

AN INTERNAL VARIABLE THEORY OF ELASTOPLASTICITY BASED ON THE MAXIMUM PLASTIC WORK INEQUALITY

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Abstract. The methods of convex analysis are used to explore in greater depth the nature of the evolution equation in internal variable formulations of elastoplasticity. The evolution equation is considered in a form in which the thermodynamic force belongs to a set defined by a multi-valued map G . It is shown that the maximum plastic work inequality together with the assumption that G is maximal responsive (a term defined in Sec. 4), is necessary and sufficient to give a theory equivalent to that proposed by Moreau. Further consequences are investigated or elucidated, including the relationship between the yield function and the dissipation function; these functions are polars of each other. Examples are given to illustrate the theory.

1. Introduction. It is now widely accepted that constitutive theories of inelastic behaviour may be successfully treated within the framework of the modern theory of continuum thermodynamics with internal variables. Early investigations along these lines have been those of Kestin and Rice [12] and Moreau [20], with Rice extending his earlier work with Kestin to the finite-strain case [23]. The work of Halphen and Nguyen [9] has also proved influential. A survey account may be found in the paper by Germain, Nguyen, and Suquet [6].

Central to the structure of any internal-variable theory of plasticity is the *evolution equation*, which provides information about the internal variable rate $\dot{\xi}$, usually in the form

$$\dot{\xi} = F(\dots) \tag{1.1}$$

where the function F and its arguments need to be specified.

There are two major issues of concern in an internal-variable approach to plasticity: one is the issue of the form which the evolution equation (1.1) should take, and the other concerns the construction of specific internal variable theories of standard models of plasticity. With regard to the former, most theories proceed on the assumption that the evolution equation can be expressed in terms of a potential (or

more correctly, a pseudopotential): Eq. (1.1) is written in the form

$$\dot{\xi} \in \partial\psi(X) \quad (1.2)$$

where X is the thermodynamic force conjugate to ξ , ψ is the potential function, and $\partial\psi(X)$ denotes its subdifferential at X . This formulation and its implications have been discussed in detail by Moreau [20] and by Halphen and Nguyen [9] using the tools of convex analysis. In particular, these authors choose as ψ the indicator function of a closed convex region K , the region of admissible values of X , whose interior is the elastic region.

The conjugate of (1.2), namely

$$X \in \partial\psi^*(\dot{\xi}), \quad (1.3)$$

where ψ^* is the Legendre-Fenchel conjugate of ψ , has received comparatively scant attention; Moreau [20], Halphen and Nguyen [9], and Mandel [16] mention it, but do not explore it further as an alternative approach (see also Germain, Nguyen, and Suquet [6]). When ψ is the indicator function of a closed convex set K , ψ^* is the support function of K , and in fact we have $\psi^*(\dot{\xi}) = X \cdot \dot{\xi}$ for any $X \in \partial\psi(\dot{\xi})$; thus ψ^* is the *dissipation function*. The formulation (1.3), with ψ^* as the dissipation function, has been used extensively by Martin and co-workers [2, 17, 18, 19], in the development of constitutive equations and in the numerical solution of the corresponding boundary-value problems.

Similar remarks apply to investigations of the construction of specific internal variable models: in contrast to other treatments (see [6] for a survey) Martin and Nappi [18], in a recent contribution, have taken as basic the formulation using a dissipation function in their construction of a formulation which yields as special cases perfect plasticity, linear kinematic hardening, and linear isotropic hardening with the von Mises yield condition.

In recent years there has been considerable activity in the area of finite strain plasticity, with much of the argument revolving around the question of how the deformation should be decomposed into elastic and plastic parts. This matter is somewhat peripheral to our aims here, so we do not attempt a full survey, and merely mention as examples of work in this area the contributions of Green and Naghdi [7], Lee [15], and Nemat-Nasser [21].

An important departure from the standard models has been the recent work of Kim and Oden [13, 14] who abandon convexity and consider, in the context of the finite-strain theory, an evolution equation of the form $\dot{\xi} \in \bar{\partial}\psi(X)$ where $\bar{\partial}\psi$ denotes the *generalised subdifferential* of the nonconvex, nondifferentiable function ψ (for the development of the theory of generalised gradients, see Clarke [3]).

The purpose of this contribution is to unify and extend existing internal variable theories of plasticity. By embedding our investigation within the framework of convex analysis we are able to establish the conditions under which different formulations of the evolution equations are equivalent, and to explore a variety of consequences. Rather than start with the formulations based on (1.2) or (1.3), we consider (1.1) in the form

$$X \in G(\dot{\xi}) \quad (1.4)$$

where G is a multi-valued map. That is, (1.4) gives the relationship between X and $\dot{\xi}$ (the X v. $\dot{\xi}$ graph) directly. We then consider the question of the minimal assumptions regarding G which, when added to the *maximum plastic work inequality*, gives a standard model of plasticity. This inequality has the form

$$(X - X') \cdot \dot{\xi} \geq 0 \quad \text{for all } X' \in K, \quad (1.5)$$

and its importance arises from the fact that it embodies the *essential structure* of plastic constitutive behaviour (see Hill [10] and Rice [24]). The inequality is a *consequence* of a formulation based on the use of the indicator function or support function as pseudopotential. We will show that the maximum plastic work inequality together with two assumptions about the map G , made precise in Sec. 4, are *necessary and sufficient* for the existence of a *dissipation function and yield surface* with the requisite properties. This result is given in Theorem 4.1.

It is worth noting here that, despite the pivotal role of (1.5) in theories of plasticity, very few authors have attempted to develop constitutive theories taking (1.5) as an axiom. Recent work by Simo [27] is an exception to this observation.

We also clarify the relationship between the yield function and the dissipation function in Sec. 5. First, we show that it is always possible to construct a positively homogeneous convex function whose level set at 1 coincides with the yield surface: this function is referred to as the *canonical* yield function. We then show that the canonical yield and dissipation functions are *polars* of each other. The concept of polar functions, and their place in solid mechanics, have also recently been explored by Hill [11], who refers to pairs of functions possessing this property as dual potentials. We pursue this topic within the broader framework considered here.

The plan of this work is as follows. In Sec. 2 we gather together results from convex analysis which will be central to our subsequent developments. Section 3 summarises the framework of continuum thermodynamics with internal variables, within which we pose the problem. The main result is presented in Sec. 4, as are various consequences of this result. In Sec. 5 we examine in detail the relationship between the dissipation function and the yield function. We conclude in Sec. 6 with a selection of concrete examples to illustrate the ideas of Secs. 4 and 5.

Since we are primarily concerned with elucidating various features of the evolution equation we confine our attention to small-strain plasticity. The extension to finite-strain plasticity will be reported elsewhere [5].

2. Some preliminary results from convex analysis. In this section we collect a summary of definitions and results from convex analysis which are of relevance to the rest of this work. Proofs are excluded; these, and other extensions may be found, for example, in the books by Aubin [1], Oden [22], and Rockafellar [25].

We will be concerned exclusively with finite-dimensional spaces (for example, spaces of vectors or tensors at a point) and functions acting on them. We denote such a finite-dimensional space by E ; E is isomorphic to (and may be identified with) \mathbb{R}^n for appropriate n . The dual space of E is denoted by E^* . Of course E^* may be identified with E , but it will be convenient to maintain this distinction. For $x^* \in E^*$ and $x \in E$, the action of x^* on x is denoted by $x^* \cdot x$.

Let S be a subset of E . The interior, closure, and boundary of S are denoted by $\text{int} S$, $\text{cl} S$, and $\text{bd} S$, respectively. S is *convex* if, for any $x, y \in S$ and $0 < \theta < 1$, $\theta x + (1 - \theta)y \in S$. The *normal cone* to a convex set S at x , denoted by $N_S(x)$, is defined by

$$N_S(x) = \{x^* \in E^* : x^* \cdot (y - x) \leq 0 \text{ for all } y \in S\} \tag{2.1}$$

(see Fig. 2.1). When $x \in \text{int} S$ we clearly have $N_S(x) = \{0\}$, whereas for $x \in \text{bd} S$, $N_S(x)$ consists of the cone of normals at x .

