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A CHARACTERIZATION OF THE ROOM INDEX BY MEANS OF FUNCTIONAL EQUATIONS

By

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Abstract. By solving a system of functional equations we characterize the classical room index which is useful in architectural problems concerning room lighting.

In order to decide the best lighting for a room most architects base their solutions upon the well-known room index

$$R(x, y, z) = \frac{cxy}{z(x+y)}, \tag{1}$$

where x and y stand for the length and width of the room (orthoedric shape), z indicates the distance form the light to the surface to be illuminated (floor, table, etc.) and c is a constant. The room index (1) has been adopted in the manuals made by factories (Philips, Osram, etc.) following the tables produced by the International Commission on Illumination in order to determine several light factors in rooms [3, 4].

Our aim in this paper is to characterize (1) as solution of a system of functional equations which represent natural conditions to be satisfied by a room index, establishing on the way a theoretical framework which justifies the empirical room index (1).

We begin with the following:

Definition. Fix a quasi-arithmetic mean F representable in the form

$$F(x, y) = f^{-1}\left(\frac{f(x)+f(y)}{2}\right), \tag{2}$$

where f is a continuous bijection from $(0, +\infty)$ into some real interval. An *F-room index* is a continuous function K from $(0, +\infty)^3$ into $(0, +\infty)$ satisfying the following four conditions for all x, x', y, z, λ in $(0, +\infty)$:

(i) $K(\lambda x, \lambda y, \lambda z) = K(x, y, z)$;
(ii) $K(\lambda x, \lambda y, z) = \lambda K(x, y, z)$;
(iii) $K(x, y, z) = K(y, x, z)$;
(iv) $K(F(x, x'), y, z) = F(K(x, y, z), K(x', y, z))$.

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Condition (i) requires K to be invariant under similitudes, i.e., the index does not depend on the particular scale used to measure the room. Condition (ii) states that if the height z is fixed, the room index is linearly sensible to the different homothetic shapes of the floor. Condition (ii) requires K to be invariant with respect to the order in which we consider length and width. Finally (iv) states that if we have two rooms (with the same y, z measures) the mean (given by F) of the corresponding indices is equal to the index of the room with measures $(F(x, x'), y, z)$.

The crucial result of this paper is contained in the following:

THEOREM. If K is an F-room index then either $F(x, y) = ((x^\alpha + y^\alpha)/2)^{1/\alpha}$ $(\alpha \neq 0)$ and $K(x, y, z) = \frac{c}{z}(x^\alpha + y^\alpha)^{1/\alpha}$, for some constant $c > 0$, or $F(x, x) = \sqrt{xy}$ and $K(x, y, z) = \frac{d\sqrt{xy}}{z}$ for some constant $d > 0$.

Proof. Let K be an F-room index. We fix y and z and introduce the function $g(t) = K(t, y, z)$. By (2) and (iv) we have

$$g\left(f^{-1}\left(\frac{f(x)+f(x')}{2}\right)\right) = f^{-1}\left(\frac{f(g(x))+f(g(x'))}{2}\right), \tag{3}$$

i.e., the continuous function $f \circ g \circ f^{-1}$ satisfies the classical Jensen equation on a real interval and we have [1] (Aczél, 1966) that $(f \circ g \circ f^{-1})(x) = ax + b$. Since y and z were fixed, in fact a and b will depend upon such variables, i.e.,

$$f(g(f^{-1}(x))) = a(y, z)x + b(y, z),$$

(with the obvious constraint that $a(y, z)x + b(y, z)$ must belong to the range of f). Therefore

$$K(x, y, z) = f^{-1}(a(y, z)f(x) + b(y, z)). \tag{4}$$

By (4), (i) and (iii) we must have

$$\begin{aligned} f^{-1}\left(a\left(\frac{x}{z}, 1\right) f\left(\frac{y}{z}\right) + b\left(\frac{x}{z}, 1\right)\right) &= f^{-1}(a(y, z)f(x) + b(y, z)) \\ &= f^{-1}\left(a\left(\frac{y}{z}, 1\right) f\left(\frac{x}{z}\right) + b\left(\frac{y}{z}, 1\right)\right) \end{aligned} \tag{5}$$

whence introducing the new variables $u = \frac{x}{z}$ and $v = \frac{y}{z}$ as well as the functions $A(t) = a(t, 1)$, $B(t) = b(t, 1)$ we can present (5) in the form

