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NONEXISTENCE OF MONOTONIC SOLUTIONS IN A MODEL OF DENDRITIC GROWTH

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Abstract. A simple model for dendritic growth is given by $\delta^2\theta''' + \theta' = \cos(\theta)$. For $\delta \approx 1$ we prove that there is no bounded, monotonic solution which satisfies $\theta(-\infty) = -\pi/2$ and $\theta(\infty) = \pi/2$. We also investigate the existence of bounded, monotonic solutions of an equation derived from the Kuramoto-Sivashinsky equation, namely $y''' + y' = 1 - y^2/2$. We prove that there is no monotonic solution which satisfies $y(-\infty) = -\sqrt{2}$ and $y(\infty) = \sqrt{2}$.

1. Introduction and statement of results. In this paper we investigate the existence of bounded, monotonic solutions of the equation

$$\delta^2\theta''' + \theta' = \cos(\theta) \tag{1}$$

where $\delta \approx 1$. Equation (1) has been proposed ([1, 2, 3]) as a simple model for two-dimensional dendritic growth of needle crystals in a supercooled liquid. Here θ is the angle between the normal to the interface of the dendrite and the direction of propagation of the interface as the crystal forms. The parameter δ represents surface tension. As shown in ([3, 5]), the physically important properties which the solutions should satisfy are given by

$$\theta(0) = \theta''(0) = 0 \tag{2}$$

$$\theta' > 0 \quad \forall \times \varepsilon(-\infty, \infty) \tag{3}$$

$$\theta(\infty) = \pi/2, \qquad \theta(-\infty) = -\pi/2. \tag{4}$$

Recently, Kruskal and Segur [5] have questioned the existence of a solution of (1)-(4) for small, positive values of δ. They obtain an expansion which indicates that "beyond all orders" there can be no solution for small $\delta > 0$. In this paper we investigate higher values of δ. Our main result is given in

THEOREM 1. There is no solution of Eq. (1) which satisfies (2)-(4) if $|\delta - 1| > 0$ is sufficiently small.

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We also investigate a problem similar to (1)–(4) which is derived from the Kuramoto-Sivashinsky equation

$$u_t + \nabla^4 u + \nabla^2 u + \tfrac{1}{2}|\nabla u|^2 = 0. \tag{5}$$

Equation (5) arises in a number of interesting physical settings. Kuramoto and Tsuzuki [4] derive Eq. (5) as a model for phase turbulence in the Belousov-Zhabotinskii reaction. Sivashinsky [7] also derives Eq. (5) in his stability studies of laminar flame propagation. Numerical experimentation by Michelson and Sivashinsky [6] suggests that solutions of Eq. (3) have the form

$$u(x, t) = -c_0^2 t + v(x) \tag{6}$$

where $c_0 \approx 1.04$. Setting $y = v'(x)$, Michelson [6] finds that $y(x)$ satisfies

$$y''' + y' = c_0^2 - y^2/2. \tag{7}$$

His computations indicate that for $c_0 \approx 1$ there are at least two periodic solutions. He conjectures that much more complicated solutions exist which are either periodic, or else satisfy

$$y(-\infty) = c_0\sqrt{2}, \qquad y(\infty) = -c_0\sqrt{2}.$$

Recently Troy [8] has investigated this problem further. For simplicity he considers the case $c_0 = 1$ and proves that Eq. (7) has at least two odd periodic solutions. Furthermore, using these solutions as a basis for a shooting argument, Troy proves that there are at least two nonmonotonic solutions which satisfy

$$y(-\infty) = \sqrt{2}, \qquad y(\infty) = -\sqrt{2}. \tag{8}$$

Further analysis of the linearizations of

$$y''' + y' = 1 - y^2/2 \tag{9}$$

around $y \equiv -\sqrt{2}$ and $y \equiv \sqrt{2}$ gives the following information:

(i) There is a two dimensional stable manifold Γ_1 of solutions leading to $(y, y', y'') = (-\sqrt{2}, 0, 0)$. All solutions which intersect Γ_1 must oscillate infinitely often as $x \to \infty$;

