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UNCOUPLING THE DIFFERENTIAL EQUATIONS ARISING FROM A TECHNIQUE FOR EVALUATING INDEFINITE INTEGRALS CONTAINING SPECIAL FUNCTIONS OR THEIR PRODUCTS

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Abstract. A previous article by Piquette and Van Buren [1] described an analytical technique for evaluating indefinite integrals involving special functions or their products. The technique replaces the integral by an inhomogeneous set of coupled first-order differential equations. This coupled set does not explicitly contain the special functions of the integrand, and any particular solution of the set is sufficient to obtain an analytical expression for the indefinite integral. It is shown here that the coupled set arising from the method always occurs in normal form. Hence, it is amenable to the method of Forsyth [6] for uncoupling such a set. That is, the solution of the set can be made to depend upon the solution of a single differential equation of order equal to the number of equations in the set. Any particular solution of this single equation is then sufficient to yield the desired indefinite integral. As examples, the uncoupled equation is given here for integrals involving (i) the product of two Bessel functions, (ii) the product of two Hermite functions, or (iii) the product of two Laguerre functions, and a tabulation of integrals of these types is provided. Examples involving products of three or four special functions are also provided. The method can be used to extend the integration capabilities of symbolic-mathematics computer programs so that they can handle broad classes of indefinite integrals containing special functions or their products.

I. Introduction. A previous article [1] presented an analytical technique for evaluating indefinite integrals of the form

$$I = \int f(x) \prod_{i=1}^{m} R_{\mu_i}^{(i)}(x)\, dx, \tag{1}$$

where $R_{\mu_i}^{(i)}(x)$ is the ith type of special function of order μ_i obeying the set of recurrence relations

$$R_{\mu+1}^{(i)}(x) = a_\mu(x) R_\mu^{(i)}(x) + b_\mu(x) R_{\mu-1}^{(i)}(x) \tag{2a}$$

$$D R_\mu^{(i)}(x) = c_\mu(x) R_\mu^{(i)}(x) + d_\mu(x) R_{\mu-1}^{(i)}(x). \tag{2b}$$

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Here a_μ, b_μ, c_μ, and d_μ are known functions corresponding to $R_\mu^{(i)}$. The superscript i on the function R denotes that the integral I may contain a mixture of special functions; i.e., each of the special functions of the product term of Eq. (1) may be of different type. The symbol D represents d/dx. The symbol $f(x)$ is used as a generic notation to denote any portion of the integrand which is not a member of the class of special functions obeying the relations of Eqs. (2). The function $f(x)$ and the product $\prod R_\mu^{(i)}$ are both assumed continuous (or with at most a finite number of discontinuities) over an interval $[x_1, x_2]$, insuring that the integral I exists in the same interval. The technique is a generalization of one used by Sonine [2] (described by Watson [3]) to evaluate indefinite integrals that involve a Bessel function. Reference [1] extended the technique to include products of special functions. These functions include most of the special functions of physics, such as Legendre functions, Hermite functions, Laguerre functions, etc. That is, the technique applies to any integral that contains one or more functions R that obey recurrence relations of the form represented by Eqs. (2). The technique replaces the integral of Eq. (1) with an inhomogeneous set of coupled first-order differential equations. The coupled set does not explicitly contain any of the special functions R, and *any particular solution* of the set is sufficient to yield an analytical expression for the integral I. In [1], the terminology "birecurrent functions" was introduced for the functions R. We will continue the use of this terminology here. (It is interesting to note that the birecurrent functions are a subset of a more general class of functions studied by Truesdell [4].)

Section II presents a review of the integration technique. The uncoupling procedure is described in Sec. III. Examples of applications of the technique to integrals containing either Bessel functions, Hermite functions, or Laguerre functions are given in Sec. IV. A comparison with previous techniques is presented in Sec. V. A summary and the conclusions are given in Sec. VI.

II. Review of the technique. The method of [1] assumes that the integral I of Eq. (1) may be represented by the expression

$$I = \sum_{p_1=0}^{1} \sum_{p_2=0}^{1} \cdots \sum_{p_m=0}^{1} A_{p_1 p_2 \cdots p_m}(x) \prod_{i=1}^{m} R_{(\mu_i+p_i)}^{(i)}(x), \tag{3}$$

where the 2^m coefficients $A_{p_1 p_2 \cdots p_m}(x)$ are functions to be determined. The technique replaces the integral I with the coupled set of differential equations

$$f(x)\delta_{0,p} = DA_p + \sum_{\{q\}} B_{pq} A_q, \tag{4}$$

where δ is a Kronecker delta defined to be zero unless $p_1 = p_2 = \cdots = p_m = 0$. In Eq. (4), the shorthand notations $A_p = A_{p_1 p_2 \cdots p_m}(x)$ and $B_{pq} = B_{p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_m}(x)$ have been used. Also, the notation $\sum_{\{q\}}$ represents the multiple summations

$$\sum_{q_1=0}^{1} \sum_{q_2=0}^{1} \cdots \sum_{q_m=0}^{1}.$$

The functions B_{pq} are related to the functions a, b, c, and d of Eqs. (2). They are given by the equation in [5]

$$B_{pq} = \sum_{j=1}^{m} \left[\left(a'_{\mu_j}(x) \prod_{\ell=1}^{m} \delta_{q_\ell,p_\ell} + c'_{\mu_j}(x) \prod_{\ell=1}^{m} \delta_{q_\ell,p_\ell+\delta_{\ell j}} \right) \delta_{p_j,0} + \left(b'_{\mu_j}(x) \prod_{\ell=1}^{m} \delta_{q_\ell,p_\ell-\delta_{\ell j}} + d'_{\mu_j}(x) \prod_{\ell=1}^{m} \delta_{q_\ell,p_\ell} \right) \delta_{p_j,1} \right], \tag{5}$$

where

$$a'_\mu(x) \equiv c_\mu(x) - \frac{a_\mu(x) d_\mu(x)}{b_\mu(x)}, \tag{6a}$$

$$b'_\mu(x) \equiv d_\mu(x)/b_\mu(x), \tag{6b}$$

$$c'_\mu(x) \equiv d_{\mu+1}(x), \tag{6c}$$

$$d'_\mu(x) \equiv c_{\mu+1}(x). \tag{6d}$$

The functions a, b, c, d of Eqs. (6) are the coefficients of the special functions occurring in the recurrence relations of Eqs. (2).

III. Uncoupling the set of differential equations. Any particular solution of the coupled set represented by Eq. (4) for the functions A is sufficient to obtain an analytical expression for the integral I of Eq. (1) via the representation of Eq. (3). However, directly obtaining a particular solution of this set can be difficult to do. It is usually more straightforward to uncouple one function A from the remainder of the set, and instead to obtain a particular solution of the resulting uncoupled equation. It is the purpose of the present section to describe a procedure that will uncouple the set represented by Eq. (4) in general. The method is due to Forsyth [6].