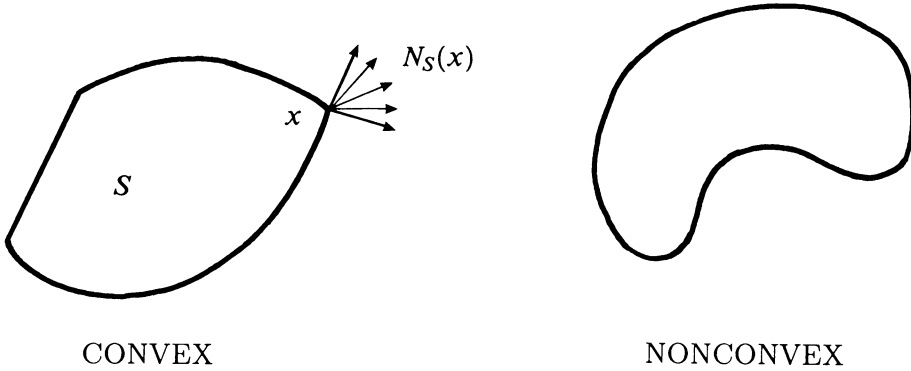


FIG. 2.1. Examples of convex and nonconvex sets in \mathbf{R}^2 , and the normal cone $N_S(x)$ to a convex set S .

Let f be a function whose domain is E and whose values are real or $\pm\infty$. The *effective domain* of f , denoted $\text{dom } f$, is defined by

$$\text{dom } f = \{x \in E : f(x) < \infty\}. \tag{2.2}$$

The *epigraph* of f , denoted by $\text{epi } f$, is the set of ordered pairs

$$\text{epi } f = \{(x, \alpha) \in E \times \mathbf{R} : f(x) \leq \alpha\}. \tag{2.3}$$

The function f is *convex* if

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y) \tag{2.4}$$

for all $x, y \in E, 0 < \theta < 1$

positively homogenous if

$$f(\alpha x) = \alpha f(x) \quad \text{for all } x \in E, 0 < \alpha \in \mathbf{R}, \tag{2.5}$$

and *lower semicontinuous* (written *lsc*) if

$$\liminf_{n \rightarrow \infty} f(x_n) \geq f(x) \tag{2.6}$$

for any sequence $\{x_n\}$ converging to x (see Fig. 2.2 for illustrations of these definitions). It is well known that f is *lsc* if and only if $\text{epi } f$ is *closed*, and that a convex

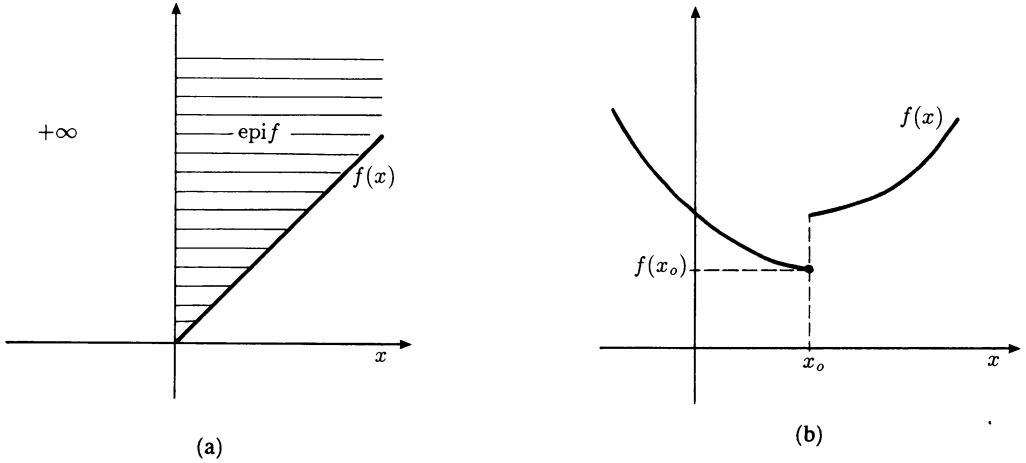


FIG. 2.2. (a) The epigraph $\text{epi } f$ of the positively homogeneous function $f(x) = x, x \geq 0, f(x) = +\infty, x < 0, \text{dom } f = [0, \infty), \text{epi } f = \{(x, \alpha) : x \geq 0; x \leq \alpha\}$. (b) A lower semicontinuous function f .

function is continuous on any open subset of E where its values are finite. A convex function f is *proper* if $f(x) < +\infty$ for at least one x and $f(x) > -\infty$ for every x .

For any set $S \subset E$, the *indicator function* I_S of S is defined on E by

$$I_S(x) = \begin{cases} 0, & x \in S, \\ +\infty, & x \notin S, \end{cases} \tag{2.7}$$

and the *support function* σ_S of S is defined on E^* by

$$\sigma_S(x^*) = \sup\{x^* \cdot x : x \in S\}. \tag{2.8}$$

Let f be a proper lsc convex function on E . The *conjugate function* f^* of f is defined by

$$f^*(x^*) = \sup\{x^* \cdot x - f(x) : x \in E\}. \tag{2.9}$$

From the definition (2.8) we see that the support function σ_S is *conjugate* to the indicator function I_S :

$$I_S^* = \sigma_S. \tag{2.10}$$

We have the further result that if f is proper, convex and lsc then so is f^* , and furthermore

$$(f^*)^* \equiv f^{**} = f. \tag{2.11}$$

In particular, I_S is proper, convex and lsc whenever S is closed and convex, so for such a set

$$I_S = \sigma_S^* = I_S^{**}. \tag{2.12}$$

Given a convex function f on E , for any $x \in E$ the *subdifferential* $\partial f(x)$ of f at x is the subset (possibly empty) of E^* defined by

$$\begin{aligned} \partial f : E &\rightarrow 2^{E^*}, \\ \partial f(x) &= \{x^* \in E^* : f(y) \geq f(x) + x^* \cdot (y - x) \text{ for all } y \in E\} \end{aligned} \tag{2.13}$$

where 2^{E^*} denotes the class of subsets of E^* . The members of $\partial f(x)$ are called subgradients. According to the definition (2.13), $\partial f(x) = \emptyset$ if $x \notin \text{dom } f$.

If f is differentiable at x then clearly

$$\partial f(x) = \{\nabla f(x)\} \quad (2.14)$$

(see Fig. 2.3).

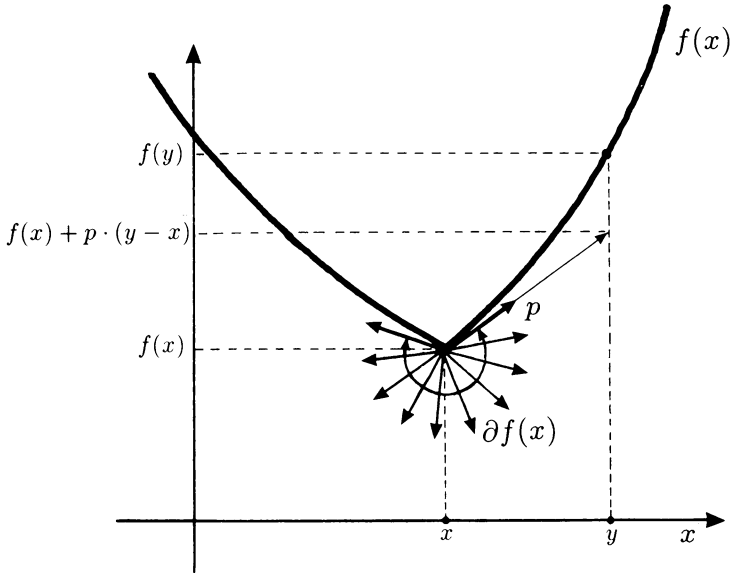


FIG. 2.3. The subdifferential $\partial f(x)$.

We have the important result that

$$x^* \in \partial f(x) \quad \text{if and only if} \quad x \in \partial f^*(x^*). \quad (2.15)$$

For the special case of the indicator function, it is evident from (2.1) and (2.13) that

$$\partial I_S(x) = N_S(x). \quad (2.16)$$

We set

$$\text{dom}(\partial f) = \{x \in E : \partial f(x) \neq \emptyset\}$$

and conclude with the following result.

LEMMA 2.1. (a) Let f be a proper, convex, lsc function on E . Then $\text{dom}(\partial f) \neq \emptyset$ and $\text{dom}(\partial f)$ is dense in $\text{dom } f$.

(b) Let f and g be proper, convex, lsc functions on E . Then

$$\partial f(x) = \partial g(x) \quad \text{for all } x \in E$$

if and only if

$$f = g + \text{const.}$$

3. Thermodynamics with internal variables. We develop our theory using the general framework of Coleman and Gurtin [4] (see also Gurtin [8] for a good summary account), focussing on the small strain theory of rate-independent plasticity.