(ii) There is a one-dimensional unstable manifold γ_1 of solutions leading from $(y, y', y'') = (\sqrt{2}, 0, 0)$ and pointing into the region $y > -\sqrt{2}, y' > 0, y'' > 0$;

(iii) There is a two-dimensional unstable manifold Γ_2 of solutions leading from $(y, y', y'') = (\sqrt{2}, 0, 0)$. Any solution which intersects Γ_2 must oscillate infinitely often as $x \to -\infty$;

(iv) There is a one-dimensional stable manifold of solution γ_2 leading to $(y, y', y'') \geq (\sqrt{2}, 0, 0)$ from the region $y < \sqrt{2}, y' > 0, y'' < 0$.

In view of (ii) and (iv), it is a natural question to determine whether there is a solution of Eq. (9) satisfying

$$y' > 0 \quad \forall x > 0 \tag{10}$$

$$y(-\infty) = -\sqrt{2}, \qquad y(\infty) = \sqrt{2}. \tag{11}$$

Furthermore, the topological shooting argument in [8] which was used to prove the existence of periodic solutions and heteroclinic orbits relied on the problem (10)–(11) having no solution. Thus we are led to

THEOREM 2. The problem (9)–(10)–(11) has no solution.

2. Proof of Theorem 1. For simplicity we set $\delta = 1$ and investigate the existence of solutions of

$$\theta''' + \theta' = \cos(\theta) \tag{12}$$

which satisfy

$$\theta' > 0 \quad \forall x \in (-\infty, \infty) \tag{13}$$

and

$$\theta(-\infty) = -\pi/2, \qquad \theta(\infty) = \pi/2. \tag{14}$$

If $\theta(x)$ solves Eq. (12) then it is easily verified that $-\theta(-x)$ also solves Eq. (12). Thus, (13) and (14) imply that $\theta(x)$ is odd and it follows that

$$\theta(0) = \theta''(0) = 0. \tag{15}$$

Next, we set $\theta'(0) = \beta$ and consider the parameter range $\beta \geq 0$ for which the solution of (12) and (15) satisfies (13) and (14). The first step in our analysis is to develop energy functionals (K and F below) which must remain negative if (13) and (14) hold. This helps us to reduce the possible range of β values to $.6 \leq \beta \leq 1.24$. For each $\beta \in [.6,\ 1.24]$ we prove either that the energy becomes positive or else $\theta = \pi/2$ before $\theta' = 0$. Since the interval $[.6,\ 1.24]$ is compact, all of our arguments will apply to values of δ sufficiently close to one, and the theorem will be proved.

We begin by defining the functional

$$K = (\theta'')^2/2 - (\theta')^2/2 - \theta'''\theta' \tag{16}$$

which satisfies

$$K' = \sin(\theta)(\theta')^2. \tag{17}$$

If (13) and (14) hold for some $\beta \geq 0$ then $(\theta, \theta', \theta'')$ lies on the one-dimensional stable manifold leading to $(\pi/2, 0, 0)$. Thus $K' > 0$ on $(0, \infty)$ and $K(\infty) = 0$. This, (13), and the definition of K imply that $\theta''' + \theta'/2 > 0$ on $(0, \infty)$ Therefore, $\theta'' + \theta/2 > 0$ on $(0, \infty)$. Multiplying this last inequality by θ' and integrating, we obtain

$$(\theta')^2 + \theta^2/2 > \beta^2 \quad \forall x > 0. \tag{18}$$

If $\theta'(\overline{x}) = 0$ at some first $\overline{x} \in (0, \infty]$ and $\theta(\overline{x}) \in (0, \pi/2]$ then it follows from (18) that

$$\beta \leq \pi/(2\sqrt{2}) \leq 1.24. \tag{19}$$

A lower bound on the possible range of values of β is obtained from the energy functional

$$F \equiv (\theta'')^2/2 + (\theta''')^2/2 + \sin(\theta)\theta'\theta'' - \cos(\theta)(\theta')^3/3 \tag{20}$$