Before proceeding, it is helpful to first rearrange Eq. (4) in the form

$$DA_p = f(x)\delta_{0,p} - \sum_{\{q\}} B_{pq} A_q. \tag{7}$$

Note that the set represented by Eq. (7) is in normal form [7].

A. *Uncoupling the set arising from an integrand containing two birecurrent functions.* Prior to considering the uncoupling of the set associated with the general case in which the integrand of Eq. (1) may contain m special functions R, it is helpful to first consider a restricted case. We will, therefore, initially restrict our attention to the case in which Eq. (1) contains a product of only two special functions (in addition to containing the generic function $f(x)$). That is, we will consider the restricted case in which

$$I = \int f(x) R^{(1)}_{\mu_1}(x) R^{(2)}_{\mu_2}(x)\, dx. \tag{8}$$

In this case, Eq. (3) generates the representation

$$I = A_{00}(x) R^{(1)}_{\mu_1}(x) R^{(2)}_{\mu_2}(x) + A_{01}(x) R^{(1)}_{\mu_1}(x) R^{(2)}_{\mu_2+1}(x) + A_{10}(x) R^{(1)}_{\mu_1+1}(x) R^{(2)}_{\mu_2}(x) + A_{11}(x) R^{(1)}_{\mu_1+1}(x) R^{(2)}_{\mu_2+1}(x). \tag{9}$$

Also, Eq. (7) will generate a coupled set of four simultaneous differential equations for the functions A. This set is of the general form

$$DA_{00}(x) = g_1(x)A_{00}(x) + g_2(x)A_{01}(x) + g_3(x)A_{10}(x) + g_4(x)A_{11}(x) + f(x), \quad (10a)$$

$$DA_{01}(x) = g_5(x)A_{00}(x) + g_6(x)A_{01}(x) + g_7(x)A_{10}(x) + g_8(x)A_{11}(x), \quad (10b)$$

$$DA_{10}(x) = g_9(x)A_{00}(x) + g_{10}(x)A_{01}(x) + g_{11}(x)A_{10}(x) + g_{12}(x)A_{11}(x), \quad (10c)$$

and

$$DA_{11}(x) = g_{13}(x)A_{00}(x) + g_{14}(x)A_{01}(x) + g_{15}(x)A_{10}(x) + g_{16}(x)A_{11}(x). \quad (10d)$$

The functions g in the set represented by Eqs. (10) have been introduced as a generic representation of the coefficients of the functions A arising in the coupled set since we are not concerned here with the specific functional forms of these coefficients.

Next, we must select which one of the functions A we wish to uncouple from the remainder of the set. It is convenient to choose to uncouple the function A_{11} (although any other of the functions A could have been selected instead).

Choosing A_{11}, then, as the function to be uncoupled, we next begin to generate a new set of coupled equations. The first member of the new set of equations is chosen to be that equation of the original set which contains the derivative of the particular function we wish to uncouple. (Since the original set always occurs in normal form, we are assured that one and only one equation of this set will contain the derivative of the particular function A we have chosen to uncouple.) Thus, based on the choice to uncouple A_{11} from the remainder of the original set, the first member of the new coupled set (note Eqs. (10)) is

$$DA_{11}(x) = g_{13}(x)A_{00}(x) + g_{14}(x)A_{01}(x) + g_{15}(x)A_{10}(x) + g_{16}(x)A_{11}(x). \quad (11)$$

To obtain the second equation of the new set, we apply the derivative operator D to each side of Eq. (11), thus obtaining the equation

$$\begin{aligned} D^2A_{11}(x) = {} & [Dg_{13}(x)]A_{00}(x) + [Dg_{14}(x)]A_{01}(x) \\ & + [Dg_{15}(x)]A_{10}(x) + [Dg_{16}(x)]A_{11}(x) \\ & + g_{13}(x)DA_{00}(x) + g_{14}(x)DA_{01}(x) + g_{15}(x)DA_{10}(x) + g_{16}(x)DA_{11}(x). \end{aligned} \quad (12)$$

The four first-order derivatives DA_{00}, DA_{01}, DA_{10}, and DA_{11} appearing on the right-hand side of Eq. (12) may now be eliminated in favor of the undifferentiated functions A_{00}, A_{01}, A_{10}, and A_{11} by substituting the expressions for each of these derivatives given by the original set (see again Eqs. (10)). The result of this series of substitutions into Eq. (12) may be denoted by the resultant equation

$$\begin{aligned} D^2A_{11}(x) = {} & h_1(x)A_{00}(x) + h_2(x)A_{01}(x) + h_3(x)A_{10}(x) \\ & + h_4(x)A_{11}(x) + h_5(x)f(x). \end{aligned} \quad (13)$$

The quantities h of Eq. (13) are again merely generic forms of the relevant functions, since we are not concerned here with the specific functional forms.

The third member of the new set of differential equations is generated by using a similar sequence of calculations as was used to generate Eq. (13). That is, if we now apply the operator D to each side of Eq. (13), and once again eliminate all first-order derivatives of the functions A occurring on the right-hand side of this new equation by using Eqs. (10), we will obtain another equation of the same general form as Eq. (13). Of course, $D^3A_{11}(x)$ will appear on the left-hand side of this new equation instead of $D^2A_{11}(x)$, and $Df(x)$ will appear on the right-hand side in addition to $f(x)$. We can apply this procedure once more, this time to the new equation which contains $D^3A_{11}(x)$, to create the fourth equation of the new set. This fourth equation will again be of the same general form as Eq. (13), except that the quantity $D^4A_{11}(x)$ will appear on the left-hand side, and $D^2f(x)$ and $Df(x)$ will appear on the right-hand side in addition to $f(x)$. This procedure will therefore produce the new set of differential equations

$$DA_{11}(x) = g_{13}(x)A_{00}(x) + g_{14}(x)A_{01}(x) + g_{15}(x)A_{10}(x) + g_{16}(x)A_{11}(x), \tag{14a}$$

$$D^2A_{11}(x) = h_1(x)A_{00}(x) + h_2(x)A_{01}(x) + h_3(x)A_{10}(x) + h_4(x)A_{11}(x) + h_5(x)f(x), \tag{14b}$$

$$\begin{aligned} D^3A_{11}(x) = {} & h_6(x)A_{00}(x) + h_7(x)A_{01}(x) + h_8(x)A_{10}(x) + h_9(x)A_{11}(x) \\ & + h_{10}(x)Df(x) + h_{11}(x)f(x), \end{aligned} \tag{14c}$$

and

$$\begin{aligned} D^4A_{11}(x) = {} & h_{12}(x)A_{00}(x) + h_{13}(x)A_{01}(x) + h_{14}(x)A_{10}(x) + h_{15}(x)A_{11}(x) \\ & + h_{16}(x)D^2f(x) + h_{17}(x)Df(x) + h_{18}(x)f(x). \end{aligned} \tag{14d}$$

