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A COMPARISON OF CERTAIN ELASTIC DISSIPATION MECHANISMS VIA DECOUPLING AND PROJECTION TECHNIQUES

By

DAVID L. RUSSELL

Virginia Polytechnic Institute and State University, Blacksburg, Virginia

Abstract. In this paper we study the Euler–Bernoulli elastic beam model, modified in a variety of ways to achieve an asymptotically linear relationship between damping rate and frequency. We review the so-called *spatial hysteresis* model and then introduce the *thermoelastic/shear diffusion* model, which is obtained by coupling the originally conservative elastic equations to two different diffusion processes. We then use a decoupling/triangulation process to project the coupled system onto the subspace corresponding to the lateral displacements and velocities and show that the projected system agrees in many significant respects with the spatial hysteresis model. The procedure also indicates some possibly desirable modifications in the elastic term of the spatial hysteresis model.

1. Internal damping mechanisms in elastic beams. Energy dissipation in elastic systems results from a wide variety of sources, both internal and external with respect to the systems themselves. It is the internal dissipation mechanisms which are of the greatest scientific interest both theoretically, because they reflect the basic structure of the medium in question, and from the point of view of projected applications, such as some proposed space structures, which do not interact with a supporting medium and are thus expected to exhibit energy losses in which internal damping should be dominant. In this article we will be concerned exclusively with internal dissipation mechanisms in the context of the Euler-Bernoulli beam equation

$$\rho \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^4}\right) = 0 \qquad (1.01)$$

for which, in the absence of additional dissipative terms, the energy

$$\mathscr{E}\left(w, \frac{\partial w}{\partial t}\right) \equiv \frac{1}{2}\int_0^L \left[\rho\left(\frac{\partial w}{\partial t}\right)^2 + EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right] dx \qquad (1.02)$$

is conserved.

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From many studies (see, e.g., [2, 9]) it has long been clear that simple "viscous" damping models such as

$$\rho \frac{\partial^2 w}{\partial t^2} + 2\gamma \frac{\partial w}{\partial t} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) = 0,$$

which produce uniform damping rates, are inadequate if experimentally observed damping properties are to be incorporated in the model. The recognition that elastic damping rates tend to increase with frequency goes back at least to Lord Kelvin in Britain and Robert Voigt, a distinguished German physicist, both working toward the end of the last century. The Kelvin-Voigt approach may be applied, in principle, to the vibrations of any linear elastic system. The Kelvin-Voigt hypothesis is that, whatever may be the linear operator describing the elastic restoring forces in an elastic body, the internal damping forces may be described with a positive multiple of that operator, but acting on the system velocity rather than displacement. Incorporated into the Euler-Bernoulli beam model, this approach yields an equation of the form

$$\rho \frac{\partial^2 w}{\partial t^2} + 2\gamma\rho \frac{\partial^3}{\partial t \partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) = 0 \tag{1.03}$$

with appropriately modified conditions at boundary points. If the fourth-order elasticity operator is denoted by

$$Aw \equiv \frac{1}{\rho} \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right),$$

then the corresponding form of Eq. (1.03) is

$$\frac{d^2 w}{dt^2} + 2\gamma A \frac{dw}{dt} + Aw = 0. \tag{1.04}$$

If the eigenvalues of the positive selfadjoint operator A are λ_k, so that the natural frequencies of the undamped system are $\omega_k = (\lambda_k)^{1/2}$, then the damped system (1.04) may be seen to have exponential solutions $e^{\sigma_k t}\varphi_k$, where φ_k is the corresponding eigenvector of A, and σ_k satisfies the quadratic equation

$$\sigma_k^2 + 2\gamma\omega_k^2\sigma_k + \omega_k^2 = 0.$$

Thus σ_k is given by

$$\sigma_k = -\gamma\omega_k^2 \pm (\gamma^2\omega_k^4 - \omega_k^2)^{1/2}. \tag{1.05}$$

Assuming γ to be small, we see that the σ_k are complex for some finite number of values of k with the damping rate proportional to the square of the frequency. Critical damping occurs for $\omega = 1/\gamma$; for ω_k larger than this the modes are overdamped with one of the values given by (1.05) going to $-\infty$ and the other tending to $-1/2\gamma$ as ω_k tends to infinity. For very small values of γ all we would expect to see would be the quadratic dependence of the damping rate on the frequency. Whether the overdamping predicted by the model has ever been observed in the laboratory is unknown to this author but it seems, on the face of it, to be unlikely.

There is no difficulty in obtaining ad hoc mathematical models exhibiting *structural damping*, i.e., a linear damping-versus-frequency relationship. Representing an arbitrary linear oscillator, without damping, by

$$\frac{d^2w}{dt^2} + Aw = 0,$$

the simplest mathematically viable example of a system exhibiting structural damping behavior, treated extensively by the author and G. Chen in [3], is provided by the so-called *square root* model

$$\frac{d^2w}{dt^2} + 2\gamma A^{1/2}\frac{dw}{dt} + Aw = 0, \tag{1.06}$$

where $A^{1/2}$ denotes the positive square root of the positive selfadjoint operator A and $\gamma > 0$. Attempting a solution of the form $e^{\sigma_k t}\varphi_k$ again, we find, assuming $|\gamma| < 1$, that

$$\sigma_k = (-\gamma \pm i(1-\gamma^2)^{1/2})\omega_k$$

so that the σ_k lie on the pair of rays in the left half-plane making an angle

$$\alpha = \tan^{-1}(\gamma/\sqrt{1-\gamma^2})$$

with the imaginary axis.

In [10], in the constant coefficient case, the set of natural boundary conditions is shown to be divisible into two classes with boundary conditions from the first class, consisting of the so-called *trigonometric cases*, being those for which the nonnegative square root of the fourth derivative operator $\frac{\partial^4 w}{\partial x^4}$ is $-\frac{\partial^2 w}{\partial x^2}$. Redefining γ, the modified Euler-Bernoulli equation in this case takes the form

$$\rho\frac{\partial^2}{\partial t^2} - 2\gamma\frac{\partial^3 w}{\partial t \partial x^2} + EI\frac{\partial^4 w}{\partial x^4} = 0. \tag{1.07}$$

In this equation the damping term is rather easy to understand from the physical point of view—it is a lateral force acting on the beam which is negatively proportional to the bending rate at the point where that force is applied. If at either of the endpoints of the interval $[0, L]$ the fourth-order operator is assigned boundary conditions from the second class, constituting the so-called *exponential cases* in [10], which includes the special instances of clamped or free endpoints, corresponding, respectively, to

$$w(0, t) = 0, \qquad \frac{\partial w}{\partial x}(0, t) = 0,$$

or

$$EI\frac{\partial^2 w}{\partial x^2}(0, t) = 0, \qquad -EI\frac{\partial^3 w}{\partial x^3}(0, t) = 0,$$

for example, the positive square root of the fourth-order operator is not the negative second-order operator and the nature of the damping term in (1.06) is such as not to admit a ready interpretation in physical terms. The apparent necessity of discarding this model for this reason is a real disappointment because the system (1.06) has very attractive mathematical properties, as outlined in [3].

The difficulties outlined in the previous paragraph led to the development of an approach which, with rather questionable accuracy, has been named the *spatial hysteresis* damping model [9] (a similar model, developed independently, appears in [6]). Assuming for the moment that the function $w(x, t)$ describing the evolution of the beam displacement is smooth, an easy calculation shows that, with $\mathscr{E}$ as in (1.02), we have

$$\frac{d}{dt}\mathscr{E}\left(w(\cdot, t), \frac{\partial w}{\partial t}(\cdot, t)\right) = \int_0^L \left[\rho \frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial t \partial x^2}\right] dx$$

(integrating the second term in the integrand by parts)

$$= \int_0^L \left[\rho \frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial t^2} - \underline{\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial^2 w}{\partial t \partial x}}\right] dx + EI\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial t \partial x}\Bigg|_0^L . \tag{1.08}$$

The presence of the angular velocity expression $\frac{\partial^2 w}{\partial t \partial x}$ in the underlined term indicates that its coefficient, $-\frac{\partial}{\partial x}(EI\frac{\partial^2 w}{\partial x^2})$, should be interpreted as a restoring *torque*, arising due to spatially variable bending of the beam. The recognition that this coefficient represents a torque aids us in interpretation of the damping term which we now introduce into the system via the definition

$$\tau_h(x, t) = 2\int_0^L h(x, \xi)\left[\frac{\partial^2 w}{\partial t \partial x}(x, t) - \frac{\partial^2 w}{\partial t \partial x}(\xi, t)\right] d\xi. \tag{1.09}$$

We think of τ_h as a torque acting on the beam at the point x due to the differential rotation, as compared with the rotation at x, of the beam at points ξ "near" x. In many cases the support of the *interaction kernel* $h(x, \xi)$ would be restricted to a thin strip in the plane, centered on the line $x = \xi$, or h would be small outside such a strip. Application of Newton's second law dictates the symmetry condition

$$h(\xi, x) = h(x, \xi). \tag{1.10}$$

Additionally, we require that $h(x, \xi)$ should have continuous partial derivatives with respect to x and ξ for $0 \le x, \xi \le L$. In constant coefficient applications it is convenient to replace $h(x, \xi)$ with a function $\gamma h(x - \xi)$, where $\gamma > 0$ is used to parametrize the strength of the damping effect and $h(\eta)$ satisfies the normalizing condition

$$\int_{-\delta}^{\delta} h(\eta)\, d\eta = 1 \tag{1.11}$$

and the evenness condition $h(\eta) = h(-\eta)$.