In local form the equations of balance of momentum, of angular momentum and of energy are, respectively,

$$\operatorname{div} \sigma + \rho b = \rho a, \quad (3.1)$$

$$\sigma^T = \sigma, \quad (3.2)$$

$$\rho \dot{e} = \sigma \cdot \dot{\varepsilon} - \operatorname{div} q. \quad (3.3)$$

Here σ is the stress tensor, ε the strain tensor, ρ the mass density, b the specific body force, a the acceleration vector, e the specific internal energy, and q the heat flux vector. Superposed dots denote derivatives with respect to time. The local form of the second law of thermodynamics (the Clausius-Duhem inequality) is

$$\rho \dot{s} \geq -\operatorname{div}(q/\theta) \quad (3.4)$$

where s is the entropy density and θ the absolute temperature. By introducing the free energy density ψ , defined by

$$\psi = e - \theta s, \quad (3.5)$$

and combining Eqs. (3.3) and (3.4), we obtain the reduced dissipation inequality

$$\rho(\dot{\psi} + s\dot{\theta}) - \sigma \cdot \dot{\varepsilon} + \theta^{-1} q \cdot \nabla \theta \leq 0. \quad (3.6)$$

We consider a class of constitutive equations of the form

$$\begin{aligned} \psi &= \bar{\psi}(\varepsilon, \theta, \nabla \theta, \xi), \\ \sigma &= \bar{\sigma}(\varepsilon, \theta, \nabla \theta, \xi), \\ s &= \bar{s}(\varepsilon, \theta, \nabla \theta, \xi), \\ q &= \bar{q}(\varepsilon, \theta, \nabla \theta, \xi), \\ p &\equiv \dot{\xi} = F(\varepsilon, \theta, \nabla \theta, \xi). \end{aligned} \quad (3.7)$$

Here ξ is an internal variable which characterises the dissipative behaviour inherent in plasticity. The number of such internal variables, as well as their tensorial nature, depends on the particular model being constructed, but we need not be specific about these attributes here. Concrete examples will, however, be given later.

It is well known that substitution of (3.7) in the reduced dissipation inequality (3.6) yields the fact that ψ , σ , and s are independent of $\nabla \theta$, and that

$$\sigma = \rho \frac{\partial \bar{\psi}}{\partial \varepsilon}, \quad s = -\frac{\partial \bar{\psi}}{\partial \theta}, \quad (3.8)$$

and

$$\rho \frac{\partial \bar{\psi}}{\partial \xi} \cdot \dot{\xi} + \theta^{-1} q \cdot \nabla \theta \leq 0. \quad (3.9)$$

We now specialise, and consider henceforth only isothermal behaviour. Consequently $\nabla \theta = 0$, $q = 0$ (see [4 or 8]) and we no longer display θ as a variable. Furthermore, we define the thermodynamic force X conjugate to the internal variable ξ by

$$X = \bar{X}(\varepsilon, \xi) \equiv -\frac{\partial \bar{\psi}}{\partial \xi}. \quad (3.10)$$

The reduced set of constitutive equations now becomes

$$\begin{aligned}\psi &= \bar{\psi}(\varepsilon, \xi), \\ \sigma &= \rho \frac{\partial \bar{\psi}}{\partial \varepsilon}, \\ X &= -\rho \frac{\partial \bar{\psi}}{\partial \xi}, \\ p &= \dot{\xi}, \quad p = F(\varepsilon, \xi),\end{aligned}\tag{3.11}$$

and (3.9) reduces to

$$X \cdot p \geq 0.\tag{3.12}$$

While it is the first three of Eqs. (3.11) which encapsulate the elastic material properties and any hardening that may be present (see the examples in Sec. 6), it is the so-called evolution law (3.11)₄ which determines the real nature of the inelastic behaviour. Naturally it is desirable that the function F be as simple as possible, yet it should contain the major features of the kind of inelastic behaviour which it is supposed to represent. For conventional, rate-independent models it is not unduly restrictive to suppose that p depends on ε and ξ through X (see [6, 16, 20]), an assumption which we adopt:

$$p = F(X).\tag{3.13}$$

We take (3.13) as basic, but will find it more convenient to work with its inverse.

Whatever the tensorial nature of p , it belongs to a finite-dimensional space E isomorphic to \mathbb{R}^n , for suitable n . We hence write $E = \mathbb{R}^n$ and $E^* = \mathbb{R}^n$, the dual space of \mathbb{R}^n . Of course \mathbb{R}^n is also isomorphic to \mathbb{R}^n , but it is useful to maintain this distinction in what follows: the internal variable rates belong to E , and the conjugate forces to E^* .

Consider a correspondence $G: p \rightarrow G(p)$ which associates with each $p \in E$ a set $G(p)$ (possibly empty for some p) in E^* . The relation inverse to (3.13) is thus of the form

$$X \in G(p).\tag{3.14}$$

Eq. (3.14) reflects the multi-valuedness of G ; Fig. 6.1 later gives a simple one-dimensional illustration, as would occur in perfect plasticity. Our first aim is to construct a theory of inelastic behaviour based on the map G .

4. Maximal responsive relations.

DEFINITION 4.1. The map G is said to be *responsive* if

$$0 \in G(0)\tag{4.1}$$

and if for any $p_0, p_1 \in E$,

$$(X_0 - X_1) \cdot p_0 \geq 0 \quad \text{and} \quad (X_1 - X_0) \cdot p_1 \geq 0\tag{4.2}$$

whenever $X_0 \in G(p_0)$, $X_1 \in G(p_1)$.

Equation (4.1) simply implies that the set of thermodynamic forces corresponding to zero dissipation contains the zero force. Equation (4.2) is a generalisation of

the *maximum plastic work* inequality: indeed, in the case of perfect plasticity $\xi = \varepsilon^p$, the plastic strain, $X = \sigma$, and (4.2) is precisely this inequality. The maximum plastic work inequality contains the essential structure of theories of plasticity in that its adoption, together with the assumption of a definite elastic range, is sufficient to deduce the major properties of plastic constitutive relations (the normality rule, convexity of the yield surface) (see [10, 24], for discussions of this viewpoint). More recently it has been postulated as an axiom in a finite-strain theory of plasticity by Simo [27].

DEFINITION 4.2. Let $G: E \rightarrow 2^{E^*}$ be a responsive map. G is said to be *maximal responsive* if there is no other responsive map whose graph properly includes the graph of G .

The notion of maximality occurs in various branches of nonlinear analysis, for example, the theory of maximal monotone operators (see, for example, [22, 26, 28]). In the present context maximal responsiveness is equivalent to the property that for any $p_0 \in E$ and $X_0 \in E^*$, the condition

$$(X_0 - X_1) \cdot p_0 \geq 0 \quad \text{and} \quad (X_1 - X_0) \cdot p_1 \geq 0 \quad \forall p_1, X_1 \in G(p_1) \quad (4.3)$$

implies that $X_0 \in G(p_0)$.

DEFINITION 4.3. A function $D: E \rightarrow [0, \infty]$ is said to be a *gauge* if

$$\begin{aligned} D(p) &\geq 0 \quad \text{for all } p \in E, \\ D(0) &= 0, \\ D &\text{ is convex, positively homogeneous and lsc.} \end{aligned}$$

REMARK. Rockafellar [25] defines a gauge to be a function with the above properties, *excluding* lower semicontinuity.

From the results in Sec. 2 we have the following

LEMMA 4.1. Let D be a *gauge* on E . Define the closed convex set K by

$$K = \{X \in E^*: X \cdot p \leq D(p) \quad \forall p \in E\}. \quad (4.4)$$

Then (a) D is the support function of K :

$$D(p) = \sup_{X \in K} X \cdot p; \quad (4.5)$$

(b) the function D^* conjugate to D is the *indicator function* of K :

$$D^*(X) = \begin{cases} 0, & X \in K, \\ +\infty, & X \notin K; \end{cases}$$

(c) $K = \partial D(0)$;

(d) $X \in \partial D(p) \Leftrightarrow p \in \partial D^*(X) = N_K(X)$.

We now come to the main result of this section.