$$A(v)f(u) + B(v) = A(u)f(v) + B(u) \tag{6}$$

and substituting $v = 1$ into (6) we determine $B(u)$:

$$B(u) = A(1)f(u) + B(1) - A(u)f(1). \tag{7}$$

Using (7) and going back to the equality (6) we obtain

$$A(v)f(u)+A(1)f(v)+B(1)-A(v)f(1) = A(u)f(v)+A(1)f(u)+B(1)-A(u)f(1) \tag{8}$$

and the substitution $v = 2$ into (8) yields that there exist two constants M and N such that

$$A(u) = Mf(u) + N. \tag{9}$$

Since by (9) A is completely determined in terms of f so will be B by means of (7), i.e., there are two constants P and Q such that

$$B(u) = Pf(u) + Q.$$

Using the representation of A and B and (6) we obtain

$$Mf(u)f(v) + Nf(u) + Pf(v) + Q = Mf(u)f(v) + Nf(v) + Pf(u) + Q,$$

whence $P = N$. Thus we can represent $K(x, y, z)$ in the form

$$K(x, y, z) = f^{-1}\left(Mf\left(\frac{x}{z}\right)f\left(\frac{y}{z}\right) + Nf\left(\frac{x}{z}\right) + Nf\left(\frac{y}{z}\right) + Q\right). \tag{10}$$

Since this function K must satisfy (iv) we have that necessarily

$$\begin{aligned} &f^{-1}\left(Mf\left(\frac{F(x, x')}{z}\right)f\left(\frac{y}{z}\right) + Nf\left(\frac{F(x, x')}{z}\right) + Nf\left(\frac{y}{z}\right) + Q\right) \\ &\quad = F(K(x, y, z), K(x', y, z)) \\ &\quad = f^{-1}\left[Mf\left(\frac{y}{z}\right)\frac{f(\frac{x}{z}) + f(\frac{x'}{z})}{2} + N\cdot\frac{f(\frac{x}{z}) + f(\frac{x'}{z})}{2} + Nf\left(\frac{y}{z}\right) + Q\right]\end{aligned}$$

and from this it is immediate to see that

$$\frac{F(x, x')}{z} = f^{-1}\left(\frac{f(\frac{x}{z}) + f(\frac{x'}{z})}{2}\right) = F\left(\frac{x}{z}, \frac{x'}{z}\right), \tag{11}$$

i.e., F is homogeneous of degree 1. It is well known [1, 2] that (11) is satisfied if, and only if, either

$$f(t) = \alpha t^\gamma + \beta \qquad (\alpha, \gamma \neq 0) \quad \text{or} \quad f(t) = \alpha \log t + \beta \qquad (\alpha \neq 0)$$

where α, β, and γ are constants. Thus by (10) the function K has two alternative representations:

$$K(x, y, z) = \left(C\left(\frac{xy}{z^2}\right)^\gamma + D\left(\frac{x}{z}\right)^\gamma + D\left(\frac{y}{z}\right)^\gamma + E\right)^{1/\gamma} \tag{12}$$

or

$$K(x, y, z) = \exp\left[C\log\left(\frac{y}{z}\right)\log\left(\frac{x}{z}\right) + D\log\left(\frac{xy}{z^2}\right) + E\right], \tag{13}$$

for some appropriate constants C, D, and E. If we impose now that K satisfies (ii) we immediately obtain that in case (12) necessarily $C = E = 0$ and in case (13) $C = 0$, $D = \frac{1}{2}$. Therefore, either

$$K(x, y, z) = \left[D\cdot\frac{x^\gamma + y^\gamma}{z^\gamma}\right]^{1/\gamma} = \frac{D^{1/\gamma}}{z}(x^\gamma + y^\gamma)^{1/\gamma}$$

or

$$K(x, y, z) = \exp\left[\frac{1}{2}\log\frac{xy}{z^2} + E\right] = \exp(E)\frac{\sqrt{xy}}{z}$$

and the theorem is proved.

COROLLARY 1. The general solution of the system of functional equations (i), (iii), and (iv) is given by (12) and (13).

COROLLARY 2. The room index R given by (1) is the unique F-room index where F is the harmonic mean $F(x, y) = \frac{2xy}{x+y}$.

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