We add the term

$$\int_0^L \tau_h(x, t)\frac{\partial^2 w}{\partial t \partial x}(x, t)\, dx$$

to both sides of (1.08) so that, with a further integration by parts,

$$\begin{aligned}\frac{d}{dt}\mathscr{E}\left(w(\cdot\,,t),\frac{\partial w}{\partial t}(\cdot\,,t)\right)&+\int_0^L \tau_h\frac{\partial^2 w}{\partial t\partial x}\,dx\\ &=\int_0^L\frac{\partial w}{\partial t}\left\{\rho\frac{\partial^2 w}{\partial t^2}+\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)-\tau_h\right]\right\}dx \qquad (1.12)\\ &+\left\{EI\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial t\partial x}-\left[\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)-\tau_h\right]\frac{\partial w}{\partial t}\right\}\Bigg|_0^L .\end{aligned}$$

Equating the separate parts of (1.12) to zero (a procedure which can be made rigorous by the principle of virtual work) yields the integro-partial differential equation

$$\begin{aligned}\rho\frac{\partial^2 w}{\partial t^2}&+\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)-\tau_h\right]\\ &=\rho\frac{\partial^2 w}{\partial t^2}-2\frac{\partial}{\partial x}\int_0^L h(x,\xi)\left[\frac{\partial^2 w}{\partial t\partial x}(x,t)-\frac{\partial^2 w}{\partial t\partial x}(\xi,t)\right]d\xi \qquad (1.13)\\ &+\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right)=0\end{aligned}$$

and the requirement that, at $x=0$ and $x=L$,

$$EI\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial t\partial x}-\left[\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)-\tau_h\right]\frac{\partial w}{\partial t}=0. \qquad (1.14)$$

Various beam configurations now lead to different sets of boundary conditions. For example, in the case where the beam is clamped at $x=0$ and free at $x=L$ (i.e., the cantilever case) we obtain

$$w(0,t)=0,\quad \frac{\partial w}{\partial x}(0,t)=0,\quad \frac{\partial^2 w}{\partial x^2}(L,t)=0, \qquad (1.15)$$

$$-\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Bigg|_{x=L}+2\int_0^L h(L,\xi)\left[\frac{\partial^2 w}{\partial t\partial x}(L,t)-\frac{\partial^2 w}{\partial t\partial x}(\xi,t)\right]d\xi=0. \qquad (1.16)$$

Equation (1.12) now becomes

$$\begin{aligned}\frac{d}{dt}\mathscr{E}&\left(w(\cdot\,,t),\frac{\partial w}{\partial t}(\cdot\,,t)\right)\\ &=-2\int_0^L\int_0^L h(x,\xi)\left[\frac{\partial^2 w}{\partial t\partial x}(\xi,t)-\frac{\partial^2 w}{\partial t\partial x}(x,t)\right]d\xi\frac{\partial^2 w}{\partial t\partial x}(x,t)\,dx=\end{aligned}$$

(since the roles of x and ξ are symmetric and $h(x,\xi)=h(\xi,x)$)

$$=-\int_0^L\int_0^L h(x,\xi)\left[\frac{\partial^2 w}{\partial t\partial x}(x,t)-\frac{\partial^2 w}{\partial t\partial x}(\xi,t)\right]^2 d\xi\,dx\le 0, \qquad (1.17)$$

provided we insist that

$$h(x, \xi) > 0, \qquad 0 \leq x, \xi \leq L. \tag{1.18}$$

Clearly (1.17) becomes an equality for motions with $\frac{\partial^2 w}{\partial t \partial x}$ constant, i.e., the inertial motions. Thus the energy is strictly decreasing whenever the bending rate is not identically zero and is conserved when the bending rate, $\frac{\partial^3 w}{\partial t \partial x^2}$, vanishes identically.

In [9] we have studied a constant coefficient, infinite length beam, for which the convolution form $h(x - \xi)$ of the interaction kernel is appropriate. It is shown that the spectrum of the generator of the related semigroup on the energy (Hilbert) space consists of complex numbers, indexed by the spatial frequency parameter τ,

$$\sigma = \sigma(\tau) = \frac{\tau^2}{\rho}[-g(\tau) \pm \sqrt{g(\tau)^2 - \rho EI}], \tag{1.19}$$

wherein, with $\hat{h}(\tau)$ the Fourier transform of the kernel $h(\eta)$,

$$g(\tau) = \gamma(1 - \hat{h}(\tau)). \tag{1.20}$$

In this context we should require $h(\eta)$ to be integrable on $(-\infty, \infty)$ with derivative, $h'(\eta)$, at least square integrable there (see [5]). From the integrability of $h(\eta)$ we know that $\hat{h}(\tau)$ is continuous and

$$\lim_{|\tau| \to \infty} \hat{h}(\tau) = 0. \tag{1.21}$$

For a relatively low level of damping we may assume $\gamma^2 < \rho EI$. Then it may be seen that for large values of τ, to first-order in $\hat{h}(\tau)$,

$$\begin{aligned} \sigma(\tau) &\simeq \frac{\tau^2}{\rho}\left[-\gamma(1 - \hat{h}(\tau)) \pm i\sqrt{\rho EI - \gamma^2}\left(1 + \frac{\gamma^2 \hat{h}(\tau) - \gamma^2(\hat{h}(\tau)^2/2) + \cdots}{\rho EI - \gamma^2}\right)\right] \\ &\simeq \frac{\tau^2}{\rho}[-\gamma \pm 1\sqrt{\rho EI - \gamma^2}]\left[1 \mp \frac{i\gamma \hat{h}(\tau)}{\sqrt{\rho EI - \gamma^2}}\right]. \end{aligned}$$

Thus, asymptotically, as $|\tau| \to \infty$, $\sigma(\tau)$ lies along the rays

$$R_{+,-} = \{z | z = r(-\gamma \pm i\sqrt{\rho EI - \gamma^2}), \ r > 0\}$$

in the left half complex plane.

Further analysis shows that as $|\tau| \to 0$,

$$\sigma(\tau) \simeq \frac{\tau^2}{\rho}[-\gamma\vartheta\tau^2 \pm i\sqrt{\rho EI - \gamma^2}], \tag{1.22}$$

where ϑ is given by

$$\vartheta = \frac{1}{2}\int_{-\infty}^{\infty} \eta^2 h(\eta)\, d\eta,$$

provided this integral exists.

From (1.22) it is clear that, as $\tau^2 \to 0$, there is a quadratic dependence of damping rate on frequency, in agreement with the Kelvin-Voigt model, while, as we have seen earlier, at the high end of the spectrum there obtains a linear damping rate versus frequency relationship of the sort provided by the square root model. In his dissertation [4] Hansen has shown, for a variety of different configurations, that the mathematical properties of the model are essentially those of the square root model as well. Additional work along these lines has been reported by Huang [5].

Nevertheless, it is only fair to add that the spatial hysteresis model bears its share of criticism. A serious objection, perhaps persuasive from the point of view of classical continuum mechanics, is that the damping term is not local and hence is not a property of the *material* of which the beam is composed as opposed to the particular *configuration* of the material in the beam. Granting the validity of that objection in the context in which it is made, we nevertheless state unequivocally that our interest lies in modelling the damping behavior of particular bodies rather than of materials.

The damping mechanisms reviewed in this section are *direct* in that they involve direct insertion of supplementary dissipation terms into the original conservative equations governing the elastic system. The methods to be studied in the sections to follow are *indirect* in the sense that they involve coupling the mechanical equations governing beam motion to related dissipative systems with additional dynamics, resulting in an overall system in which mechanical energy is dissipated. Two types of coupled dissipative systems are discussed, the overall processes being described as the Euler-Bernoulli beam with *thermoelastic* damping and with *shear diffusion* damping. Our main result pertains to the situation wherein both types of damping are simultaneously present. These are introduced in Sec. 2 and their eigenvalue distributions are studied, for certain cases, in Sec. 3.