where

$$F(0) = (1-\beta)^2/2 - \beta^3/3 \tag{21}$$

and

$$F' = \sin(\theta)((\theta')^4/3 + (\theta'')^2). \tag{22}$$

Again, if there is a first $\overline{x} \in (0, \infty]$ for which $\theta'(\overline{x}) = 0$ and $0 < \theta(\overline{x}) \le \pi/2$ then $F(\overline{x}) \ge 0$, and $F' > 0$ on $(0, \overline{x})$. This and (21) imply that

$$(1-\beta)^2/2 - \beta^3/3 < 0. \tag{23}$$

An elementary calculation shows that (23) is possible only if $\beta \ge .6$. Thus we restrict our attention to the interval $.6 \le \beta \le 1.25$.

In order to proceed with our analysis, we need to obtain the equations satisfied by $\theta^{(4)}$ and $\theta^{(5)}$. These are

$$\theta^{(4)} = -\theta'' - \sin(\theta)\theta' \tag{24}$$

$$\theta^{(5)} = -\theta''' - \sin(\theta)\theta'' - \cos(\theta)(\theta')^2. \tag{25}$$

The initial conditions for θ''', $\theta^{(4)}$, and $\theta^{(5)}$ are

$$\theta'''(0) = 1 - \beta, \qquad \theta^{(4)}(0) = 0, \qquad \theta^{(5)}(0) = 1 - \beta - \beta^2 < 0. \tag{26}$$

Multiplying Eq. (24) by θ''' and integrating, we observe that $((\theta''')^2 + (\theta'')^2)' < 0$ while $\theta''' > 0$. This and (26) imply that $\theta''' < 1 - \beta$ while $\theta''' > 0$ and it follows that

$$\theta'' < (1-\beta)x, \qquad \theta' < (1-\beta)x^2/2 + \beta, \qquad \theta < (1-\beta)x^3/6 + \beta x. \tag{27}$$

Next, we observe that

$$\theta''' = -\theta' + \cos(\theta) \ge -\theta' + 1 - \theta^2/2 \quad \forall \theta > 0. \tag{28}$$

Substitution of (27) into (28) leads to

$$\theta''' \ge (1-\beta) - (1-\beta+\beta^2)x^2/2 - \beta(1-\beta)x^4/6 - \frac{(1-\beta)^2 x^6}{72}$$

$$\theta'' \ge (1-\beta)x - (1-\beta+\beta^2)x^3/6 - \beta(1-\beta)x^5/30 - (1-\beta)^2x^7/504 \tag{29}$$

$$\begin{aligned} \theta' \ge \beta &+ (1-\beta)x^2/2 - (1-\beta+\beta^2)x^4/24 \\ &- \beta(1-\beta)x^6/180 - (1-\beta)^2x^8/4032 \end{aligned} \tag{30}$$

$$\begin{aligned} \theta \ge \beta x &+ (1-\beta)x^3/6 - (1-\beta+\beta^2)x^5/120 \\ &- \beta(1-\beta)x^7/1260 - (1-\beta)^2x^9/36288. \end{aligned} \tag{31}$$

We now use the estimates obtained in (29), (30), and (31) to help reduce the range of β values still further. We consider several subcases. The first is

(i) $.6 < \beta < .75$. It follows from (24), (29), (30), and (31) that $\theta''(x) > 0$ and $\theta'''' < 0$ on $(0, 1.3)$, $\theta'(1.3) > .895$, and $\theta(1.3) > .8$. If for some $\beta \in [.6, .75]$ there were a first $x_0 < 1.3$ for which $\theta''(x_0) = 0$ then

$$\theta'(x_0) + \theta(x_0) \ge 1.6. \tag{32}$$

In order to obtain a contradiction, we define $u \equiv \theta''/\theta'$. It follows from Eq. (1.3) that u satisfies

$$u' + u^2 + 1 = \cos(\theta)/\theta' > 0 \tag{33}$$

on (x_0, ∞) as long as $\theta' > 0$.