Note that the three functions A_{00}, A_{01}, and A_{10} only occur algebraically in this new set. That is, no derivatives of these functions appear in Eqs. (14). This means that these three functions can be expressed in terms of the function A_{11} (and its first, second, and third derivatives), and in terms of the function $f(x)$ (and its first derivative), as well as in terms of the functions g and h, by algebraic methods. This can be done by solving the first three of Eqs. (14) as a simultaneous set of *algebraic* equations in the unknowns A_{00}, A_{01}, and A_{10}. These algebraic solutions will provide representations of the functions A_{00}, A_{01}, and A_{10} in terms of the function A_{11}. If these representations are then substituted into Eq. (14d), a differential equation in the single unknown A_{11} will result. Note that this differential equation has the properties that it is ordinary, linear, inhomogeneous, and of fourth order. Any particular solution of this uncoupled equation, when combined with the algebraically-obtained solutions to the first three equations of the set, would therefore represent a particular solution to the original set represented by Eqs. (10). This particular solution would then provide, via the representation given by Eq. (9), an analytical expression for the integral of Eq. (8).

B. *Generalizing the uncoupling procedure for an integrand containing m birecurrent functions.* We next generalize the discussion of the preceding subsection so that it will apply to the case in which Eq. (1) contains an arbitrary number of birecurrent

functions R. For the sake of compactness of notation, we define a quantity C_p such that

$$C_p \equiv f(x)\delta_{0,p} - \sum_{\{q\}} B_{pq} A_q, \tag{15}$$

so that the coupled set represented by Eq. (7) becomes

$$DA_p = C_p. \tag{16}$$

The first step in the uncoupling process is to identify which one of the unknown functions A is to be uncoupled from the remainder. This function may be any one of the functions A_p. (However, the complexity of the final uncoupled equation can vary with this choice.) We will denote by A_ℓ the particular function of the set A_p to be uncoupled from the remainder. That is, we use the subscript ℓ to denote the particular set of subscripts p that represents the particular member of the set A_p that we wish to uncouple from the remainder.

To generalize the uncoupling procedure, outlined above for the particular case in which the coupled set contains four equations, it is convenient to follow the notation used by Forsyth [6], and to introduce an operator θ where

$$\theta \equiv \left.\frac{\partial}{\partial x}\right|_A + \sum_{\{q\}} C_q \left.\frac{\partial}{\partial A_q}\right|_x. \tag{17}$$

The sum in Eq. (17) again represents a shorthand notation which denotes multiple summations over all possible subscripts associated with the functions A (see remarks following Eqs. (4)). In applying the operator θ, the notation $\frac{\partial}{\partial x}|_A$ denotes that the functions A are to be treated as constants. Similarly, explicit appearances of the variable x are to be treated as constants in applying the operator $\frac{\partial}{\partial A}|_x$.

The operator θ summarizes both the effect of differentiation with D, and the elimination of first-order derivatives in favor of undifferentiated functions A via the original set represented by Eq. (16). The resulting new set can be expressed in a compact notation as

$$D^k A_\ell = \theta^{k-1} C_\ell \qquad (k = 1, 2, \ldots, n). \tag{18}$$

Here, $n = 2^m$, where m is the number of special functions appearing in the integral of Eq. (1). In the new set represented by Eq. (18) the only member of the set A that appears underneath the differential operator D^k is the particular function A_ℓ. The remaining $n - 1$ functions A_p appear algebraically. Therefore, the first $n - 1$ equations of the set represented by Eq. (18) may be solved by algebraic methods for all functions A except A_ℓ. The resulting solutions may then be substituted into the nth equation of the set represented by Eq. (18) to produce a differential equation in the single unknown A_ℓ. Any particular solution of this equation can be combined with the algebraic solutions of the first $n - 1$ equations to yield a particular solution of the set represented by Eq. (16). This particular solution will result in an analytical expression for the desired integral I of Eq. (1) via the representation given by Eq. (3). Hence, the process of obtaining the antiderivative of Eq. (1) is reduced to the process of finding any particular solution of the differential equation for the single

function A_ℓ. Since the special functions R of the integral of Eq. (1) do not appear in this equation, the problem of finding a particular solution is frequently not difficult to do.

It is important to note that, since the coupled set resulting from the present integration technique always occurs in normal form, the method of Forsyth can always be used to uncouple the set, regardless of the particular special functions R occurring in the integrand. Thus, even the coupled set occurring in the somewhat unusual case of an integrand containing a *mixed* product of special functions (say a Laguerre function times a Bessel function) can be uncoupled by this method.

We will next consider several examples.

IV. Examples.

A. *Integrals containing the product of two Bessel functions* Consider integrals of the form

$$I = \int Z_\mu(x)\overline{Z}_\nu(x)f(x)\,dx, \tag{19}$$

where Z and $\overline{Z}$ denote any solutions of Bessel's differential equation. For this case, the representation of Eq. (3) becomes

$$\begin{aligned} I = {} & A_{00}(x)Z_\mu(x)\overline{Z}_\nu(x) + A_{01}(x)Z_\mu(x)\overline{Z}_{\nu+1}(x) \\ & + A_{10}(x)Z_{\mu+1}(x)\overline{Z}_\nu(x) + A_{11}(x)Z_{\mu+1}(x)\overline{Z}_{\nu+1}(x). \end{aligned} \tag{20}$$

Equation (7) generates the coupled set

$$DA_{00} = -\frac{(\mu+\nu)}{x}A_{00} - A_{01} - A_{10} + f(x), \tag{21a}$$

$$DA_{01} = A_{00} - \frac{(\mu-\nu-1)}{x}A_{01} - A_{11}, \tag{21b}$$

$$DA_{10} = A_{00} - \frac{(\nu-\mu-1)}{x}A_{10} - A_{11}, \tag{21c}$$

and

$$DA_{11} = A_{01} + A_{10} + \frac{(2+\mu+\nu)}{x}A_{11}. \tag{21d}$$

Following the procedure outlined in Sec. III.A, we choose to uncouple the function A_{11} from the remaining functions A. In this case the resulting uncoupled differential equation is

$$\begin{aligned} & x^4D^4A_{11} - 2x^3D^3A_{11} + x^2(7-2\mu^2-2\nu^2+4x^2)D^2A_{11} \\ & \quad - 3x(5-2\mu^2-2\nu^2)DA_{11} + [(2-\mu)^2-\nu^2][(2+\mu)^2-\nu^2]A_{11} \\ & \quad = 2x^4Df + 2x^3f. \end{aligned} \tag{22}$$