The mechanisms referred to above enjoy the advantage that they are motivated and derived from simple physical considerations, a project which is carried out in a separate paper [8]. The resulting coupled systems are not excessively complicated but they are of higher order, or dimension, than the original undamped systems are, with states which include temperature and shear distributions as well as displacements of the beam itself. Such a model may be excessively complicated for use in many types of simulations. A second objective, described in Sec. 4, is to introduce and study a decoupling process which projects the dynamics of the coupled system onto the subspace of lateral displacements and velocities. Notably, and fortuitously for the future of the spatial hysteresis model, this projection, applied to the combined thermoelastic/shear diffusion process in certain of the constant coefficient cases, yields a system with damping term consistent with spatial hysteresis for a particular choice of the interaction kernel $h(x-\xi)$, as we demonstrate in Sec. 5. The projection process also results in modification of the elastic term of the system; the implications of this modification are discussed in Sec. 6.

2. Combined thermoelastic/shear diffusion dissipation in an elastic beam. The models which we will introduce here are *indirect* in the sense that they involve coupling of the mechanical equations governing beam motion to related dissipative systems with their own dynamics, resulting in an overall system in which mechanical energy is dissipated. The "derivations" which we offer here are of an ad hoc character to be used for information and motivation only; more rigorous derivations will appear in [8]. We will restrict attention to constant coefficient cases here.

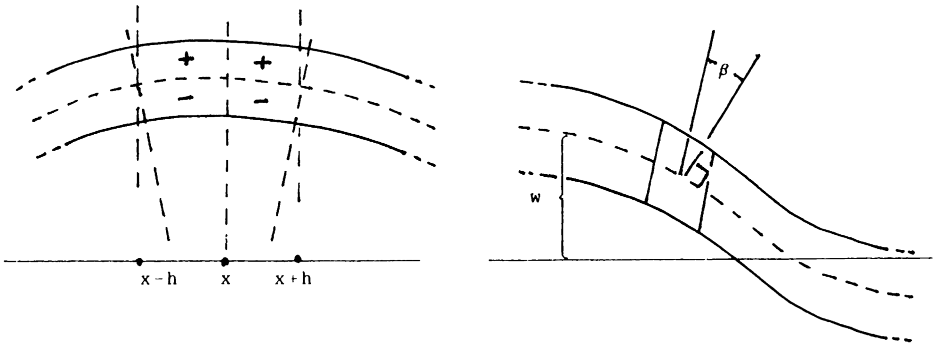

FIG. 1. Schematic representation for thermoelastic and shear diffusion damping.

We begin by discussing thermoelastic dissipation in an elastic beam. Starting again with the Euler-Bernoulli model for beam motion (cf. (1.01)), we ask the reader to consider a cross-section of the beam as shown in Fig. 1. A curvilinear trapezoid is bounded by the lines orthogonal to the elastic axis at $x-h$ and $x+h$, where h is a small positive number, and by the upper and lower lateral surfaces of the beam. That curvilinear trapezoid is, in turn, divided into upper and lower halves by the elastic axis itself. For this expository treatment it is enough to suppose that the absolute temperature in the upper trapezoidal region can be adequately approximated in mean by a function $T^+(x, t)$ and in the lower region by $T^-(x, t)$, neglecting any more complex variations in the transverse direction. We probably should call the difference $\delta T(x, t)$ but, for notational simplicity, let us just write

$$T(x, t) = T^+(x, t) - T^-(x, t).$$

In the absence of other influences, and assuming the beam to be sufficiently thin so that transverse conduction occurs much more rapidly than that due to any longitudinal variations in T to be considered, we should expect such a temperature variation to decay according to the simple law

$$\frac{\partial T}{\partial t}(x, t) = -kT(x, t).$$

However, the laws of thermoelasticity indicate a coupling of this conduction process with the purely mechanical processes taking place in the beam, which we now proceed to elucidate.

Taking the thickness of the beam to be $2d$, $d > 0$ but small, we may assume that for small h the volumes of the upper and lower trapezoids are represented by $V^+(x, t)\,dh$ and $V^-(x, t)\,dh$. From the elementary principles of thermoelasticity, as presented in [1], e.g., where further references are indicated, we conclude that in adiabatic deformation in the absence of conduction we should have a relationship

$$T(x, t) = \hat{K}\left(\frac{1}{V^+(x, t)} - \frac{1}{V^-(x, t)}\right).$$

Letting $V(x, t) = V^+(x, t) - V^-(x, t)$ and supposing the variations in $V^+(x, t)$ and $V^-(x, t)$ to be very small relative to their mean values, which we take to be very nearly constant both in space and time, we can see that we should have an approximate relationship

$$\frac{\partial T}{\partial t}(x, t) = -\tilde{K}\frac{\partial V}{\partial t}(x, t).$$

Now allowing for this thermoelastic effect and conductivity at the same time, the complete relationship becomes

$$\frac{\partial T}{\partial t}(x, t) = -kT(x, t) - \tilde{K}\frac{\partial V}{\partial t}(x, t). \tag{2.01}$$

But, for small h, $V(x, t)$ is nearly proportional to $-\frac{\partial^2 w}{\partial x^2}(x, t)$, so we may replace (2.01) by

$$\frac{\partial T}{\partial t}(x, t) = -kT(x, t) + K\frac{\partial^3 w}{\partial t \partial x^2}(x, t). \tag{2.02}$$

Complementary to these temperature effects of mechanical motion, we now suppose that the total force couple about the elastic axis is modified from its usual Euler-Bernoulli value of $EI\frac{\partial^2 w}{\partial x^2}(x, t)$ to $EI\frac{\partial^2 w}{\partial x^2}(x, t) + \theta KT(x, t)$, for some $\theta > 0$. The appropriate modification of the Euler-Bernoulli energy expression (1.02) is now

$$\hat{\mathscr{E}}\left(w, \frac{\partial w}{\partial t}, T\right) = \frac{1}{2}\int_0^L \left(\rho\left(\frac{\partial w}{\partial t}\right)^2 + EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \theta T^2\right) dx. \tag{2.03}$$

Proceeding in the usual way, we see that if we assume (2.02) and the equation of motion for the beam,

$$\rho\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} + \theta K\frac{\partial^2 T}{\partial x^2} = 0, \tag{2.04}$$

we will obtain, for smooth solutions of these coupled equations, the energy dissipation relation

$$\frac{d\hat{\mathscr{E}}}{dt} = -\theta k \int_0^L T(x, t)^2 \, dx \tag{2.05}$$

provided that the boundary form

$$\left.\left(EI\frac{\partial^2 w}{\partial x^2} + \theta KT\right)\frac{\partial^2 w}{\partial t \partial x} - \left(EI\frac{\partial^3 w}{\partial x^3} + \theta K\frac{\partial T}{\partial x}\right)\frac{\partial w}{\partial t}\right|_0^L$$

vanishes.

The second indirect mechanism which we will study here is the *shear diffusion* process. It takes as its starting point the Timoshenko model [12]. We choose here, in our expression of that model, to use as variables the lateral deflection w and the shear angle, β, rather than w and the local rotation angle ψ, because the dissipation mechanism to be developed is particularly dependent on β. In the variables w and

β the Timoshenko beam energy is

$$\mathscr{T} = \frac{1}{2}\int_0^L \left\{ \rho\left(\frac{\partial w}{\partial t}\right)^2 + \tau\beta^2 + I_\rho\left(\frac{\partial \beta}{\partial t} + \frac{\partial^2 w}{\partial t \partial w}\right)^2 + EI\left(\frac{\partial \beta}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)^2 \right\} dx \tag{2.06}$$

(cf. [13] for definition of the (positive and constant) coefficients introduced here), from which we obtain a modified form of the Timoshenko equations

$$\rho\frac{\partial^2 w}{\partial t^2} - I_\rho\left(\frac{\partial^3 \beta}{\partial t^2 \partial x} + \frac{\partial^4 w}{\partial t^2 \partial x^2}\right) + EI\left(\frac{\partial^3 \beta}{\partial x^3} + \frac{\partial^4 w}{\partial x^4}\right) = 0, \tag{2.07}$$

$$I_\rho\left(\frac{\partial^2 \beta}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial x}\right) + \tau\beta - EI\left(\frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^3 w}{\partial x^3}\right) = 0. \tag{2.08}$$

In (2.08) all terms except the first can be regarded as *shearing* forces to which the shear angle β responds dynamically. Now suppose that a further *viscous* force affects the evolution of β:

$$I_\rho\left(\frac{\partial^2 \beta}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial x}\right) + 2\sigma\frac{\partial \beta}{\partial t} + \tau\beta - EI\left(\frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^3 w}{\partial x^3}\right) = 0. \tag{2.09}$$