Therefore $du/dx \geq -(u^2 + 1)$ and an integration leads to

$$\theta''(x)/\theta'(x) \geq -\tan(x - x_0). \tag{34}$$

Integrating (34) from $x = x_0$ to x, we obtain

$$\theta'(x) \geq \theta'(x_0)\cos(x - x_0). \tag{35}$$

Finally, one last integration leads to

$$\theta(x) \geq \theta'(x_0)\sin(x - x_0) + \theta(x_0). \tag{36}$$

Setting $x = x_0 + \pi/2$ in (36), we conclude from (32) and (35) that $\theta' > 0$ and that $\theta(x_0 + \pi/2) \geq 1.6 > \pi/2$, a contradiction of (13) and (14). Next, we consider the case

(ii) $.75 < \beta < .85$. This case proceeds in exactly the same fashion as case (i) above. The only difference is that we are forced to restrict x to $0 \leq x \leq 1$. It follows from (29), (30), (31) that $\theta'' > 0$ and $\theta'''' < 0$ on $[0, 1]$, $\theta'(1) \geq .83$, and $\theta(1) \geq .784\ \forall \beta \in [.75, .85]$.

Therefore, if there is a first value $x_0 > 1.10$ where $\theta''(x_0) = 0$ then $\theta' > 0$ on $(1, x_0)$, hence

$$\theta(x_0) + \theta'(x_0) > 1.6. \tag{37}$$

It then follows as in case (i) that (34)–(36) hold. Thus, $\theta(x_0 + \pi/2) > \pi/2$. Again this contradicts the conditions (13) and (14).

(iii) $.85 < \beta < 1.0$. As was noted earlier, if a solution satisfies (13) and (14) for some $\beta \in [.85, 1.0)$ then $\theta''' + \theta'/2 > 0$ and $\theta'' + \theta/2 > 0\ \forall x > 0$. It then follows from (24) that $\theta'''' \leq \theta/2 - \sin(\theta)\ \theta' \leq \theta/2 - 2\theta\theta'/\pi$ for all $x > 0$ since $\sin(\theta) \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$. Thus, for $x > 0$, as long as $\theta' > \pi/4$ then $\theta'''' < 0$, $\theta''' < 1 - \beta$, and (29)–(31) must hold. It easily follows from (30) that $\theta' \geq .85$ for all $x \in [0, 1]$, and all $\beta \in [.85, 1.0]$. Furthermore, at $x = 1$, and elementary calculation in (29)–(31) shows that

$$\theta(1) \geq .864, \qquad \theta'(1) \geq .88, \qquad \theta''(1) \geq -.168 \tag{38}$$

for all $\beta \in [.85, 1.0]$

Again we analyze Eq. (33) for $x > 1$ under the assumption that there is a β value $\beta \in [.85, 1.0]$ for which (13) and (14) hold. Then $u(1) = \theta''(1)/\theta'(1) \geq -.198$, $\tan^{-1}(u(1)) \geq -.198$, and $u' + u^2 + 1 > 0\ \forall x \geq 1$. Integrating, we obtain

$$\theta''(x)/\theta'(x) \geq -\tan(x + .802) \tag{39}$$

for all $x \geq 1$. Further integration leads to

$$\theta'(x) \geq \theta'(1)\cos(x - .802)/\cos(.198) \tag{40}$$

and

$$\theta(x) \geq \theta(1) + \theta'(1)(\sin(x - .802) - \sin(.198))/\cos(.198). \tag{41}$$

Let $\overline{x} = \pi/2 + (40)$. From (38), (40), and (41) it follows that $\theta' > 0$ on $(1, \overline{x})$ and $\theta(\overline{x}) \geq +(.819)\ \theta(1) \geq 1.58 > \pi/2$, contradicting (13) and (14). It remains to consider the case

(iv) $1 < \beta < 1.24$. This case proceeds in the same fashion as case (iii) above. First, it follows from (30) that

$$\theta'(x) \geq .94 \quad \forall x \in [0, 1], \quad \forall \beta \in [1, 1.24] \tag{42}$$

and therefore, as in case (iii), the approximations (29)–(31) are valid. Furthermore, from (29) and (31) we observe

$$\theta''(1) \geq -.41 \quad \text{and} \quad \theta(1) \geq .99 \quad \forall \beta \in [1, 1.24]. \tag{43}$$