THEOREM 4.1. Let $G: E \rightarrow 2^{E^*}$ be a multivalued mapping. The following are equivalent

- (a) G is maximal responsive;
- (b) There exists a gauge D on E with the property

$$G(p) = \partial D(p) \quad \text{for all } p \in E.$$

Furthermore, when G is maximal responsive it determines D *uniquely* and in terms of the set $\text{dom } G = \{p \in E: G(p) \neq \emptyset\}$ satisfies

$$D(p) = \begin{cases} X \cdot p & \text{for all } X \in G(p), p \in \text{dom } G, \\ +\infty & \text{for } p \notin \text{cl dom } G. \end{cases} \quad (4.6)$$

REMARK. The above theorem is similar in nature to results which connect maximal cyclic monotone maps with the subgradients of convex lsc functions (see Rockafellar [26], Zeidler [28]).

Proof of Theorem 4.1.

(b) \Rightarrow (a). Assume that the conditions of (b) hold. It follows from Lemma 4.1(d) that

$$G(p) = \{X \in K: X \cdot p \geq Y \cdot p \quad \forall Y \in K\} \quad (i)$$

since $G = \partial D$, where K is defined by (4.4). In particular, (4.2) holds. Property (4.1) follows from the observation that $0 \in K = \partial D(0)$ (Lemma 4.1 (c)). To show that G is maximal, consider any pair (\bar{p}, \bar{X}) such that

$$(X - \bar{X}) \cdot p \geq 0 \quad \text{and} \quad (\bar{X} - X) \cdot \bar{p} \geq 0 \quad (ii)$$

for all $p \in E, X \in G(p)$. We must verify that $\bar{X} \in G(\bar{p})$. We have, from Lemma 4.1 (d),

$$\bar{X} \cdot p \leq D(p) \quad \text{for all } p \in E$$

so that $\bar{X} \in K$. The second part of (ii) then implies by (i) that $\bar{X} \in G(\bar{p})$, as required.

(a) \Rightarrow (b). We show that

$$G(0) \supset G(p) \quad \text{for all } p \in E. \quad (iii)$$

Consider any \bar{p} and $\bar{X} \in G(\bar{p})$. From (4.2),

$$(X - \bar{X}) \cdot p \geq 0 \quad \text{and} \quad (\bar{X} - X) \cdot \bar{p} \geq 0$$

whenever $X \in G(p)$. Hence the pair $(0, \bar{X})$ has the property that

$$(X - \bar{X}) \cdot p \geq 0 \quad \text{and} \quad (\bar{X} - X) \cdot 0 \geq 0$$

whenever $X \in G(p)$, and so $(0, \bar{X})$ could be added to the graph of G without violating (4.2). Since G is maximal responsive, we must have $\bar{X} \in G(0)$, whence (iii).

The above argument actually establishes that $G(0)$ coincides with the set

$$K = \{\bar{X} \in E^*: (X - \bar{X}) \cdot p \geq 0 \quad \text{for all } p \in E, X \in G(p)\}. \quad (iv)$$

This set is closed and convex, and contains 0, by property (4.1). From (iii) and (iv),

$$\bar{X} \in G(\bar{p}) \Rightarrow \bar{X} \in G(0) = K \Rightarrow \bar{p} \in N_K(\bar{X}). \quad (v)$$

Let D be the support function of K . Since D is the support function of a closed convex set containing 0 it is a gauge (see [25], Ch. 15) and

$$\bar{X} \in \partial D(\bar{p}) \Leftrightarrow \bar{p} \in N_K(\bar{X}).$$

Moreover ∂D is a responsive map. Furthermore, (v) implies that the graph of G is included in the graph of ∂D . Inasmuch as G is maximal responsive, we may conclude that $G = \partial D$, whence part (a).

To establish the uniqueness of D , we recall (Lemma 2.1) that two lsc proper convex functions have the same subdifferential if and only if they differ by an additive constant. We fix this constant by the requirement that $D(0) = 0$, thereby defining D uniquely.

To establish (4.6), we note that D is the support function of K , defined by (4.4), so that from (i) $D(p) = X \cdot p$ when $X \in G(p)$. Since $\text{cl dom } G = \text{cl dom } \partial D = \text{cl dom } D$ (Lemma 2.1) we also have $D(p) = +\infty$ when $p \notin \text{cl dom } G$, whence (4.6). \square

We recognise D to be what is known in classical plasticity as the “dissipation function”; $D(p)$ gives the rate of plastic (dissipative) work. The assumption of non-negativity in the definition is dictated by the second law of thermodynamics (see (3.12)).

We see that for a maximal responsive map G , (3.14) may be written as

$$X \in \partial D(p), \quad (4.7)$$

which identifies D as a *pseudopotential* for X (see Moreau [20]). What is interesting here, of course, is the equivalence of properties (a) and (b) in the theorem: maximal responsiveness of G is *necessary* and *sufficient* for the existence of D with the requisite properties.

We also note that the assumption of maximal responsiveness *implies* the existence of a region K in which all achievable values X will be found. The interior of this region is commonly known as the elastic region since

$$X \in \text{int } K \Rightarrow N_K(X) = \{0\},$$

so that

$$p = 0 \quad \text{for } X \in \text{int } K. \quad (4.8)$$

The boundary $\text{bd } K$ of K is known as the *yield surface*: it is only those X which lie on $\text{bd } K$ which have associated with them nonzero values of p . Indeed, we have

$$X \in \text{bd } K \Rightarrow p \in N_K(X) \neq \{0\} \quad (4.9)$$

which is simply the classical *normality rule*.

Since D^{**} , the dual of the indicator function D^* , is equal to D , we have embodied in Theorem 4.1 and Lemma 4.1 three *equivalent* formulations of the evolution

equation, summarised below:

$$\begin{aligned} &G \text{ maximal responsive} \\ &X \in G(p) \end{aligned} \tag{I}$$

\Updownarrow

$$\begin{aligned} &D \text{ convex, pos hom, lsc,} \\ &D(p) \geq 0, \quad D(0) = 0 \\ &X \in \partial D(p) \end{aligned} \tag{II}$$

\Updownarrow

$$\begin{aligned} &K \text{ closed, convex, contains zero} \\ &D^* = \text{indicator function of } K \\ &p \in \partial D^*(X). \end{aligned} \tag{III}$$

The formulation (III) is well known, and goes back to Moreau [20]. Formulation (II) is sometimes mentioned as a consequence of (III) but is seldom given prominence in its own right. An exception is the work by Martin [17], in which the evolution equation is assumed to be of the form (II). Formulation (I), which we have taken as the starting point, is new, though there is some connection with the work of Rice [24] and of Hill [10], who regard the maximum plastic work inequality (4.2) as a fundamental property of a plastic material.

These three formulations show clearly the *minimal* assumptions which need to be made if an acceptable classical theory is to emerge. In particular, we see that (I) and (II) do not require the assumption of an elastic region and yield surface: these are *consequences*. Practical considerations would dictate which of these formulations would be most appropriate for the problem at hand. For example, (III) is most often used, in one guise or another; (II) has been used in [2, 5, 17, and 18]; (I) may have limitations in that it is not simple or natural to formulate evolution equations in this form, except perhaps for one-dimensional problems. The major benefit of (I), though, is that it resolves the issue of how much information needs to be added to the assumption of the maximum plastic work inequality, in order that it form the basis of an internal variable theory of plasticity.

5. The yield function and its relation to the dissipation function. It is common in engineering practice to identify the region K by defining it to be a level set of a given function:

$$K = \{X: f(X) \leq \text{const}\}. \tag{5.1}$$

By a slight abuse of terminology we refer to f as the yield function: generally it is the function $(f - \text{const})$ (where const is prescribed) which is given this name. We show in this section that, given K closed, convex, nonempty, it is always possible to

construct a yield function f which is a *gauge*. Thus $\text{epi } f$ is a closed convex cone containing the origin.

We also investigate the dual relationship between f and D . So far we know that D and the indicator function D^* of K are *conjugate*, in the sense that

$$\zeta \cdot q \leq D^*(\zeta) + D(q) \quad \text{for all } \zeta \in E^*, q \in E, \tag{5.2}$$

with equality when ζ is related to q through

$$\begin{aligned} D^*(\zeta) &= \sup_{r \in E} \{\zeta \cdot r - D(r)\} \\ &= \zeta \cdot q - D(q). \end{aligned} \tag{5.3}$$

We will show that f and D are also dual in a different sense: in the language of convex analysis they are *polars* of each other. This type of relationship between pairs of functions has also been investigated by Hill [11], who refers to such pairs as *dual potentials*. Rather than invoke the theory of polar functions [25], we develop these ideas from first principles, making use of the general framework established in Secs. 2 and 4.