We now further assume I_ρ to be small relative to the other constants present and we neglect the first term. What then remains is a diffusion process for β,

$$2\sigma\frac{\partial \beta}{\partial t} + \tau\beta - EI\left(\frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^3 w}{\partial x^3}\right) = 0; \tag{2.10}$$

thus the name *shear diffusion* for the process. Combining (2.10) with the similarly modified Eq. (2.07), i.e.,

$$\rho\frac{\partial^2 w}{\partial t^2} + EI\left(\frac{\partial^3 \beta}{\partial x^3} + \frac{\partial^4 w}{\partial x^4}\right) = 0, \tag{2.11}$$

taking $I_\rho = 0$ in (2.06) to obtain $\hat{\mathscr{T}}$, and assuming the vanishing of an appropriate boundary form ((2.16) below with $K = 0$), we have

$$\frac{d\hat{\mathscr{T}}}{dt} = -2\sigma\int_0^L \frac{\partial \beta}{\partial t}(x, t)^2\, dx. \tag{2.12}$$

The shear diffusion model is not simply a special case of the classical model of linear thermoelasticity. The natural extension of our model in that direction would result from including the effects of thermal conductivity in the longitudinal direction of the beam in Eq. (2.02). That is not done here because we believe those effects should be small in comparison to those due to transverse conduction and the shear effects. The shear diffusion model differs from the thermoelastic model in a very significant *mathematical* respect; namely, the shear rate equation (2.10) couples to the beam equation via (a spatial derivative of) the beam *displacement* rather than via the beam *velocity* as in the case of (2.02) or any other similar equation based on thermoelastic considerations. This is discussed further in [12].

While it is not entirely obvious, it will be plausible that the combined effect of thermoelastic and shear diffusion dissipation acting simultaneously corresponds to a system involving all four of Eqs. (2.02), (2.04), (2.10), (2.11), ((2.04) and (2.11) combined in one equation):

$$\begin{gathered}\rho\frac{\partial^2 w}{\partial t^2}+EI\left(\frac{\partial^3\beta}{\partial x^3}+\frac{\partial^4 w}{\partial x^4}\right)+\theta K\frac{\partial^2 T}{\partial x^2}=0,\\ \frac{\partial T}{\partial t}+kT-K\frac{\partial^3 w}{\partial t\partial x^2}=0,\\ 2\sigma\frac{\partial\beta}{\partial t}+\tau\beta-EI\left(\frac{\partial^2\beta}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)=0.\end{gathered}\tag{2.13}$$

A more formal derivation of these equations will appear in [8].

Using (2.13) we may verify that if we define (cf. (2.03), (2.06))

$$\mathscr{S}=\frac{1}{2}\int_0^L\left\{\rho\left(\frac{\partial w}{\partial t}\right)^2+\tau\beta^2+EI\left(\frac{\partial\beta}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)^2+\theta T^2\right\}dx,\tag{2.14}$$

and assume smooth solutions, we have

$$\frac{d\mathscr{S}}{dt}=-\int_0^L\left(\theta kT(x,t)^2+2\sigma\left(\frac{\partial\beta}{\partial t}(x,t)\right)^2\right)dx\tag{2.15}$$

provided that the boundary form

$$\begin{aligned}&\left(EI\left(\frac{\partial\beta}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+\theta KT\right)\frac{\partial^2 w}{\partial t\partial x}-\left(EI\left(\frac{\partial^2\beta}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)+\theta K\frac{\partial T}{\partial x}\right)\frac{\partial w}{\partial t}\\ &\quad+EI\left(\frac{\partial\beta}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial\beta}{\partial t}\end{aligned}\tag{2.16}$$

vanishes.

In much of the work to follow we will be concerned with those cases in which the boundary conditions used to annihilate (2.16), along with the system equations (2.13), admit solutions of the form $e^{\lambda(\nu)t}e^{i\nu x}(\hat{w}(\nu),\hat{T}(\nu),\hat{\beta}(\nu))$ for a certain set of values of the wave length parameter ν. This corresponds, e.g., to the trigonometric cases discussed in [10], the case of periodic boundary conditions on a finite interval and the case of an infinite interval with conditions at infinity sufficient to imply that the expression (2.16) has limit zero as $|x|\to\infty$.

The bending equation, i.e., the first equation in (2.13), and the thermal equation, which is the second equation there, clearly decouple as $K\to 0$. In what follows we will be assuming that K is positive but sufficiently small. The shear equation, the third member of the triple (2.13), becomes independent of w if we assume that σ and β are large relative to EI and we can expect β to be small, and thus a relatively insignificant term in the first equation of (2.13) if β is large relative to σ. Assumptions of this form will be made in the sequel without further discussion.

3. Eigenvalue/eigenvector analysis of constant coefficient, trigonometric cases. We again consider the coupled system (2.13). As already noted in the previous section, with the assumptions indicated there, we may look for solutions of the form

$$\begin{aligned} w(x,t) &= \hat{w}(\nu)e^{\lambda(\nu)t}e^{i\nu x}, \\ T(x,t) &= \hat{T}(\nu)e^{\lambda(\nu)t}e^{i\nu x}, \\ \beta(x,t) &= \hat{\beta}(\nu)e^{\lambda(\nu)t}e^{i\nu x}, \end{aligned} \tag{3.01}$$

for various values of ν depending on the spatial interval involved and the specific form of the boundary conditions. Substituting these solution forms into (2.13) we obtain an eigenvalue problem for λ, parametrized by ν (in which we suppress ν as an argument):

$$\rho\lambda^2\hat{w} + EI(-i\hat{\beta}\nu^3 + \hat{w}\nu^4) - \theta K\hat{T}\nu^2 = 0, \tag{3.02}$$

$$\lambda\hat{T} + k\hat{T} + K\hat{w}\lambda\nu^2 = 0, \tag{3.03}$$

$$2\sigma\hat{\beta}\lambda + \tau\hat{\beta} + EI(\hat{\beta}\nu^2 + i\hat{w}\nu^3) = 0. \tag{3.04}$$

Solving (3.03) and (3.04) for $\hat{T}$ and $\hat{\beta}$, respectively, in terms of $\hat{w}$, we have

$$\begin{aligned} \hat{T} &= -(\lambda + k)^{-1}K\hat{w}\nu^2, \\ \hat{\beta} &= -(2\sigma\lambda + \tau + EI\nu^2)^{-1}EI\hat{w}i\nu^3. \end{aligned}$$

Substituting these into (3.02) and removing the common factor $\hat{w}$ from the resulting equation, we obtain

$$\rho\lambda^2 + EI\{\nu^4 - (2\sigma\lambda + \tau + EI\nu^2)^{-1}EI\nu^6\} + (\lambda + k)^{-1}\theta K^2\lambda\nu^4 = 0. \tag{3.05}$$

PROPOSITION 3.1. *If K, $EI\rho/\sigma^2$, and σ/τ are sufficiently small, (3.05) has two negative real solutions and a conjugate pair of complex solutions for every nonzero real ν. Moreover, the complex solutions have asymptotic representations*

$$\begin{aligned} \lambda(\nu) = {} & \left(-EI/4\sigma \pm i\sqrt{EI/\rho - (EI/4\sigma)^2} + \mathscr{O}(K^2)\right)\nu^2 \\ & + \mathscr{O}(1), \qquad \nu^2 \to \infty, \end{aligned} \tag{3.06}$$

$$\begin{aligned} \lambda(\nu) = {} & -\left(1 + \frac{1}{\rho}\right)\frac{2\sigma\theta K^2}{\tau k}\nu^4 + \mathscr{O}(\nu^6) \\ & \pm i(2(EI/\rho)^{1/2}\nu^2 + \mathscr{O}(\nu^4)), \qquad \nu^2 \to 0. \end{aligned} \tag{3.07}$$

REMARK. Thus for small values of ν the damping exponent is proportional to the square of the frequency, in agreement with both the Kelvin-Voigt and "spatial hysteresis" models. On the other hand, as $\nu^2 \to \infty$, (3.06) shows the (negative) real part of λ to be, asymptotically, a multiple of the imaginary part so that the damping exponent is proportional to frequency at that end of the spectrum, in agreement with the "square root" and other "structural damping" approaches. The negative real roots correspond to thermal and shear relaxation modes of the coupled system.