The remaining functions A are expressed in terms of the function A_{11} by the equations

$$\begin{aligned} A_{00} = {} & \frac{x}{2(\mu+\nu)}D^3A_{11} + \frac{(3+\mu+\nu)}{2(\mu+\nu)}D^2A_{11} \\ & + \frac{(-7-3\mu-3\nu-2\mu\nu+\mu^2+\nu^2-4x^2)}{2x(\mu+\nu)}DA_{11} \\ & + \frac{(-2-\mu-\nu)(-4-2\mu\nu+\mu^2+\nu^2-2x^2)}{2x^2(\mu+\nu)}A_{11} + \frac{xf}{\mu+\nu}, \end{aligned} \tag{23}$$

$$A_{01} = \frac{-x^2}{2(\mu^2-\nu^2)}D^3A_{11} + \frac{3x}{2(\mu^2-\nu^2)}D^2A_{11} - \frac{(7-3\mu^2-\nu^2+4x^2)}{2(\mu^2-\nu^2)}DA_{11} + \frac{(4+\mu\nu^2-3\mu^2-\mu^3-\nu^2+2x^2)}{x(\mu^2-\nu^2)}A_{11} + \frac{x^2 f}{(\mu^2-\nu^2)}, \tag{24}$$

and

$$A_{10} = \frac{x^2}{2(\mu^2-\nu^2)}D^3A_{11} - \frac{3x}{2(\mu^2-\nu^2)}D^2A_{11} + \frac{(7-\mu^2-3\nu^2+4x^2)}{2(\mu^2-\nu^2)}DA_{11} - \frac{(4+\mu^2\nu-\mu^2-3\nu^2-\nu^3+2x^2)}{x(\mu^2-\nu^2)}A_{11} - \frac{x^2 f}{(\mu^2-\nu^2)}. \tag{25}$$

A particular solution of the differential equation of Eq. (22) can be found for a number of functions f. For example, if $f(x) = 1/x$, the right-hand side of Eq. (22) vanishes. Hence, $A_{11} = 0$ is a particular solution for this case. Setting $A_{11} = 0$ in Eqs. (23)–(25) yields the remaining functions A which, when substituted into Eq. (20), results in

$$\int \frac{Z_\mu(z)\overline{Z}_\nu(x)}{x}\,dx = \frac{1}{\mu+\nu}\left[Z_\mu\overline{Z}_\nu + \frac{xZ_\mu\overline{Z}_{\nu+1}}{(\mu-\nu)} - \frac{xZ_{\mu+1}\overline{Z}_\nu}{(\mu-\nu)}\right]. \tag{26}$$

An integral equivalent to that presented in Eq. (26) was first obtained by Lommel [8] by studying a generalization of Bessel's equation.

It is also relatively straightforward to obtain a particular solution to Eq. (22) in each of the cases $f(x) = 1/x^2$ and $f(x) = 1/x^3$. Substituting the solutions for A_{11} for each of these cases into Eqs. (23)–(25), and substituting the resulting functions A into Eq. (20), yields the integrals

$$\int \frac{Z_\mu(x)\overline{Z}_\nu(x)\,dx}{x^2} = \frac{-(1+\mu+\nu+2\mu\nu+\mu\nu^2+\mu^2\nu-\mu^2-\mu^3-\nu^2-\nu^3+2x^2)Z_\mu\overline{Z}_\nu}{x(-1+\mu-\nu)(-1+\mu+\nu)(1+\mu-\nu)(1+\mu+\nu)} + \frac{Z_\mu\overline{Z}_{\nu+1}}{(1-\mu-\nu)(1-\mu+\nu)} + \frac{Z_{\mu+1}\overline{Z}_\nu}{(1-\mu-\nu)(1+\mu-\nu)} - \frac{2xZ_{\mu+1}\overline{Z}_{\nu+1}}{(1-\mu-\nu)(1-\mu+\nu)(1+\mu-\nu)(1+\mu+\nu)}, \tag{27}$$

and

$$\begin{aligned}
&\int \frac{Z_\mu(x)\overline{Z}_\nu(x)\,dx}{x^3} \\
&\quad = \frac{-(2+\mu+\nu)(4\mu+4\nu+\mu\nu^2+\mu^2\nu-\mu^3-\nu^3+4x^2)}{x^2(\mu+\nu)[\nu^2-(2-\mu)^2][\nu^2-(2+\mu)^2]} Z_\mu \overline{Z}_\nu \\
&\qquad - \frac{(4\mu\nu^2+2\mu^2\nu^2-4\mu^2-4\nu^3-\mu^4+4\nu^2-\nu^4+8x^2)}{x(\mu^2-\nu^2)[\nu^2-(2-\mu)^2][\nu^2-(2+\mu)^2]} Z_\mu \overline{Z}_{\nu+1} \\
&\qquad + \frac{4\mu^2\nu+2\mu^2\nu^2+4\mu^2-\mu^4-4\nu^2-4\nu^3-\nu^4+8x^2}{x(\mu^2-\nu^2)[\nu^2-(2-\mu)^2][\nu^2-(2+\mu)^2]} Z_{\mu+1} \overline{Z}_\nu \\
&\qquad - \frac{4}{[-\nu^2+(2-\mu)^2][-\nu^2+(2+\mu)^2]} Z_{\mu+1} \overline{Z}_{\nu+1}.
\end{aligned} \tag{28}$$

Results equivalent for Eqs. (27) and (28) were previously obtained by Maximon [9], and were also obtained by Luke [10].

A particular solution to Eq. (22) is also relatively straightforward to obtain for $f(x) = 1/x^4$. In this case, we obtain the solutions

$$\begin{aligned}
A_{11} &= \frac{-6}{x[-\nu^2+(-3+\mu)^2][-\nu^2+(3+\mu)^2]} \\
&\quad + \frac{48x}{[-\nu^2+(-3+\mu)^2][-\nu^2+(-1+\mu)^2][-\nu^2+(1+\mu)^2][-\nu^2+(3+\mu)^2]},
\end{aligned} \tag{29}$$

$$\begin{aligned}
A_{01} &= \frac{1}{x^2[-\nu^2+(-3+\mu)^2]} \\
&\quad - \frac{24}{[-\nu^2+(-3+\mu)^2][-\nu^2+(-1+\mu)^2][-\nu^2+(3+\mu)^2]},
\end{aligned} \tag{30}$$

$$A_{10} = \frac{1}{x^2[-\mu^2+(3-\nu)^2]} + \frac{24}{[\mu^2-(1-\nu)^2][-\nu^2+(-3+\mu)^2][-\nu^2+(3-\mu)^2]}, \tag{31}$$

and

$$\begin{aligned}
A_{00} &= \frac{1}{x^3(\mu+\nu-3)} + \frac{6}{x[-\nu^2+(3-\mu)^2[1-\mu-\nu][3+\mu-\nu]} \\
&\quad + \frac{48x}{[-\nu^2+(-3+\mu^2)][-\nu^2+(-1+\mu)^2][-\nu^2+(1+\mu)^2][-\nu^2+(3+\mu)^2]}.
\end{aligned} \tag{32}$$

Substitution of Eqs. (29)–(32) into Eq. (20) will produce an analytical expression for the integral

$$\int \frac{1}{x^4} Z_\mu(x)\overline{Z}_\nu(x)\,dx.$$

Particular solutions to Eq. (22) have also been obtained for $f(x) = 1/x^5$ and for $f(x) = 1/x^6$. However, the resulting expressions for the associated indefinite integrals are quite unwieldy, so they will not be displayed.