Given the closed convex set K defined by (4.4), we define a function f on E^* , called the *canonical yield function*, by

$$f(X) = \inf\{\mu > 0: X \in \mu K\} \tag{5.4}$$

where $\mu K = \{\mu Y: Y \in K\}$. From (5.4) it is evident that f is a *nonnegative, positively homogeneous convex* function whose level set at 1 is equal to K (see Fig. 5.1), that is, f is a *gauge*. Note that $f(X)$ can take the value $+\infty$ (when $X \notin \mu K$ for any $\mu > 0$).

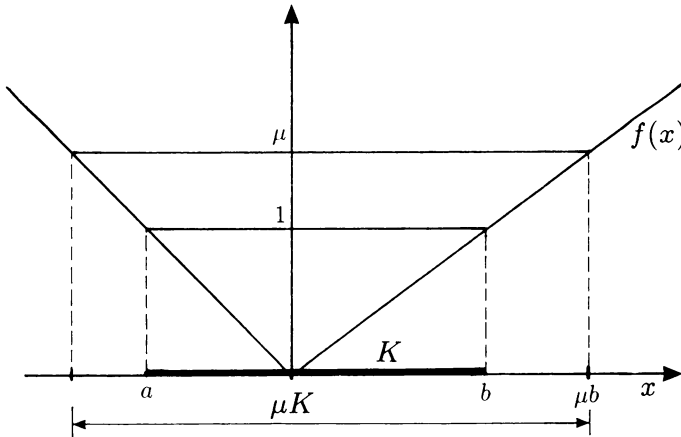


FIG. 5.1. The gauge function of the set $K = [a, b]$.

From the definitions (4.4), (5.4) of K and of f , an alternative form for f is

$$f(X) = \inf\{\mu > 0: X \cdot q \leq \mu D(q) \text{ for all } q\}. \tag{5.5}$$

Now assume that $D(q) = 0$ if and only if $q = 0$ (this assumption is consistent with the proposed application); then f and D are related by

$$f(X) = \sup_{\substack{q \in \text{dom } D \\ q \neq 0}} \frac{X \cdot q}{D(q)}; \quad (5.6)$$

in other words, we have the inequality

$$X \cdot q \leq f(X)D(q) \quad \text{for all } X \in E, q \in \text{dom } D \quad (5.7)$$

(we adopt the convention $0 \cdot \infty = \infty \cdot 0 = 0$, thus avoiding ambiguities in the event that the right-hand side of (5.7) takes one of these forms). Consider $X \in \text{bd } K$: then

$$\sup_{\substack{q \neq 0 \\ q \in \text{dom } D}} \frac{X \cdot q}{D(q)} \leq 1 \quad (5.8)$$

and the supremum is achieved when $q = p$, say, and p is conjugate to X in the sense of Lemma 4.1 (d). Thus, for $X \in K$ and $X \in \partial D(p)$, $p \neq 0$,

$$X \cdot p = f(X)D(p). \quad (5.9)$$

Thus whereas D and D^* are *conjugate* in the sense of (5.2), (5.3), D and f are *polar* in the sense of (5.7), (5.9). Furthermore, just as $D^{**} = D$, it can be shown [25] that if $f = D^0$, then $D^{00} = D$ (for this we need the lower semicontinuity of D).

The introduction of the canonical yield function f allows us to express the normality rule (4.9) in another form. For this we need

LEMMA 5.1. Let g be nonnegative and convex, with $g(0) = 0$ and x a point in the interior of $\text{dom } g$ such that $g(x) > 0$. Set $C = \{z: g(z) \leq g(x)\}$. Then $x^* \in N_C(x)$ if and only if there exists $\lambda \geq 0$ such that $x^* \in \lambda \partial g(x)$.

REMARK. Lemma 5.1 appears in [25] as a Corollary (see Cor. 23.7.1). We give a simple proof here.

Proof. Assume first that $x^* \in \lambda \partial g(x)$ for some $\lambda \geq 0$. Then $x^* = \lambda y^*$ for $y^* \in \partial g(x)$, and

$$g(z) \geq g(x) + y^* \cdot (z - x).$$

For $z \in C$, $g(z) \leq g(x)$, so that

$$y^* \cdot (z - x) \leq 0 \Rightarrow x^* \cdot (z - x) \leq 0$$

for all $z \in C$, hence $x^* \in N_C(x)$.

Now assume that $x^* \in N_C(x)$. We want to show that there exists $\lambda \geq 0$ such that $x^* = \lambda z^*$ and

$$g(y) \geq g(x) + z^* \cdot (y - x) \quad (i)$$

for all y . If $x^* \neq 0$ and $x^* \cdot (y - x) = 0$ then $x \in \text{bd } C$ and since C is convex, $y \in (\text{int } C)^0$, the complement of $\text{int } C$. Hence $g(y) \geq g(x)$, so that (i) holds. Assume next that $x^* \cdot (y - x) < 0$. Then for $y \in E \setminus C$ we have $g(y) - g(x) \geq 0$ so that (i) holds for any $\lambda \geq 0$. Finally suppose that $y \in C$. Then for some $\mu > 0$

$$g(y) - g(x) - \mu x^* \cdot (y - x) = \left\{ \frac{g(y) - g(x)}{x^* \cdot (y - x)} - \mu \right\} x^* \cdot (y - x)$$

which is nonnegative provided that $\mu > \max\{(g(y) - g(x))/(x^* \cdot (y - x)): y \in C\}$. The result follows with $\lambda = \mu^{-1}$. \square

Various results follow from the lemma. First, by setting $g = f$ and $x^* = X$ where $f(X) = 1$, so that $C = K$, it is found that

$$p \in \lambda \partial f(X). \tag{5.10}$$

If f is differentiable at X we recover the normality rule in its classical form

$$p = \lambda \frac{\partial f}{\partial X}, \quad \lambda \geq 0. \tag{5.11}$$

It is also possible to characterise λ by using the properties of f . Indeed, from (5.10) we have

$$p = \lambda p' \quad \text{where } p' \in \partial f(X),$$

that is,

$$f(Y) - f(X) - p' \cdot (Y - X) \geq 0 \quad \text{for all } Y \in E^*.$$

Since $f(X) = 1$ and $f(0) = 0$, we have $X \cdot p' \geq 1$. Furthermore, setting $Y = 2X$, since $f(2X) = 2f(X)$ we obtain $X \cdot p' \leq 1$. In other words,

$$X \cdot p' = 1$$

or

$$\lambda = \lambda X \cdot p' = X \cdot p = D(p). \tag{5.12}$$

Thus the scalar multiplier λ associated with p has the same value as $D(p)$.

Lemma 4.1 may also be applied to the dissipation function; setting $g = D$ and defining

$$C = \{q: D(q) \leq D(p)\}$$

for a given $p \neq 0$, we have immediately, for X related to p through (4.7),

$$X \in \partial D(p) \Rightarrow X \in N_C(p). \tag{5.13}$$

There is also the converse result

$$X \in N_C(p) \Rightarrow X \in \lambda \partial D(p) \quad \text{for some } \lambda > 0$$

(we exclude the possibility $\lambda = 0$ since $X \neq 0$).

This situation is illustrated in Fig. 5.2: in E^* the conjugate pair (X, p) is such that p lies in the normal cone to K (the level set $f(Y) \leq 1$) at X , while in E we find that X lies in the normal cone to C (the level set $D(q) \leq D(p)$) at p .