Proof. We observe, first of all, that we can rewrite (3.05) as

$$\frac{2\sigma\rho}{\tau}\lambda^3 + \left(\rho + \frac{\rho EI}{\tau}\nu^2\right)\lambda^2 + \frac{2\sigma EI}{\tau}\nu^4\lambda + EI\nu^4 + \left(\frac{2\sigma}{\tau} + 1 + \frac{EI}{\tau}\nu^2\right)\frac{\lambda}{\lambda+k}\theta K^2\nu^4 = 0.$$

Dividing by the leading coefficient, we have

$$\lambda^3 + \left(\frac{\tau}{2\sigma} + \frac{EI}{2\sigma}\nu^2\right)\lambda^2 + \frac{EI}{\rho}\nu^4\lambda + \frac{EI\gamma}{2\sigma\rho}\nu^4 + \left(\frac{1}{\rho} + 1 + \frac{EI}{2\sigma\rho}\nu^2\right)\frac{\lambda}{\lambda+k}\theta K^2\nu^4 = 0.$$

Setting $\delta = EI/4\sigma$, $\alpha^2 = EI/\rho$, and $\varepsilon = \tau/2\sigma$ we obtain the form

$$\lambda^3 + 2\delta\nu^2\lambda^2 + \alpha^2\nu^4\lambda + \varepsilon(\lambda^2 + \alpha^2\nu^4) + \left(1 + \frac{1}{\rho} + \frac{2\delta}{\rho}\nu^2\right)\frac{\lambda}{\lambda+k}\theta K^2\nu^4 = 0, \tag{3.08}$$

where all coefficients shown are positive. Letting $\lambda = \tilde{\lambda}\nu^2$ and dividing the resulting equation by ν^4, Eq. (3.08) may, in turn, be replaced by

$$\tilde{\lambda}^2 + 2\delta\tilde{\lambda} + \alpha^2 = \frac{2\delta\varepsilon\tilde{\lambda}}{\nu^2\tilde{\lambda} + \varepsilon} - \left(1 + \frac{1}{\rho} + \frac{2\delta}{\rho}\nu^2\right)\frac{\theta K^2\tilde{\lambda}\nu^2}{(\tilde{\lambda}\nu^2 + \varepsilon)(\tilde{\lambda}\nu^2 + k)}. \tag{3.09}$$

Since the coefficients of δ and K^2 on the right-hand side of (3.09) are uniformly bounded for $0 \le \nu < \infty$, it is easy to see that there are solutions having the form, again uniformly for $0 \le \nu < \infty$,

$$\tilde{\lambda} = \pm i\alpha + \mathscr{O}(\delta) + \mathscr{O}(K^2),$$
$$\lambda = \tilde{\lambda}\nu^2 = \pm i\alpha\nu^2 + \nu^2(\mathscr{O}(\delta) + \mathscr{O}(K^2)).$$

The energy dissipation results of the preceding section guarantee that the real part of λ is negative in all cases. For large values of ν we can obtain more information by noting that (3.09) takes the asymptotic form, as $\nu \to \infty$,

$$\tilde{\lambda}^3 + 2\delta\tilde{\lambda}^2 + \alpha^2\tilde{\lambda} + 2\delta\theta K^2/\rho = \frac{1}{\nu^2}f(\tilde{\lambda}, \nu^2), \tag{3.10}$$

where f is uniformly bounded and has uniformly bounded partial derivatives with respect to $\tilde{\lambda}$ and ν^2 for all large ν and $\tilde{\lambda}$ bounded away from zero. We conclude that (3.10) has a solution

$$\begin{aligned}\tilde{\lambda} &= -\delta \pm i\sqrt{\alpha^2 - \delta^2} + cK^2 + \mathscr{O}(\nu^{-2}) + \mathscr{O}(K^4)\\ &= \tilde{\lambda}_0 + cK^2 + \mathscr{O}(\nu^{-2}) + \mathscr{O}(K^4),\end{aligned} \tag{3.11}$$

provided that

$$c = \frac{-2\delta\theta}{\rho(3\tilde{\lambda}_0^2 + 4\delta\tilde{\lambda}_0 + \alpha^2)} = \frac{\delta\theta}{\rho\alpha^2} - \frac{\delta^2\theta i}{\rho\alpha^2(\alpha^2 - \delta^2)^{1/2}}. \tag{3.12}$$

For small values of λ, on the other hand, we can see that

$$\begin{aligned}\frac{2\delta\varepsilon\tilde{\lambda}}{\nu^2\tilde{\lambda}+\varepsilon} &= 2\delta\tilde{\lambda} - \frac{2\delta\nu^2\tilde{\lambda}^2}{\varepsilon} + \frac{2\delta\nu^4\tilde{\lambda}^3}{\varepsilon^2} + \cdots \\ &= 2\delta\tilde{\lambda} - \frac{2\delta\nu^2\tilde{\lambda}^2}{\varepsilon} + \nu^4 g(\tilde{\lambda}, \nu),\end{aligned}$$

and

$$\left(1+\frac{1}{\rho}+\frac{2\delta}{\rho}\nu^2\right)\frac{\theta K^2\tilde{\lambda}\nu^2}{(\tilde{\lambda}\nu^2+\varepsilon)(\tilde{\lambda}\nu^2+k)} = \left(1+\frac{1}{\rho}\right)\frac{\theta K^2}{\varepsilon k}\tilde{\lambda}\nu^2 + \nu^4 h(\tilde{\lambda}, \nu),$$

where $g(\tilde{\lambda}, \nu)$ and $h(\tilde{\lambda}, \nu)$ are uniformly bounded with bounded partial derivatives as long as ν and $\tilde{\lambda}$ are bounded. Thus (3.09) becomes

$$\left(1+\frac{2\delta\nu^2}{\varepsilon}\right)\tilde{\lambda}^2 + \left(1+\frac{1}{\rho}\right)\frac{\theta K^2}{\varepsilon k}\nu^2\tilde{\lambda} + \alpha^2 = \nu^4(g(\tilde{\lambda}, \nu) + h(\tilde{\lambda}, \nu))$$

and consequently, as $\nu^2 \to 0$,

$$\lambda = \tilde{\lambda}\nu^2 = -\left(1+\frac{1}{\rho}\right)\frac{\theta K^2}{\varepsilon k}\nu^4 + \mathscr{O}(\nu^6) \pm i(2\alpha\nu^2 + \mathscr{O}(\nu^4)), \tag{3.13}$$

where both "$\mathscr{O}$" terms are real. Going back to the original coefficients, (3.11) and (3.13) give us (3.06) and (3.07), respectively.

Now let us rewrite (3.05) in the form

$$2\sigma\lambda + \tau + EI\nu^2\left(1 - \frac{EI\nu^4}{\rho\lambda^2 + EI\nu^4 + \theta K^2\lambda\nu^4/(\lambda+k)}\right) = 0.$$

Setting $\lambda = \mu/\sigma$ we obtain

$$2\mu + \tau + EI\nu^2\left(1 - \frac{\sigma^2 EI\nu^4}{\rho\mu^2 + \sigma^2 EI\nu^4 + \sigma^2\theta K^2\mu\nu^4/(\mu+\sigma k)}\right) = 0,$$

which we can rewrite as

$$\begin{aligned}&(2\mu + \tau + 2EI\nu^2)(\rho\mu^2(\mu+\sigma k) + \sigma^2 EI\nu^4(\mu+\sigma k) + \sigma^2\theta K^2\mu\nu^4) \\ &\quad - \sigma^2(EI)^2\nu^6(\mu+\sigma k) = 0.\end{aligned} \tag{3.14}$$

Let us agree to restrict attention to the case $\tau > 2\sigma k$. Then (3.14) is negative for each $\nu \neq 0$, when $\mu = -\sigma k$. But it clearly must become positive as $\mu \to -\infty$. It follows that for each $\nu \neq 0$ there is some real $\mu(\nu) < -\sigma k$, clearly reducing to $-\sigma k$ when $\nu = 0$, for which (3.14) vanishes. Consequently (3.14), and therefore also (3.05), has a negative real solution $\hat{\lambda}(\nu) = \hat{\mu}(\nu)/\sigma < -k$ when $\tau > 2\sigma k$. Since we have already seen that there is a pair of complex conjugate solutions when $\alpha^2 > \delta^2$, i.e., when $EI\rho < 16\sigma^2$, it follows that there are two real solutions, both negative in view of (2.18). With this the proof is complete.

4. Projection onto the space of lateral beam motions. Taking $l = \theta K/(\rho EI)^{1/2}$, $r = K(\rho/EI)^{1/2}$, $\alpha = (EI/\rho)^{1/2}$, $s = \rho/2\sigma$, $\varepsilon = \tau/2\sigma$, A to be the nonnegative selfadjoint operator $(EI/\rho)\frac{\partial^4}{\partial x^4}$, and $A^{1/2}$ to be its nonnegative square root, as in the previous section, an elastic beam incorporating thermoelastic and shear diffusion effects may be represented in 4×4 matrix operator form as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & I & 0 & 0 \\ -A & 0 & lA^{1/2} & \alpha A^{1/2} \\ 0 & -rA^{1/2} & -kI & 0 \\ sA & 0 & 0 & -\varepsilon I - s\alpha A^{1/2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}, \tag{4.01}$$

wherein x_1 and x_2 stand for w and $\partial w/\partial t$, and y_1 and y_2 for T and $\partial \beta/\partial x$.

