QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLVII, NUMBER 3
SEPTEMBER 1989, PAGES 545-546

NOTE ON THE FUNCTIONAL DIFFERENTIAL EQUATION $\sum_{m=-\infty}^{\infty} a_m y'(x+m\alpha) = f(x)$*

By

LL. G. CHAMBERS

University College of North Wales, Bangor, Wales, United Kingdom

1. The solution of functional differential equations of the form

$$\sum_{m=-\infty}^{\infty} a_m y'(x+m\alpha) = f(x) \tag{1}$$

does not appear to have been considered in the literature. It is easy to show however that a very simple particular solution exists. Let

$$F(x) = \int_c^x f(\xi)\,d\xi \tag{2}$$

where c is some convenient arbitrary constant. The arbitrariness of c corresponds to the fact that y is indefinite by an arbitrary constant. Let

$$A(t) = \sum_{m=-\infty}^{\infty} a_m t^m. \tag{3}$$

It is clear that if

$$\sum_{m=-\infty}^{\infty} a_m (e^{mk\alpha}) = 0 \tag{4}$$

that is k is a root of

$$A(e^{k\alpha}) = 0 \tag{5}$$

there will also be a further set of complementary functions e^{kx} in the solution.

2. Look for a solution of the form

$$y(x) = \sum_{n=-\infty}^{\infty} b_n F(x+n\alpha). \tag{6}$$

Substitution of this expression into the original differential equation gives

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_m b_n f(x+\overline{m+n}\alpha) = f(x)$$

that is

$$\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_m b_{l-m} f(x+l\alpha) = f(x). \tag{7}$$

*Received November 15, 1988.

©1989 Brown University

Equation (7) is satisfied by

$$\sum_{m=-\infty}^{\infty} a_m b_{l-m} = 1 \qquad l = 0 \tag{8a}$$

$$= 0 \qquad l \neq 0. \tag{8b}$$

If

$$B(t) = \sum_{m=-\infty}^{\infty} b_m t^m \tag{9}$$

the relations (8) are equivalent to the statement

$$A(t)B(t) = 1$$

that is

$$B(t) = [A(t)]^{-1} \tag{10}$$

and the b_n will be the coefficients of the corresponding powers of t in the Laurent series expansion of the reciprocal of $A(t)$.

3. If we consider the example

$$y'(x) + y'(x+\alpha) = f(x) \qquad \alpha > 0 \tag{11}$$

it may be verified that a particular solution

$$y(x) = \sum_{n=0}^{\infty} (-1)^n \int_{n\alpha}^{x+n\alpha} f(\xi)\, d\xi \tag{12}$$

exists, provided that the series converges. If, for example

$$|f(x)| \leq K e^{-\beta x} \qquad \beta > 0 \tag{13}$$

it may be verified that this is so.