We conclude this section by recovering the plastic constitutive equations in their conventional classical form: to do this it is necessary to add a further axiom, the *consistency condition*, which states that at $X \in \text{bd } K$,

$$\lambda \dot{f} = 0 \tag{5.14}$$

where \dot{f} is the change in f associated with the change in X accompanying $\dot{\xi}$, i.e., p . Thus either

$$\lambda > 0, \quad \dot{f} = 0$$

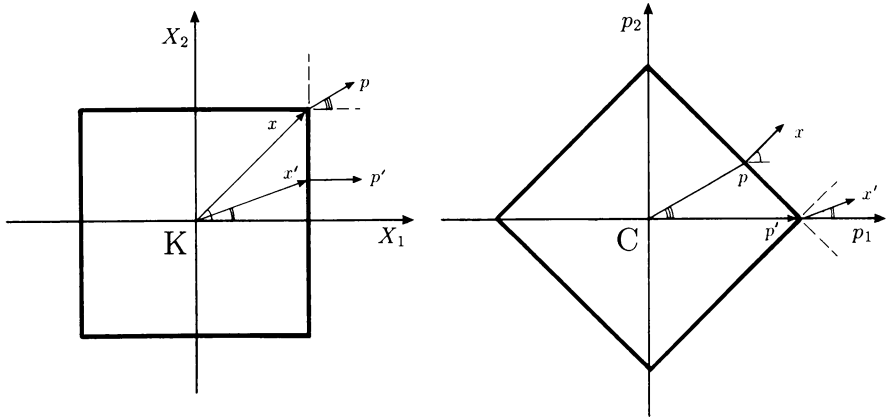


FIG. 5.2. Conjugate pairs (X, p) as normals to $K = \{Y: f(Y) \leq 1\}$ and $C = \{q: D(q) \leq D(p)\}$.

or

$$\dot{f} < 0, \quad \lambda = 0$$

($\dot{f} > 0$ is not permitted since we require $X \in K$ always). Then one form of the complete set of equations prescribing the evolution of ξ is

$$p \in \lambda \partial f(X)$$

where

$$\lambda \geq 0 \quad \text{if } f(X) = 1, \dot{f} = 0$$

and

$$\begin{aligned} \lambda = 0 \quad &\text{if } f(X) = 1, \dot{f} < 0 \\ &\text{or } f(X) < 1. \end{aligned} \tag{5.15}$$

Furthermore $\lambda = D(p)$.

6. Examples. We first give some one-dimensional examples in which formulation (I) of Sec. 4 is used to obtain a full description of plastic behaviour. We then investigate the application of the theory of two nontrivial higher-dimensional examples.

6.1. *Single slip plane or uniform rod extension.* A single internal variable ξ is introduced which characterizes the rigid plastic extension of a uniform rod (see Fig. 6.1 (a)). This same system could be interpreted as the slip on a given plane [20]. It is assumed that changes to ξ take place only when the conjugate force X reaches a magnitude of X_0 . This assumption is sufficient to establish the set-valued map $G(p)$. This is illustrated in Fig. 6.1 (b). The form of the dissipation function is easily obtained from $G(p) = \partial D(p)$ and is shown in Fig. 6.1 (c).

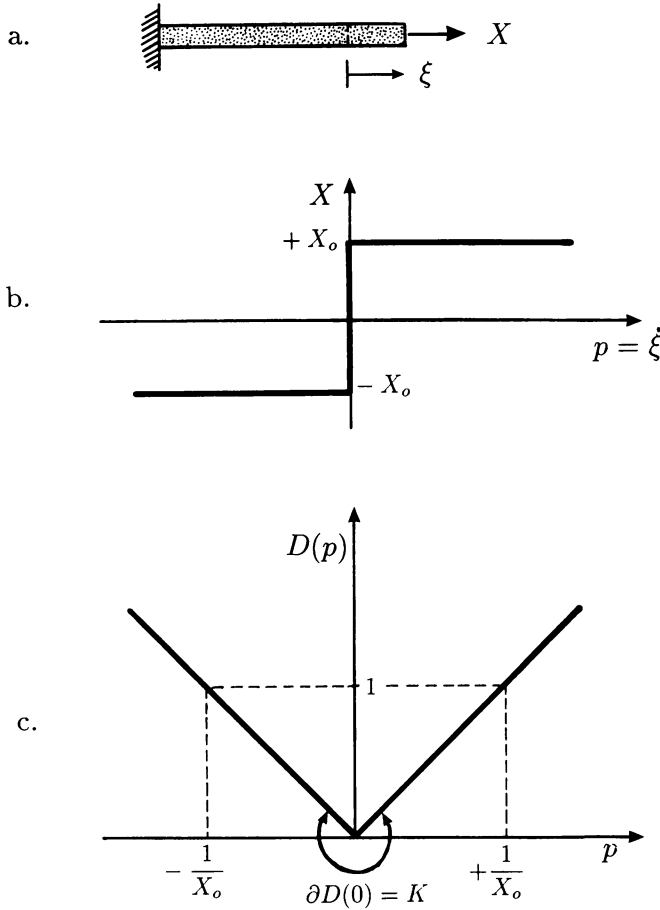


FIG. 6.1. Uniform rod. (a) Geometric interpretation. (b) $G(p) = \{X_0 \operatorname{sgn} p\}$, $p \neq 0$; $G(0) = [-X_0, X_0]$. (c) The dissipation function $D(p) = X_0|p|$.

6.2. *Constrained slip system or rod.* This example is very similar to the previous one, but here negative forces X are unable to bring about any change in ξ . A geometric interpretation of this system is shown in Fig. 6.2 (a).

Again the set-valued map $G(p)$ is established directly. This example is of particular interest since the map is defined only for nonnegative values of p , i.e., $\operatorname{dom} D = [0, \infty)$.

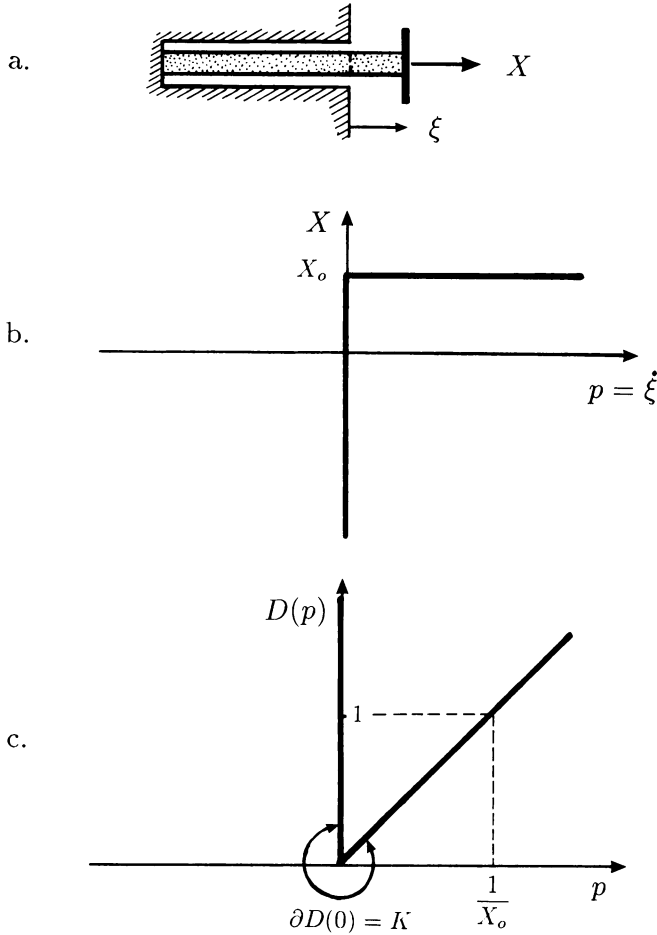


FIG. 6.2. Constrained rod. (a) Geometric interpretation. (b) $G(p) = \emptyset, p < 0; G(0) = (-\infty, X_0]; G(p) = \{X_0\}, p > 0$. (c) The dissipation function $D(p) = +\infty, p < 0; D(p) = X_0 p, p \geq 0$. $\text{dom } D = [0, +\infty)$.

6.3. *Unidirectional dissipation.* This example is an interesting special case of a rate-independent plasticity theory. Here we model a material which shows plastic, dissipative behaviour only in one direction, under compression here. In this direction changes in the internal variables occur when the force X attains a value of magnitude $X_0 < 0$. In the other direction, the material offers no resistance to change in internal variables. Such behaviour may model materials like concrete or soil where there is negligible tensile strength. The map $G(p)$ and the dissipation function are shown in Fig. 6.3.