The system (4.01) belongs to a general class of coupled systems which we can write in the form

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{4.02}$$

Our primary interest, let us say, lies in the "X" part of the state, which would satisfy $\dot{X} = AX$ were it not coupled to the second system involving Y. We are really interested in how the coupling of the X system to the Y system affects the X dynamics. To explore this, we make the transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix} \begin{pmatrix} X \\ Z \end{pmatrix}, \tag{4.03}$$

wherein Q is as yet indeterminate. Then we can easily see that the resulting system in the X, Z variables is

$$\begin{aligned} \begin{pmatrix} \dot{X} \\ \dot{Z} \end{pmatrix} &= \begin{pmatrix} I & 0 \\ -Q & I \end{pmatrix} \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{pmatrix} \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix} \begin{pmatrix} X \\ Z \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{A} + \mathbb{B}\mathbb{Q} & \mathbb{B} \\ \mathbb{D}Q - Q\mathbb{A} - Q\mathbb{B}Q + \mathbb{C} & \mathbb{D} - Q\mathbb{B} \end{pmatrix} \begin{pmatrix} X \\ Z \end{pmatrix}. \end{aligned}$$

If we can find a solution Q of the quadratic operator equation

$$\mathbb{D}Q - Q\mathbb{A} - Q\mathbb{B}Q + \mathbb{C} = 0 \tag{4.04}$$

and use that solution in (4.03), we clearly achieve a system in upper triangular form

$$\begin{pmatrix} \dot{X} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} \mathbb{A} + \mathbb{B}Q & \mathbb{B} \\ 0 & \mathbb{D} - Q\mathbb{B} \end{pmatrix} \begin{pmatrix} X \\ Z \end{pmatrix}. \tag{4.05}$$

A particular class of solutions is obtained for which $Z = 0$, i.e.,

$$Y = QX, \tag{4.06}$$

and the X dynamics, conditioned on this restriction, are described by

$$\dot{X} = (\mathbb{A} + \mathbb{B}Q)X. \tag{4.07}$$

With appropriate care (the solution Q of (4.04) is not unique in general) we can interpret (4.07) as describing the coupled behavior of X, with Y entrained to X via (4.06). The "appropriate care" just referred to involves the question of which of the eigenvalues of the operator matrix appearing on the right-hand side of (4.02) are to be assigned to the operator $\mathbb{A} + \mathbb{B}Q$ of the "decoupled" system (4.07) and which are to be assigned to the operator $\mathbb{D} - Q\mathbb{B}$. This done, we observe that if φ is an eigenvector of $\mathbb{A} + \mathbb{B}Q$ corresponding to the eigenvalue λ, then

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \varphi \\ Q\varphi \end{pmatrix} \tag{4.08}$$

is an eigenvector of the operator in (4.02), also corresponding to the eigenvalue λ.

Returning to (4.01), we introduce a transformation of the form (4.03), now written in the expanded form

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ P_1 & P_2 & I & 0 \\ Q_1 & Q_2 & 0 & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{pmatrix}. \tag{4.09}$$

An elementary, but rather lengthy, computation shows that if we define the operator expressions E, F, G, H by

$$E(P_1, P_2) = P_2A - lP_2A^{1/2}P_1 - kP_1, \tag{4.10}$$

$$F(P_1, P_2) = -P_1 - rA^{1/2} - lP_2A^{1/2}P_2 - kP_2, \tag{4.11}$$

$$G(Q_1, Q_2) = Q_2A + sA - \alpha Q_2A^{1/2}Q_1 - (\varepsilon I + s\alpha A^{1/2})Q_1, \tag{4.12}$$

$$H(Q_1, Q_2) = -Q_1 - \alpha Q_2A^{1/2}Q_2 - (\varepsilon I + s\alpha A^{1/2})Q_2, \tag{4.13}$$

the new system in the x_1, x_2, z_1, and z_2 variables becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & I & 0 & 0 \\ -A + lA^{1/2}P_1 & lA^{1/2}P_2 & lA^{1/2} & \alpha A^{1/2} \\ \begin{pmatrix} E(P_1, P_2) \\ -\alpha P_2A^{1/2}Q_1 \end{pmatrix} & \begin{pmatrix} F(P_1, P_2) \\ -\alpha P_2A^{1/2}Q_2 \end{pmatrix} & -kI - lP_2A^{1/2} & -\alpha P_2A^{1/2} \\ \begin{pmatrix} G(Q_1, Q_2) \\ -lQ_2A^{1/2}P_1 \end{pmatrix} & \begin{pmatrix} H(Q_1, Q_2) \\ -lQ_2A^{1/2}P_2 \end{pmatrix} & -lQ_2A^{1/2} & \begin{pmatrix} -\alpha Q_2A^{1/2} \\ -\varepsilon I - s\alpha A^{1/2} \end{pmatrix} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{pmatrix}. \tag{4.14}$$

A decoupling of the lateral deflection variables x_1 and x_2 from the variables z_1 and z_2 is effected by setting the operator expressions in the ${}^3_1$, ${}^3_2$, ${}^4_1$, and ${}^4_2$ positions equal to zero. With the resulting equations listed in that order, if the values for P_1 and Q_1 obtained from the second and fourth are substituted into the first and third, respectively, we obtain two operator equations:

$$\begin{aligned} P_2A + r(lP_2A^{1/2} + k)A^{1/2} + (lP_2A^{1/2} + k)^2P_2 \\ + \alpha(lP_2A^{1/2} + kI)P_2A^{1/2}Q_2 \\ + \alpha P_2A^{1/2}(\alpha Q_2A^{1/2}Q_2 + (\varepsilon I + s\alpha A^{1/2})Q_2 + lQ_2A^{1/2}P_2) \\ \equiv B(P_2, Q_2) = 0, \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} Q_2A + sA + (\alpha Q_2A^{1/2} + \varepsilon + s\alpha A^{1/2})^2Q_2 \\ + l(\alpha Q_2A^{1/2} + (\varepsilon I + s\alpha A^{1/2}))Q_2A^{1/2}P_2 \\ + lQ_2A^{1/2}(rA^{1/2} + lP_2A^{1/2}P_2 + kP_2 + \alpha P_2A^{1/2}Q_2) \\ \equiv C(P_2, Q_2) = 0. \end{aligned} \tag{4.16}$$

Once we obtain solutions for these equations, values for P_1 and Q_1 are obtained from the $\frac{3}{2}$ and $\frac{4}{2}$ equations, as indicated previously.

Under our standing assumptions for this paper (restriction to "trigonometric" cases, cf. Sec. 1) all operators shown commute; they are scalar operators on the coefficients of the expansion of the state in the eigenfunctions $w_k(x) = e^{i\nu_k x}$ of A, or on the Fourier transform (series) of $w(x, \nu)$ in the infinite interval and periodic cases, respectively. In that context we can represent P_j and Q_j by scalar functions $p_j(\nu)$ and $q_j(\nu)$, respectively, A by ν^4, and $A^{1/2}$ by ν^2. Making these modifications in (4.15) and (4.16) we arrive at the following algebraic equations for $p_2(\nu)$ and $q_2(\nu)$ (in which we suppress ν as the argument of p_2 and q_2):

$$\begin{aligned} &\nu^4 p_2 + r(l\nu^2 p_2 + k)\nu^2 + (l\nu^2 p_2 + k)^2 p_2 + \alpha(l\nu^2 p_2 + k)\nu^2 p_2 q_2 \\ &\quad + \alpha\nu^2 p_2(\alpha\nu^2 q_2^2 + (\varepsilon I + s\alpha\nu^2)q_2 + l\nu^2 p_2 q_2) \equiv b(p_2, q_2, \nu) = 0, \end{aligned} \tag{4.17}$$

$$\begin{aligned} &\nu^4 q_2 + s\nu^4 + (\alpha\nu^2 q_2 + \varepsilon + s\alpha\nu^2)^2 q_2 + (\alpha\nu^2 q_2 + (\varepsilon + s\alpha\nu^2))l\nu^2 p_2 q_2 \\ &\quad + l\nu^2 q_2(r\nu^2 + l\nu^2 p_2^2 + k p_2 + \alpha\nu^2 p_2 q_2) \equiv c(p_2, q_2, \nu) = 0. \end{aligned} \tag{4.18}$$

Strictly speaking, it is not necessary to show that these equations have the desired real solutions. This already follows from the eigenvalue/eigenvector analysis of the preceding section. The 4×4 matrix $M(\nu)$ obtained from (4.01) by replacing A with ν^4 and $A^{1/2}$ with ν^2 is reduced to upper block triangular form via a similarity transformation involving the 4×4 matrix obtained from (4.09) by replacing P_j by $p_j(\nu)$ and Q_j by $q_j(\nu)$, $j = 1, 2$. Results on matrix quadratic equations proved in [7] and redeveloped in [11] show that if the real and imaginary parts of eigenvectors corresponding to the complex eigenvalues of $M(\nu)$ (a pair of such eigenvalues was shown to exist, under our assumptions, in the preceding section) are used to form the columns of a 4×2 matrix $\binom{X(\nu)}{Y(\nu)}$, then

$$\begin{pmatrix} p_1(\nu) & p_2(\nu) \\ q_1(\nu) & q_2(\nu) \end{pmatrix} = Y(\nu)X(\nu)^{-1}.$$