B. *Integrals containing the product of two Hermite functions.* Consider integrals of the form

$$\int H_\mu(x)\overline{H}_\nu(x)f(x)\,dx, \tag{33}$$

where H and $\overline{H}$ are any solutions of the Hermite differential equation. Equation (3) produces the representation

$$\begin{aligned} I = {} & A_{00}(x)H_\mu(x)\overline{H}_\nu(x) + A_{01}(x)H_\mu(x)\overline{H}_{\nu+1}(x) \\ & + A_{10}(x)H_{\mu+1}(x)\overline{H}_\nu(x) + A_{11}(x)H_{\mu+1}(x)\overline{H}_{\nu+1}(x). \end{aligned} \tag{34}$$

Applying the uncoupling procedure to the coupled set generated by Eq. (7) for this case yields the differential equation

$$\begin{aligned} & D^4A_{11} + 8xD^3A_{11} + 4(4+\mu+\nu+5x^2)D^2A_{11} + 4x(13+4\mu+4\nu+4x^2)DA_{11} \\ & \quad + 4(4+2\mu+2\nu-2\mu\nu+4\mu x^2+4\nu x^2+\mu^2+\nu^2+8x^2)A_{11} \\ & \quad = 4xf + 2Df. \end{aligned} \tag{35}$$

The remaining functions A are expressed in terms of the function A_{11} by the equations

$$A_{00} = \frac{1}{2}D^2A_{11} + xDA_{11} + (2+\mu+\nu)A_{11}, \tag{36}$$

$$\begin{aligned} A_{01} = {} & \frac{1}{4}\frac{D^3A_{11}}{(\mu-\nu)} + \frac{3}{2}\frac{xD^2A_{11}}{(\mu-\nu)} + \frac{(5+3\mu+\nu+4x^2)}{2(\mu-\nu)}DA_{11} \\ & + \frac{2x(2+\mu+\nu)}{(\mu-\nu)}A_{11} - \frac{f}{2(\mu-\nu)}, \end{aligned} \tag{37}$$

and

$$\begin{aligned} A_{10} = {} & -\frac{1}{4}\frac{D^3A_{11}}{(\mu-\nu)} - \frac{3}{2}\frac{xD^2A_{11}}{(\mu-\nu)} - \frac{(5+\mu+3\nu+4x^2)}{2(\mu-\nu)}DA_{11} \\ & - \frac{2x(2+\mu+\nu)}{(\mu-\nu)}A_{11} + \frac{f}{2(\mu-\nu)}. \end{aligned} \tag{38}$$

A particular solution of the differential equation represented by Eq. (35) is relatively straightforward to obtain for the cases in which $f(x) = e^{-x^2}$, $f(x) = xe^{-x^2}$, and $f(x) = x^2e^{-x^2}$. (Note that e^{-x^2} is the usual weighting function used with the Hermite function.) We thus obtain the integrals

$$\int e^{-x^2}H_\mu(x)\overline{H}_\nu(x)\,dx = \frac{e^{-x^2}}{2(\mu-\nu)}[-H_\mu(x)\overline{H}_{\nu+1}(x) + H_{\mu+1}(x)\overline{H}_\nu(x)], \tag{39}$$

$$\begin{aligned} & \int xe^{-x^2}H_\mu(x)\overline{H}_\nu(x)\,dx \\ & \quad = \frac{e^{-x^2}}{2}\left[-\frac{H_\mu(x)\overline{H}_\nu(x)(1+\mu+\nu)}{(1-\mu+\nu)(1+\mu-\nu)} + H_{\mu+1}(x)\overline{H}_\nu(x)\frac{x}{(1+\mu-\nu)}\right. \\ & \qquad \left. + H_\mu(x)\overline{H}_{\nu+1}(x)\frac{x}{(1-\mu+\nu)} - \frac{H_{\mu+1}(x)\overline{H}_{\nu+1}(x)}{(1-\mu+\nu)(1+\mu-\nu)}\right], \end{aligned} \tag{40}$$

and

$$\int x^2 e^{-x^2} H_\mu(x)\overline{H}_\nu(x)\,dx$$
$$= e^{-x^2}\left[\frac{-H_\mu(x)\overline{H}_\nu(x)x(\mu+\nu)}{(2-\mu+\nu)(2+\mu-\nu)}\right.$$
$$+H_{\mu+1}(x)\overline{H}_\nu(x)\frac{(2+\mu+3\nu+2\mu x^2-2\nu x^2-\mu^2x^2-\nu^2x^2+2\mu\nu x^2)}{2(\mu-\nu)(2-\mu+\nu)(2+\mu-\nu)}$$
$$-H_\mu(x)\overline{H}_{\nu+1}(x)\frac{(2+3\mu+\nu-2\mu x^2+2\nu x^2-\mu^2x^2-\nu^2x^2+2\mu\nu x^2)}{2(\mu-\nu)(2-\mu+\nu)(2+\mu-\nu)}$$
$$\left.-H_{\mu+1}(x)\overline{H}_{\nu+1}(x)\frac{x}{(2-\mu+\nu)(2+\mu-\nu)}\right], \tag{41}$$

by substituting the solutions for A_{11} into Eqs. (36)–(38), and by further substituting these results into Eq. (34). Integrals of the form of Eqs. (39)–(41) arise in quantum mechanical problems, for example in molecular spectroscopy [11]. The integral of Eq. (40) arises in computing the transition probability between harmonic oscillator states μ and ν, and the integral of Eq. (41) arises in computing the mean-squared displacement of such an oscillator.

