given for systems with equal diffusion coefficients, so long as the Lyapunov structure of Secs. 3 and 4 is present. Furthermore, in this case it should be apparent that the separability assumption on H is not necessary. It remains to be seen whether the estimates (4.6) and (4.10) can be improved to ones similar to (4.14) and (4.16) when unequal diffusion is present.

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