However, a more precise analysis is required for our purposes. We will prove

PROPOSITION 4.1. Under the assumptions of Proposition 3.1, the equations (4.17), (4.18) have solutions $p_2(\nu)$, $q_2(\nu)$ for all real values of ν which have the asymptotic forms (wherein the repeated factor Δ has the value $\theta K^2/EI + (\rho EI)^{1/2}/2\sigma$)

$$\begin{aligned} p_1(\nu) &= \Delta\mathscr{O}(\nu^4), \qquad \nu \to 0, \\ q_1(\nu) &= (\rho/\tau)\nu^4 + \Delta\mathscr{O}(\nu^6), \qquad \nu \to 0, \\ p_2(\nu) &= -(K\rho^{1/2}/kEI^{1/2})\nu^2 + \Delta\mathscr{O}(\nu^4), \qquad \nu \to 0, \\ q_2(\nu) &= -(2\sigma\rho/\tau^2)\nu^4 + \Delta\mathscr{O}(\nu^6), \qquad \nu \to 0. \end{aligned} \tag{4.19}$$

$$
\begin{aligned}
p_1(\nu) &= -K(\rho/EI)^{1/2}\nu^2 + \mathscr{O}(1), \qquad \nu^2 \to \infty, \\
q_1(\nu) &= \rho\tau/(4\sigma^2 + \tau^2\nu^{-4}) + \mathscr{O}(\Delta), \qquad \nu^2 \to \infty, \\
p_2(\nu) &= (-kK(\rho EI)^{1/2}/(EI + \theta K^2))\nu^{-2} + \Delta\mathscr{O}(\nu^{-4}), \qquad \nu^2 \to \infty, \\
q_2(\nu) &= -\rho(4\sigma^2 + \tau^2\nu^{-4}) + \Delta\mathscr{O}(\nu^{-2}), \qquad \nu^2 \to \infty.
\end{aligned} \tag{4.20}
$$

REMARK. The hypotheses of Proposition 3.1 as regards K and $EI\rho/\sigma^2$ are clearly equivalent to the condition that Δ should be sufficiently small.

Proof. If we set $p_2 = r\tilde{p}_2$, $q_2 = s\tilde{q}_2$, and divide the first equation by r and the second by s, elementary calculations show that, as $\nu \to 0$,

$$k\nu^2 + (\nu^4(1 + lr) + k^2)\tilde{p}_2 + (lr + s\alpha)\mathscr{O}(|\tilde{p}_2|^2 + |\tilde{p}_2\tilde{q}_2|) = 0, \tag{4.21}$$

$$\nu^4 + (\nu^4 + (\varepsilon + s\alpha\nu^2)^2)\tilde{q}_2 + (lr + s\alpha)\mathscr{O}(|\tilde{q}_2|^2 + |\tilde{p}_2\tilde{q}_2|) = 0. \tag{4.22}$$

Comparing with the notation of the preceding section, we see that

$$lr = \theta K^2/EI, \qquad s\alpha = 2(\delta/\alpha) = (\rho EI)^{1/2}/2\sigma,$$

both of which were assumed small in the hypotheses of Proposition 3.1. Since $lr+s\alpha$ appears repeatedly in the formulae to follow, we economize notation by setting

$$\Delta = lr + s\alpha,$$

in agreement with the value assigned to Δ in the statement of the proposition.

The terms indicated by "$\mathscr{O}$" are sufficiently regular, with respect to ν as $\nu \to 0$, so that standard iterative techniques allow us to see that there are unique solutions $\tilde{p}_2$ and $\tilde{q}_2$ with

$$
\begin{aligned}
\tilde{p}_2(\nu) &= -k^{-1}\nu^2 + \Delta\mathscr{O}(\nu^4), \qquad \nu \to 0, \\
\tilde{q}_2(\nu) &= -\frac{\nu^4}{\varepsilon^2} + \Delta\mathscr{O}(\nu^6), \qquad \nu \to 0.
\end{aligned}
$$

From this we have

$$p_2(\nu) = -rk^{-1}\nu^2 + r\Delta\mathscr{O}(\nu^4), \qquad \nu \to 0, \tag{4.23}$$

$$q_2(\nu) = -s\nu^4/\varepsilon^2 + s\Delta\mathscr{O}(\nu^6), \qquad \nu \to 0. \tag{4.24}$$

To carry out the analysis as $\nu^2 \to \infty$ Eqs. (4.17) and (4.18) are divided by ν^4, after which we set $\mu = \nu^{-1}$. Then setting

$$p_2 = r\mu^2\hat{p}_2, \qquad q_2 = s(-1 + \mu^2\hat{q}_2),$$

and dividing both resulting equations by μ^2, we arrive at

$$
\begin{aligned}
&\hat{p}_2 + (lr\hat{p}_2 + k) + \mu^4(lr\hat{p}_2 + k)^2\hat{p}_2 \\
&\quad + s\alpha\hat{p}_2(-1 + \mu^2\hat{q}_2)(2lr\mu^2\hat{p}_2 + s\alpha\mu^2\hat{q}_2 + (k + \varepsilon)\mu^2) = 0,
\end{aligned} \tag{4.25}
$$

$$
\begin{aligned}
&\hat{q}_2 + \mu^2(s\alpha\hat{q}_2 + \varepsilon)^2(-1 + \mu^2\hat{q}_2) + lr\hat{p}_2(-1 + \mu^2\hat{q}_2) \\
&\quad \times (2s\alpha\mu^2\hat{q}_2 + lr\mu^2\hat{p}_2 + (k + \varepsilon)\mu^2 + r - s\alpha) = 0,
\end{aligned} \tag{4.26}
$$

from which we obtain solutions of the form

$$\hat{p}_2(\nu) = -k/(1+lr) + \Delta\mathscr{O}(\mu^2), \qquad \mu^{-2} = \nu^2 \to \infty,$$

$$\hat{q}_2(\nu) = \frac{-klr/(1+lr)(r-s\alpha) + \mu^2\varepsilon^2}{(1 - 2s\alpha\varepsilon\mu^2 + \mu^4\varepsilon^2)} + \Delta\mathscr{O}(\mu^2), \qquad \mu^{-2} = \nu^2 \to \infty.$$

Then

$$\begin{aligned} p_2(\nu) &= r((-k/(1+lr))\mu^2 + \Delta\mathscr{O}(\mu^4)) \\ &= -(kr/(1+lr))\nu^{-2} + \Delta\mathscr{O}(\nu^{-4})), \qquad \mu^{-2} = \nu^2 \to \infty, \end{aligned} \tag{4.27}$$

$$\begin{aligned} q_2(\nu) &= s\left(-1 + \frac{\mu^4\varepsilon^2}{(1+\mu^4\varepsilon^2)} + \Delta\mathscr{O}(\mu^2)\right) \\ &= s\left(\frac{-1}{(1+\nu^{-4}\varepsilon^2)} + \Delta\mathscr{O}(\nu^{-2})\right), \qquad \mu^{-2} = \nu^2 \to \infty. \end{aligned} \tag{4.28}$$

Equations (4.21) and (4.22) also allow us to see that if Δ is sufficiently small then there are unique solutions of the form

$$p_2(\nu) = -r\left(\frac{kEI\nu^2}{(\nu^4(EI+\theta K^2)+k^2)} + \mathscr{O}(\Delta)\right),$$

$$q_2(\nu) = -s\left(\frac{\nu^4}{\nu^4 + (\varepsilon + 2(\delta/\alpha)\nu^2)^2} + \mathscr{O}(\Delta)\right),$$

valid on any compact ν-interval. These solutions match up continuously with the solutions (4.23), (4.24) and (4.27), (4.28) as $\nu \to 0$ and as $\nu \to \infty$.