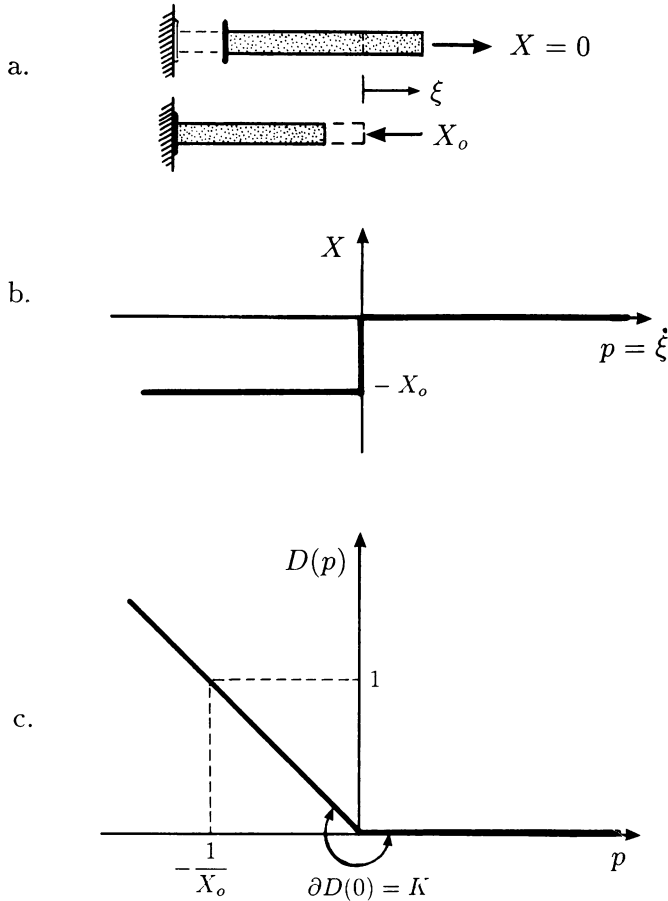


FIG. 6.3. No-tension example. (a) Geometric interpretation. (b) $G(p) = -X_0, p < 0; G(0) = [-X_0, 0]; G(p) = 0, p > 0$. (c) The dissipation function $D(p) = X_0|p|, p < 0; D(p) = 0, p \geq 0$. $\text{dom } D(p) = (-\infty, 0]$.

6.4. *The contact problem.* In this extreme example we have an internal variable model for frictionless contact. The internal variable ξ measures distance from a rigid obstacle and so $\xi \geq 0$. The conjugate force X is the reaction force experienced when contact takes place. We have

$$X \leq 0 \quad \text{if } \dot{\xi} = 0$$

and

$$X = 0 \quad \text{if } \dot{\xi} > 0.$$

This situation and the resulting function G and dissipation function D are shown in Fig. 6.4.

A point worth emphasizing is that, in the complete set (3.11) of equations characterizing the material behaviour, the evolution equation (3.11)₄ describes the nature of the irreversible behaviour, but effects such as hardening are to be found in the free

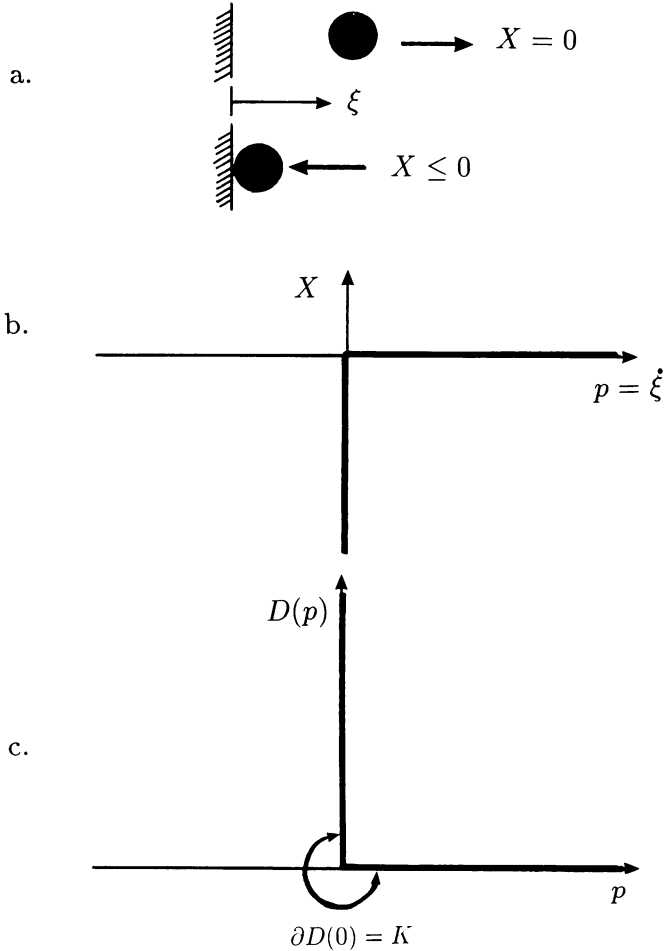


FIG. 6.4. A contact problem. (a) Geometric interpretation. (b) $G(p) = \emptyset$, $p < 0$; $G(p) = \{0\}$, $p \geq 0$. (c) The dissipation function $D(p) = +\infty$, $p < 0$; $D(p) = 0$, $p \geq 0$. $\text{dom } D = [0, +\infty)$.

energy function ψ . This contrasts with the methods and terminology of, for example, Simo [27] and Kim and Oden [13]. To clarify what we mean consider the following well known example.

6.5. *Elastoplasticity with the von Mises yield criterion and linear kinematic hardening.* Let the internal variable ξ be identified with the plastic strain tensor ε^p . We define the free energy function $\bar{\psi}$ by

$$\rho \bar{\psi}(\varepsilon, \xi) = \frac{1}{2}(\varepsilon - \xi^D) \cdot C(\varepsilon - \xi^D) + \frac{1}{2}h|\xi^D|^2,$$

where C is the fourth-order tensor of elastic moduli, $h \geq 0$ is the hardening constant, and $\xi^D = \xi - \frac{1}{3}(\text{tr } \xi)I$ denotes the deviator of ξ . The density ρ is assumed constant.

We obtain

$$\begin{aligned}\sigma &= \rho \frac{\partial \bar{\psi}}{\partial \varepsilon} = C(\varepsilon - \xi^D), \\ X &= -\rho \frac{\partial \bar{\psi}}{\partial \xi} = [C(\varepsilon - \xi)]^D - h\xi^D\end{aligned}$$

so the conjugate force is

$$X = (\sigma - h\xi)^D.$$

The von Mises canonical yield function is

$$f(X) = k^{-1}|X| = k^{-1}(X \cdot X)^{1/2}$$

where k is a prescribed constant so the yield surface is a sphere of radius k . The flow law in the form (5.11) is then

$$\dot{\xi} = \lambda \frac{\partial f}{\partial X} = \lambda \frac{X}{|X|}, \quad \lambda \geq 0, \quad (6.1)$$

(the constant k^{-1} has been absorbed into λ) when $f(X) = 1$, and from (6.1) we find that $\text{tr} \dot{\xi} = \text{tr} \dot{\varepsilon}^p = 0$, so that no volume change accompanies plastic deformation (indeed, this constraint motivates the particular choice of free energy involving ξ^D).

The dissipation function D is found from

$$D(p) = \sup_{X \in K} X \cdot p = k|p|$$

and the function G from

$$G(p) = \partial D(p) = \begin{cases} \{p/k|p|\}, & p \neq 0, \\ \{\tau: \tau \cdot q \leq D(q) \forall q\}, & p = 0. \end{cases}$$

The example of a von Mises yield criterion with linear isotropic hardening may be treated in a similar manner, though the corresponding details are less straightforward to derive (see Martin and Nappi [18]).