Thus it follows that

$$u(1) = \theta''(1)/\theta'(1) \geq -.437, \qquad \tan^{-1}(u(1)) \geq -.42. \tag{44}$$

If (13) and (14) hold for some $\beta \in (1, 1.25]$ then $u' + u^2 + 1 > 0$ for all $x \in [1, \infty)$. This and (44) lead to

$$\theta''(x)/\theta'(x) \geq -\tan(x - .58) \quad \forall x > 1. \tag{45}$$

Two further integrations show that

$$\theta'(x) \geq \theta'(1)\cos(x - .58)/\cos(.42) \tag{46}$$

and

$$\theta(x) \geq \theta(1) + \frac{\theta'(1)}{\cos(.42)}(\sin(x - .58) - \sin(.42)). \tag{47}$$

Let $\overline{x} = \pi/2 + .58$. Then (42), (43), and (46) imply that $\theta' > 0$ on $(1, \overline{x})$ and $\theta(\overline{x}) \geq .99 + (.94/\cos(.42))(1 - \sin(.42) \geq 1.59$, again contradicting (13) and (14).

3. Proof of Theorem 2. It follows exactly as in the proof of Theorem 1 that $y(x)$ must be odd if (10) and (11) hold. Thus we investigate the existence of solutions of the initial value problem

$$y''' + y' = 1 - y^2/2 \tag{48}$$

$$y(0) = y''(0) = 0, \qquad y'(0) = \beta \geq 0 \tag{49}$$

which satisfy

$$y' > 0 \quad \forall x \in (-\infty, \infty) \tag{50}$$

and

$$y(-\infty) \geq -\sqrt{2}, \qquad y(\infty) = \sqrt{2}. \tag{51}$$

Our goal, as in the proof of Theorem 1 is to prove that there is no value of $\beta \geq 0$ for which (50) and (51) hold. Again, we define two energy functionals which help to reduce the range of possible β values. The first is given by

$$H \equiv (y'')^2/2 - (y')^2/2 - y'y''' \tag{52}$$

where

$$H' = +y(y')^2/2. \tag{53}$$

If there is a β value for which (50) and (51) hold then (y, y', y'') lies on the one-dimensional stable manifold leading to $(\sqrt{2}, 0, 0)$. Thus $y' \to 0$, $y'' \to 0$, and

$H \to 0$ as $x \to \infty$. This and (53) imply that $H < 0$ for all $x > 0$. Thus, from (52) we conclude that $y''' + y'/2 > 0$ and

$$y'' + y/2 > 0 \tag{54}$$

for all $x > 0$. Multiplying (54) by y' and integrating from $x = 0$ to $x = \infty$, we conclude that $\beta \leq 1$. Next, define

$$G = (y'')^2/2 + (y''')^2/2 + yy'y'' - (y')^3/3 \tag{55}$$

where

$$G(0) = (1 - \beta)^2/2 - \beta^3/3 \tag{56}$$

and

$$G' = y((y')^4/3 + (y'')^2). \tag{57}$$

Again, as in the proof of Theorem 1, it follows that $G(0) \leq 0$ so that $\beta \geq .6$. Thus, we only need consider values of β in the range $.6 \leq \beta \leq 1.0$. We now proceed exactly as in the proof of Theorem 1 and obtain the same polynomial estimates as are found in (29), (30), and (31). The details for the case in which $.6 \leq \beta \leq .85$ are the same as those in (i) and (ii) of Sec. 2 and are therefore eliminated. For $\beta \in [.85, 1.0]$ we again let $u = y''/y'$ which is found to satisfy $u' + u^2 + 1 = (1 - y^2/2)/y'$. This equation is used in exactly the same fashion as in (iii) of Section 2 to eliminate the values $.85 \leq \beta \leq 1$. Therefore, for the sake of brevity we omit the details. This completes our proof of Theorem 2.

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