C. *Integrals containing the product of two Laguerre functions.* Consider integrals of the form

$$\int L_\mu(x)\overline{L}_\nu(x)f(x)\,dx, \tag{42}$$

where L and $\overline{L}$ are any solutions of the Laguerre differential equation. Equation (3) generates the representation

$$\begin{aligned} I = {} & A_{00}(x)L_\mu(x)\overline{L}_\nu(x) + A_{01}(x)L_\mu(x)\overline{L}_{\nu+1}(x) \\ & + A_{10}(x)L_{\mu+1}(x)\overline{L}_\nu(x) + A_{11}L_{\mu+1}(x)\overline{L}_{\nu+1}(x). \end{aligned} \tag{43}$$

Applying the uncoupling procedure to the coupled set generated by Eq. (7) for this case yields the differential equation

$$\begin{aligned} & x^2D^4A_{11} + x(5+4x)D^3A_{11} + (4+17x+2\mu x+2\nu x+5x^2)D^2A_{11} \\ & + (11+3\mu+3\nu+16x+4\mu x+4\nu x+2x^2)DA_{11} \\ & + (6+3\mu+3\nu+4x-2\mu\nu+2\mu x+2\nu x+\mu^2+\nu^2)A_{11} \\ & = 2(1+\mu)(1+\nu)f + 2(1+\mu)(1+\nu)Df, \end{aligned} \tag{44}$$

where the remaining functions A are expressed in terms of the function A_{11} by the equations

$$\begin{aligned} A_{00} = {} & \frac{-x^2}{2(1+\mu)(1+\nu)}D^3A_{11} - \frac{x(3+2x)}{2(1+\mu)(1+\nu)}D^2A_{11} \\ & - \frac{(1+5x+\mu x+\nu x+x^2)}{2(1+\mu)(1+\nu)}DA_{11} \\ & + \frac{(\mu+\nu-2x+2\mu\nu-\mu x-\nu x)}{2(1+\mu)(1+\nu)}A_{11} + f, \end{aligned} \tag{45}$$

$$
\begin{aligned}
A_{01} = & -\frac{x^2}{2(1+\mu)(\mu-\nu)}D^3A_{11} - \frac{3}{2}\frac{x(1+x)}{(1+\mu)(\mu-\nu)}D^2A_{11} \\
& - \frac{(1+8x+3\mu x+\nu x+2x^2)}{2(1+\mu)(\mu-\nu)}DA_{11} \\
& - \frac{(2+3\mu-\nu+4x-2\mu\nu+2\mu x+2\nu x+2\mu^2)}{2(1+\mu)(\mu-\nu)}A_{11} + \frac{1+\nu}{\mu-\nu}f,
\end{aligned} \tag{46}
$$

and

$$
\begin{aligned}
A_{10} = & \frac{x^2}{2(1+\nu)(\mu-\nu)}D^3A_{11} + \frac{3}{2}\frac{x(1+x)}{(1+\nu)(\mu-\nu)}D^2A_{11} \\
& + \frac{(1+8x+\mu x+3\nu x+2x^2)}{2(1+\nu)(\mu-\nu)}DA_{11} \\
& + \frac{(2-\mu+3\nu+4x-2\mu\nu+2\mu x+2\nu x+2\nu^2)}{2(1+\nu)(\mu-\nu)}A_{11} - \frac{1+\mu}{\mu-\nu}f.
\end{aligned} \tag{47}
$$

We consider only two examples in this class. Namely, $f(x) = e^{-x}$ and $f(x) = xe^{-x}$. These examples yield the integrals

$$
\begin{aligned}
\int e^{-x}L_\mu(x)\overline{L}_\nu(x)\,dx = e^{-x}\Bigg[& L_\mu(x)\overline{L}_\nu(x) + \frac{(1+\nu)}{(\mu-\nu)}L_\mu(x)\overline{L}_{\nu+1}(x) \\
& - \frac{(1+\mu)}{(\mu-\nu)}L_{\mu+1}(x)\overline{L}_\nu(x)\Bigg],
\end{aligned} \tag{48}
$$

and

$$
\begin{aligned}
& \int xe^{-x}L_\mu(x)\overline{L}_\nu(x)\,dx \\
& = e^{-x}\Bigg[-\frac{(1+\mu+\nu-x+2\mu\nu+\mu^2x+\nu^2x-2\mu\nu x)}{(1-\mu+\nu)(1+\mu-\nu)}L_\mu(x)\overline{L}_\nu(x) \\
& \qquad + \frac{(1+\nu)(1+2\mu-\mu x+\nu x)}{(\mu-\nu)(1-\mu+\nu)}L_\mu(x)\overline{L}_{\nu+1}(x) \\
& \qquad - \frac{(1+\mu)(1+2\nu+\mu x-\nu x)}{(\mu-\nu)(1+\mu-\nu)}L_{\mu+1}(x)\overline{L}_\nu(x) \\
& \qquad - \frac{2(1+\mu)(1+\nu)}{(1-\mu+\nu)(1+\mu-\nu)}L_{\mu+1}(x)\overline{L}_{\nu+1}(x)\Bigg].
\end{aligned} \tag{49}
$$

Note that in the examples given above the results appear to become invalid for certain combinations of subscripts μ and ν. That is, the analytical expressions for the given examples contain denominators that vanish for certain combinations of subscripts. These cases can be handled by evaluating appropriate limits with the help of l'Hôpital's rule. However, this process will result in expressions containing derivatives with respect to the order of the special functions. Those cases in which Eqs. (19), (33), or (42) contain the square of the special function in question are special cases that have been considered previously [12]. The case in which the integrand contains the product of two Legendre functions was also discussed in [12], so further discussion of this class of integrals will be avoided here.

The integrals (39) and (48) can also be evaluated using the same technique that is usually employed to obtain the orthogonality integral arising in the Sturm-Liouville

problem, if the usual limits are omitted from the integrals (see [13], especially the formula at the top of p. 339). (Although this derivation is usually done for integer special function orders, it is based on the differential equation of the special function, so that the integer nature of the order need not be invoked.) Formulas (40), (41), and (49) can then be derived using integration by parts. These same integrals can also be evaluated using the technique of [9], provided that H and L are appropriately expressed in terms of confluent hypergeometric functions (see, for example, [14]). An appropriate change of variables must also be made in the cases involving Hermite functions.