The relations (5.15) can be written in more conventional form if we set

$$f(X) = f(X(\sigma, \xi)) = \hat{f}(\sigma, \xi).$$

Then for $\lambda \geq 0$ we have

$$0 = \dot{f}(X) = \frac{\partial \hat{f}}{\partial \sigma} \cdot \dot{\sigma} + \frac{\partial \hat{f}}{\partial \xi} \cdot \dot{\xi}.$$

But $\partial \hat{f} / \partial \xi = -hk^{-1}X/|X|$ and $\dot{\xi} = \lambda \partial f / \partial X = \lambda k^{-1}X/|X|$, so that

$$\frac{\partial \hat{f}}{\partial \xi} \cdot \dot{\xi} \leq 0.$$

Hence

$$\lambda \geq 0 \quad \text{if } f(X) = 1 \quad \text{and} \quad \frac{\partial \hat{f}}{\partial \sigma} \cdot \dot{\sigma} \geq 0.$$

Similarly,

$$\begin{aligned}\lambda &= 0 \quad \text{if } f(X) < 1 \\ &\text{or } f(X) = 1 \quad \text{and} \quad \frac{\partial \hat{f}}{\partial \sigma} \cdot \dot{\sigma} < 0.\end{aligned}$$

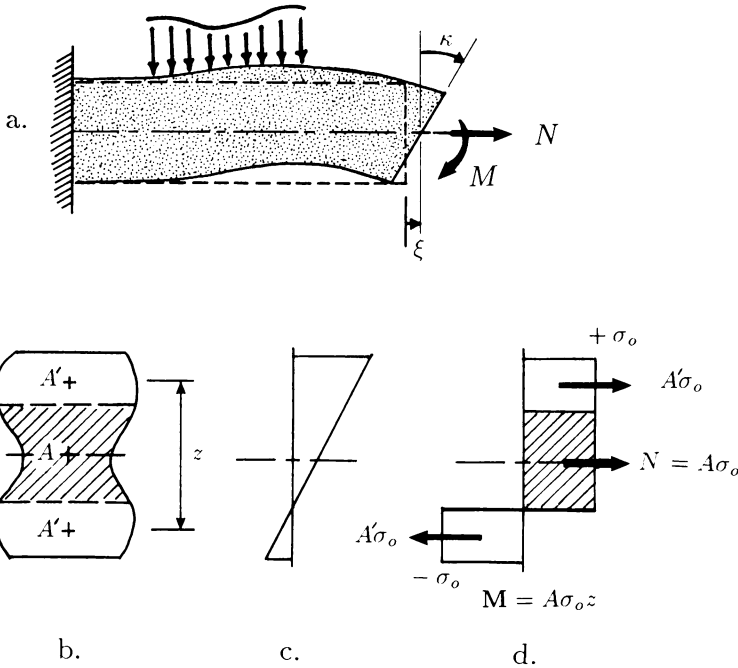


FIG. 6.5. (a) A uniform rigid-plastic beam; internal forces and displacements at an arbitrary cross-section are shown. (b) An arbitrary cross-section. (c) Strain distribution. (d) Stress distribution.

6.6 *Bending and extension of the beam.* The theory of Secs. 4 and 5 applies in general situations, and is not dictated by any physical assumptions other than those embodied in the properties possessed by the sets and functions appearing there. In particular, while our prime motivation has been evolution laws for continuous media, there is no restriction in applying the results to more specialised situations, such as those arising from theories for particular geometrical types such as beams, plates and shells. For example consider a rigid-perfectly plastic beam of arbitrary cross-section subject to bending and extension. The Kirchhoff assumption is imposed, and the two internal variables are the axial extension ξ at the neutral axis and the cross-sectional rotation κ (see Fig. 6.5 (a)). Under these circumstances it is a straightforward matter to show that the region K of admissible forces consists of those values of bending moment M and axial force N satisfying

$$\pm aM \geq b^2 N^2 - 1 \tag{6.2}$$

where a and b are constants depending on the cross-sectional geometry and yield stress (see Fig. 6.6). The conjugate forces corresponding to κ and ξ are thus M and N .

According to Sec. 5 it is possible to express the region K as a level set $\{X: f(X) \leq 1\}$ where f is a gauge and $X = (M, N)$. To do this we rewrite (6.2) as

$$\pm aM + 1 \geq b^2 N^2$$

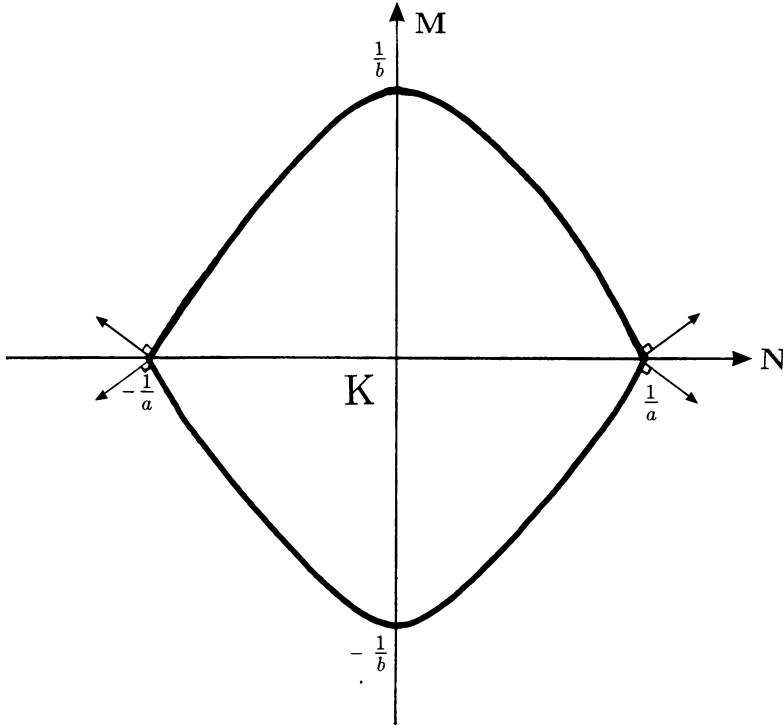


FIG. 6.6. Yield function for beam example.

and complete the square to get

$$(1 \pm aM)^2 \geq b^2 N^2 + \frac{a^2}{4} M^2$$

whence the canonical yield function is

$$f(X) \equiv \mp \frac{a}{2} M + (b^2 N^2 + \frac{a^2}{4} M^2)^{1/2} \leq 1.$$

The dissipation function is then found from

$$\begin{aligned} D(p) &= \sup_{X \in K} X \cdot p \\ &= \sup_{f(M,N) \leq 1} (Mp_1 + Np_2) \\ &= \sup_{f(M,N)=1} (Mp_1 + Np_2). \end{aligned}$$

Consider any p satisfying $p \in \lambda \partial f(X)$ for X such that f is differentiable, that is, for the set

$$\{(M, N): |N| < 1/b, M = \pm \frac{1}{a}(1 - b^2 N^2)\}.$$

For these values we have $p = \lambda \partial f / \partial X$, from which we find after some manipulation,

$$D(\dot{\kappa}, \dot{\xi}) = \frac{a}{4b} (\dot{\xi}^2 / \dot{\kappa}) + \frac{1}{a} \dot{\kappa}.$$

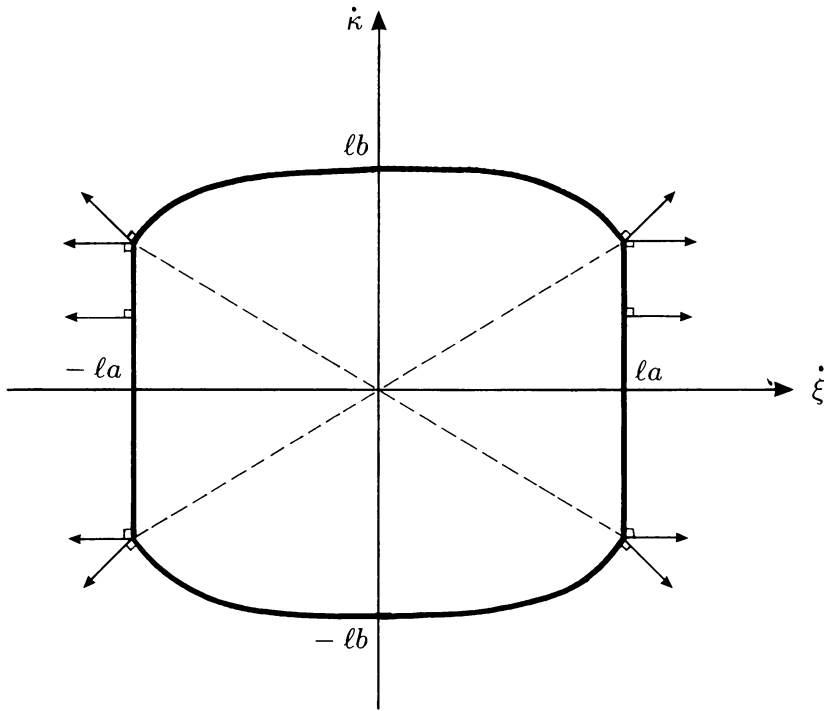


FIG. 6.7. The l -level set of the dissipation function for the beam example.

When $f(X) = 1$ and $M = 0$, $N = \pm 1/b$, f is not differentiable. At these points

$$D(\dot{\kappa}, \dot{\xi}) = \pm \frac{1}{b} \dot{\xi}.$$

The level set $\{q: D(q) \leq D(p)\}$ (see Eq. (5.13) and preceding equation) is shown in Fig. 6.7.

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