D. *Examples of integrals containing more than two birecurrent functions.* Next, we consider integrals of the form

$$I = \int f(x) Z_\nu^4(x)\, dx, \tag{50}$$

where Z_ν is again any solution of Bessel's differential equation. The assumed form generated by (3) in this case yields

$$\begin{aligned} I = {} & A_{0000}(x) Z_\nu^4(x) + A_{1111}(x) Z_{\nu+1}^4(x) \\ & + 4A_{0001}(x) Z_\nu^3(x) Z_{\nu+1}(x) + 6A_{0011}(x) Z_\nu^2(x) Z_{\nu+1}^2(x) \\ & + 4A_{0111}(x) Z_\nu(x) Z_{\nu+1}^3(x), \end{aligned} \tag{51}$$

where we have taken advantage of the obvious symmetries of the problem. The function $A_{1111}(x)$ can be uncoupled from the set produced by (7), using the method described in Sec. III, to give

$$\begin{aligned} & x^5 D^5 A_{1111}(x) - 10x^4 D^4 A_{1111}(x) - 5x^3(-13 + 4\nu^2 - 4x^2) D^3 A_{1111}(x) \\ & \quad + 15x^2(-19 + 12\nu^2 - 8x^2) D^2 A_{1111}(x) \\ & \quad + x(781 - 128\nu^2 x^2 - 740\nu^2 + 64\nu^4 + 392x^2 + 64x^4) D A_{1111}(x) \\ & \quad - 64(16 - 6\nu^2 x^2 - 20\nu^2 + 4\nu^4 + 9x^2 + 2x^4) A_{1111}(x) \\ & \qquad = 24x^5 f(x). \end{aligned} \tag{52}$$

The remaining functions A are expressed in terms of $A_{1111}(x)$ by the equations

$$\begin{aligned} A_{0000}(x) = {} & \frac{1}{24} D^4 A_{1111}(x) - \frac{(5+2\nu)}{12x} D^3 A_{1111}(x) \\ & - \frac{(-55 - 36\nu + 4\nu^2 - 16x^2)}{24x^2} D^2 A_{1111}(x) \\ & + \frac{(-175 - 148\nu - 40\nu x^2 + 28\nu^2 + 16\nu^3 - 84x^2)}{24x^3} D A_{1111}(x) \\ & + \frac{(32 + 32\nu + 16\nu x^2 - 8\nu^2 - 8\nu^3 + 20x^2 + 3x^4)}{3x^4} A_{1111}(x), \end{aligned} \tag{53}$$

$$\begin{aligned} A_{0001}(x) = {} & \frac{1}{24} D^3 A_{1111}(x) - \frac{(3+2\nu)}{8x} D^2 A_{1111}(x) \\ & + \frac{(37 + 42\nu + 8\nu^2 + 10x^2)}{24x^2} D A_{1111}(x) \\ & - \frac{(8 + 12\nu + 3\nu x^2 + 4\nu^2 + 4x^2)}{3x^3} A_{1111}(x), \end{aligned} \tag{54}$$

$$A_{0011}(x) = \frac{1}{12}D^2A_{1111}(x) - \frac{(7+6\nu)}{12x}DA_{1111}(x) + \frac{(4+6\nu+2\nu^2+x^2)}{3x^2}A_{1111}(x), \tag{55}$$

and

$$A_{0111}(x) = \frac{1}{4}DA_{1111}(x) - \frac{(1+\nu)}{x}A_{1111}(x). \tag{56}$$

In view of the complexity of the differential equation (52), it is difficult to obtain solutions for arbitrary orders ν. Therefore, in this case, we restrict our attention to examples of particular orders.

As one example in this category, we consider $f(x) = 1/x$, $\nu = 1$. A particular solution to (52) for this case is $A_{1111}(x) = x^2/4$. This produces the integral

$$\begin{aligned}\int \frac{Z_1^4(x)}{x}\,dx = {}& \frac{x^2}{4}Z_2^4(x) + \left(\frac{3}{4}+\frac{x^2}{4}\right)Z_1^4(x) \\ & - \frac{3x}{2}Z_1(x)Z_2^3(x) \\ & + 6\left(\frac{1}{2}+\frac{x^2}{12}\right)Z_1^2(x)Z_2^2(x) \\ & + 4\left(-\frac{3x}{8}-\frac{1}{2x}\right)Z_1^3(x)Z_2(x),\end{aligned} \tag{57}$$

when $A_{1111} = x^2/4$ and $f = 1/x$ are substituted into (53)–(56), and the resulting expressions are substituted into (51). As a second example, we let $f(x) = 1/x^3$ and $\nu = 3$. A particular solution to (52) can also be obtained for this case, thus yielding the integral

$$\begin{aligned}\int \frac{Z_3^4(x)}{x^3}\,dx = {}& \left(\frac{1}{24}+\frac{1}{2x^2}+\frac{2}{x^4}+\frac{x^2}{378}\right)Z_3^4(x) \\ & + \left(\frac{5}{216}+\frac{2}{27x^2}+\frac{x^2}{378}\right)Z_4^4(x) \\ & + 4\left(-\frac{x}{108}-\frac{5}{54x}-\frac{1}{3x^3}\right)Z_3(x)Z_4^3(x) \\ & + 6\left(\frac{7}{216}+\frac{1}{3x^2}+\frac{4}{3x^4}+\frac{x^4}{1134}\right)Z_3^2(x)Z_4^2(x) \\ & + 4\left(-\frac{x}{108}-\frac{1}{8x}-\frac{1}{x^3}-\frac{4}{x^5}\right)Z_3^3(x)Z_4(x).\end{aligned} \tag{58}$$

By applying similar methods to integrands containing Legendre functions P_ν, we can also deduce the results

$$\begin{aligned}\int x[P_{1/3}(x)]^3\,dx = {}& \left(\frac{125x^4}{12}-\frac{14}{3}x^2-\frac{5}{12}\right)[P_{1/3}(x)]^3 \\ & + (-4+20x^2)P_{1/3}(x)[P_{4/3}(x)]^2 \\ & + (9x-25x^3)[P_{1/3}(x)]^2P_{4/3}(x) \\ & - \frac{16}{3}x[P_{4/3}(x)]^3,\end{aligned} \tag{59}$$

and,

$$\begin{aligned}\int x[P_{1/2}(x)]^4\,dx = &\left(-\frac{5}{16}-\frac{19}{4}x^2\right)[P_{1/2}(x)]^4\\ &+\frac{81}{4}xP_{1/2}(x)[P_{3/2}(x)]^3\\ &+6\left(-\frac{9}{16}-\frac{9}{2}x^2\right)[P_{1/2}(x)]^2[P_{3/2}(x)]^2\\ &+4\left(\frac{33x}{16}+3x^3\right)[P_{1/2}(x)]^3P_{3/2}(x)\\ &-\frac{81}{16}[P_{3/2}(x)]^4.\end{aligned} \tag{60}$$

V. Comparison with previous methods. Indefinite integration of special functions has also been the object of previous efforts by others [9, 15, 16, 17]. As previously mentioned, the present technique is a generalization of a method proposed previously by Sonine [2]. Similarly, the method of Muller [16] is a generalization of a technique proposed by Lommel [18] for integrals involving a single Bessel function. Muller's method is similar in spirit and approach to the method developed here. However, Muller's method is restricted to integrands containing a single special function. It is a generalization of Lommel's method in that it is not restricted to integrals containing a Bessel function.

Filippov has included a technique for special function integration in [17]. Unfortunately, no English translation of this Russian source seems, as yet, to be available. Hence, no comparison of the present method with Filippov's method will be considered.

The method of Maximon and Morgan [9, 15] is quite different from the method presented here. Their technique is based on analyzing certain general formulas (see (12), (28), and (30) of [15]). The primary results of their method are summarized in formulas (28) and (30) of [15]. In what follows, these formulas will be referred to as Formula I and Formula II, respectively. We will also use the notations y_0 and Y_1, which are used in [15] to refer to the terms that contain the special functions of the integrand of interest. Although Formulas I and II can be used to evaluate many of the integrals that are also amenable to the method given here, the current method has certain advantages which we will examine presently.

In order to obtain useful results from Formula I, the functions y_0 and Y_1 are restricted. The restriction arises from the fact that Formula I contains combinations of certain auxiliary functions which usually must be chosen to have zero values to obtain a useful expression from this formula, and this requirement can only be satisfied for certain choices for the functions y_0 and Y_1. (See the statement at the top of p. 83 of [15].) Thus, for an integral having the full generality of Eq. (1) here, Formula I would frequently not produce a useful expression.

Formula II is applicable to a far more general class of special function integrands than is Formula I. This is due to the fact that y_0 is not required to satisfy a specific differential equation, as it is in order for Formula I to be applicable. In fact, in Formula II, y_0 is permitted to be, essentially, completely general. However, Formula

II contains the sum

$$\sum_{i=0}^{2} c_i \frac{d^i y_0}{dx^i}.$$

Here, c_i are the coefficient functions from the differential equation *which* Y_1 *is required to satisfy.* (See (29) of [15]. Please note there is a minor misprint in this equation.) This sum appears underneath the integral sign of Formula II, multiplied by the function Y_1. This product is further multiplied by two other factors which involve the functions c_i. One of these factors involves an exponential of an integral over the ratio of two of the functions c_i. Thus, in view of the intricate combination of functions appearing in Formula II, an application of this formula will also frequently not produce a useful expression, despite the generality permitted in the definition of y_0. It would seem to be particularly difficult to obtain a useful expression from this formula if the integral of interest contains the product of three or more special functions, since this would require that y_0 or Y_1 include the product of at least two special functions.

Due to the nature of Formulas I and II, a straightforward application of them usually produces a relation between special function integrals, rather than directly producing a fully integrated expression (see (2), (10), and (26) of [9], as examples). This happens whenever a sufficient number of the auxiliary functions arising in the method is not, or cannot be chosen, to be zero. Thus, in the general case, Formulas I and II are similar to the integration-by-parts formula of elementary calculus in the sense that this formula also produces a relationship between integrals. In order to obtain a fully integrated expression using the method of Maximon and Morgan when such a relation between integrals is produced, additional transformations, such as the additional application of recurrence relations, frequently must be performed on the expressions that result from a direct evaluation of Formulas I and II. In contradistinction, the present method directly produces a fully integrated expression, via Eq. (3), when the required particular solution to the attendant differential equation can be found.

In summary, in applying Formula I, a judicious choice of the functions y_0 and Y_1 usually must be made to produce useful results. This can be a nontrivial procedure, and this would seem to restrict the applicability of Formula I to integrands that contain the product of no more than two special functions. The utility of Formula II is also restricted since, due to its structure, it is likely to produce a relation between special function integrals, rather than producing a fully integrated expression. Obtaining fully integrated expressions from the expressions that result from Formulas I and II usually requires the use of further transformations. Although these additional transformations are frequently not difficult to perform, their implementation for the general case is certainly not mechanical; i.e., it is not definable beforehand for a general integrand of the form of Eq. (1). On the other hand, as has been demonstrated here, the present method always applies to an integral of the form of Eq. (1), and the required transformations are implemented in a completely mechanical way. The mechanical nature of these transformations is not affected by the complexity of the nonbirecurrent function $f(x)$ appearing in the integrand of (1). For example, in

generating the coupled set (4), using Eqs. (5) and (6), note that use is made only of the known recurrence relation coefficient functions a, b, c, d of Eqs. (2). Also, no function choices of a special nature are involved in implementing the uncoupling process detailed in Sec. III, and the required transformations are straightforward regardless of the complexity of the function $f(x)$. Thus, a user of the present technique can perform the required transformations without having to make any special choices of auxiliary functions. Of course, after performing these transformations, it is still necessary to obtain a particular solution of the resulting differential equation. However, this is frequently easy to do, and can even be trivial. Recall, in this regard, the example that produced Eq. (26).

It is because of the completely mechanical nature of the transformations involved in implementing the present method (i.e., requiring no human intervention) that the present method is currently being incorporated as the basis of a special function integration package in the commercially available computer mathematics program *Mathematica*™, by Wolfram Research, Inc. [19].

VI. Summary and conclusions. A technique for evaluating indefinite integrals containing one or more special functions has been described. It has been demonstrated that the coupled set arising from the application of the technique can always be uncoupled using the method of Forsyth, since the coupled set generated by the technique always occurs in normal form. Several examples were presented to illustrate the method.

It should be noted that Eqs. (4) and (5) can be used to algorithmically generate the required coupled set for any integral of the form of Eq. (1). The method of Forsyth, as described in Sec. III, can be applied in a mechanical manner to uncouple any function A_ℓ from the set A_p, resulting in the new set represented by Eq. (18). The first $n-1$ equations of this set can be solved algebraically for all functions of the set except A_ℓ. These algebraically-obtained solutions may then be substituted into the nth equation of the new set to obtain an nth-order, ordinary, inhomogeneous, linear differential equation in the single unknown A_ℓ. Any particular solution of this equation can be substituted into the algebraically-determined solutions to yield a particular solution for the entire set of unknown functions A_p. This particular solution, when substituted into Eq. (3), yields an analytical expression for the desired indefinite integral of Eq. (1). Thus, the problem of evaluating the indefinite integral of Eq. (1) is reduced to the problem of finding any particular solution to the uncoupled equation in the single unknown A_ℓ. Since this equation does not involve any of the special functions of the integral of Eq. (1), finding this particular solution can be much simpler than attempting to directly obtain the desired antiderivative.

It is interesting to note that the search for a particular solution of the uncoupled equation, as well as the implementation of the entire integration technique described herein, is particularly well suited to the capabilities of symbolic-mathematics computer programs such as MACSYMA™ and SMP™. In fact, the author has implemented this algorithm using the SMP program. This implementation permits automatic symbolic evaluation of integrals of the form represented by Eq. (1). If

such an algorithm were incorporated into the integration operator of such a symbolic-mathematics computer program, it would extend the integration capabilities to include the class of indefinite integrals represented by Eq. (1). As previously mentioned, this is in fact currently being done for the program *Mathematica* [19].

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