If we now substitute the asymptotic values for $p_2(\nu)$ and $q_2(\nu)$ shown in (4.23) and (4.24) into the equation for $p_1(\nu)$ resulting from setting the $^3_2$ entry of (4.14) equal to zero we obtain

$$\begin{aligned} p_1(\nu) &= -r\nu^2 - kr(-k^{-1}\nu^2 + \Delta\mathscr{O}(\nu^4)) - l\nu^2 r^2\tilde{p}_2(\nu)^2 - lrs\nu^2\tilde{p}_2(\nu)\tilde{q}_2(\nu) \\ &= r\Delta\mathscr{O}(\nu^4), \qquad \nu \to 0. \end{aligned} \tag{4.29}$$

On the other hand, if the indicated asymptotic values for $p_2(\nu)$ and $q_2(\nu)$ are substituted into the equation for $q_1(\nu)$ resulting from setting the $^4_2$ entry of (4.14) equal to zero we obtain

$$\begin{aligned} q_1(\nu) &= -\varepsilon\nu^2 q_2(\nu)^2 - (\varepsilon + s\alpha\nu^2)q_2(\nu) - \varepsilon\nu^2 p_2(\nu)q_2(\nu) \\ &= s\varepsilon^{-1}\nu^4 + \Delta\mathscr{O}(\nu^6)), \qquad \nu \to 0. \end{aligned} \tag{4.30}$$

The expressions (4.27) and (4.28), substituted into the equation for $p_1(\nu)$, give

$$p_1(\nu) = -r[\nu^2 + \mathscr{O}(1)], \qquad \nu \to \infty, \tag{4.31}$$

and, substituted into the equation for $q_1(\nu)$,

$$q_1(\nu) = \frac{s\varepsilon}{(1+\nu^{-4}\varepsilon^2)} + \mathscr{O}(\Delta), \qquad \nu \to \infty. \tag{4.32}$$

Expressing the coefficients which appear in the equations shown here in terms of the original coefficients, we have the stated results (4.19), (4.20) and the proof is complete.

5. The projected lateral motion equations. From (4.07) and (4.14), the projected equations of mechanical motion, expressed in terms of the spatial frequency variable ν, are

$$\begin{pmatrix} \dot{\xi}_1(t,\nu) \\ \dot{\xi}_2(t,\nu) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \begin{pmatrix} -\nu^4 + l\nu^2 p_1(\nu) \\ +\alpha\nu^2 q_1(\nu) \end{pmatrix} & l\nu^2 p_2(\nu) + \alpha\nu^2 q_2(\nu) \end{pmatrix} \begin{pmatrix} \xi_1(t,\nu) \\ \xi_2(t,\nu) \end{pmatrix}. \quad (5.01)$$

Using the formulae for $p_j(\nu)$ and $q_j(\nu)$ developed in Sec. 4, we are now able to show the asymptotic forms taken by this system as $\nu \to 0$ and as $\nu^2 \to \infty$. All work in this section is carried out under the assumption that Δ (cf. (4.22)ff.) is sufficiently small.

PROPOSITION 5.1. Expressed in terms of the spatial frequency variable ν, the system (5.01) assumes the following asymptotic forms.

As $\nu \to 0$:

$$\begin{pmatrix} \dot{\xi}_1(t,\nu) \\ \dot{\xi}_2(t,\nu) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \begin{pmatrix} -\nu^4 + (\rho EI)^{1/2}\nu^6/\tau \\ +\Delta^2\mathcal{O}(\nu^6) \end{pmatrix} & \begin{pmatrix} -\theta K^2\nu^4/kEI \\ -2\sigma(\rho EI)^{1/2}\nu^6/\tau^2 + \Delta^2\mathcal{O}(\nu^6) \end{pmatrix} \end{pmatrix} \times \begin{pmatrix} \xi_1(t,\nu) \\ \xi_2(t,\nu) \end{pmatrix}. \quad (5.02)$$

As $\nu^2 \to \infty$:

$$\begin{pmatrix} \dot{\xi}_1(t,\nu) \\ \dot{\xi}_2(t,\nu) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \begin{pmatrix} -(1+\theta K^2/EI)\nu^4 \\ +\tau(\rho EI)^{1/2}\nu^2/4\sigma^2 + \Delta\mathcal{O}(1) \end{pmatrix} & \begin{pmatrix} -2\sigma(\rho EI)^{1/2}\nu^2/4\sigma^2 + \tau^2\nu^{-4} \\ -k\theta K^2/(EI+\theta K^2) + \Delta^2\mathcal{O}(1) \end{pmatrix} \end{pmatrix} \times \begin{pmatrix} \xi_1(t,\nu) \\ \xi_2(t,\nu) \end{pmatrix}. \quad (5.03)$$

Proof. Substituting the formulae of Proposition 4.1 into the system (5.01), we see that the asymptotic forms of this system are

$$\begin{pmatrix} \dot{\xi}_1(t,\nu) \\ \dot{\xi}_2(t,\nu) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \begin{pmatrix} -\nu^4 + s\alpha\varepsilon^{-1}\nu^6 \\ +(lr+s\alpha)^2\mathcal{O}(\nu^6) \end{pmatrix} & \begin{pmatrix} -lrk^{-1}\nu^4 - s\alpha\varepsilon^{-2}\nu^6 \\ +(lr+s\alpha)^2\mathcal{O}(\nu^6) \end{pmatrix} \end{pmatrix} \times \begin{pmatrix} \xi_1(t,\nu) \\ \xi_2(t,\nu) \end{pmatrix}, \quad \nu \to 0, \quad (5.04)$$

$$\begin{pmatrix} \dot{\xi}_1(t,\nu) \\ \dot{\xi}_2(t,\nu) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \begin{pmatrix} -(1+lr)\nu^4 + \varepsilon s\alpha\nu^2 \\ +(lr+s\alpha)\mathcal{O}(1) \end{pmatrix} & \begin{pmatrix} -s\alpha\nu^2/(1+\nu^{-4}\varepsilon^2) - lrk/(1+lr) \\ +(lr+s\alpha)^2\mathcal{O}(1) \end{pmatrix} \end{pmatrix} \times \begin{pmatrix} \xi_1(t,\nu) \\ \xi_2(t,\nu) \end{pmatrix}, \quad \nu \to \infty. \quad (5.05)$$

We then use the coefficient relationships introduced at the beginning of Sec. 4 and the definition of Δ in Sec. 4 to obtain the formulae (5.02), (5.03). This completes the proof.

Now we are in a position to demonstrate a degree of consistency with the "spatial hysteresis" model, introduced in Sec. 2, at least in the periodic and infinite interval cases where the meaning and properties of the convolution operator are clear.

THEOREM 5.2. The interaction kernel $h(x-\xi)$ of the constant coefficient spatial hysteresis model on $(-\infty, \infty)$ can be selected so as to agree with the term in the $^2_2$ position of (5.04), (5.05). If the resulting kernel is used in the spatial hysteresis model, the Euler-Bernoulli energy is monotone decreasing.

Proof. The spatial hysteresis model on $(-\infty, \infty)$ is embodied in the integro-partial differential equation

$$\rho\frac{\partial^2 w}{\partial t^2} - 2\gamma\frac{\partial}{\partial x}\int_{-\infty}^{\infty} h(x-\xi)\left(\frac{\partial^2 w}{\partial t\partial x}(x, t) - \frac{\partial^2 w}{\partial t\partial x}(\xi, t)\right) d\xi + EI\frac{\partial^4 w}{\partial x^4} = 0. \quad (5.06)$$

With $\eta = x-\xi$, and taking into account the earlier assumption $\int_{-\infty}^{\infty} h(\eta)\, d\eta = 1$, we have

$$\rho\frac{\partial^2 w}{\partial t^2} - 2\gamma\frac{\partial}{\partial x}\int_{-\infty}^{\infty} h(\eta)\left(\frac{\partial^2 w}{\partial t\partial x}(x, t) - \frac{\partial^2 w}{\partial t\partial x}(x-\eta, t)\right) d\eta + EI\frac{\partial^4 w}{\partial x^4}$$
$$= \rho\frac{\partial^2 w}{\partial t^2} - 2\gamma\frac{\partial^3 w}{\partial t\partial x^2} + 2\gamma\int_{-\infty}^{\infty} h(\eta)\,\frac{\partial^3 w}{\partial t\partial x^2}(x-\eta, t)\, d\eta + EI\frac{\partial^4 w}{\partial x^4} = 0. \quad (5.07)$$

Defining

$$\xi_1(t, \nu) = \hat{w}(t, \nu) = \int_{-\infty}^{\infty} e^{-ia\nu x} w(x, t)\, dx, \qquad \xi_2(t, \nu) = \frac{\partial \hat{w}}{\partial t}(t, \nu),$$

wherein $a = (\rho/EI)^{1/4}$, the transformed equations constitute a ν-parametrized set of ordinary differential equations

$$\frac{d^2\hat{w}}{dt^2} + \frac{2\gamma a^2}{\rho}\nu^2(1-\hat{h}(\nu))\frac{d\hat{w}}{dt} + \nu^4\hat{w} = 0. \quad (5.08)$$

In first-order, two-dimensional form, these become $(\cdot \equiv d/dt)$

$$\begin{pmatrix} \dot{\xi}_1(t, \nu) \\ \dot{\xi}_2(t, \nu) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\nu^4 & -2\gamma a^2\rho^{-1}\nu^2(1-\hat{h}(\nu)) \end{pmatrix} \begin{pmatrix} \xi_1(t, \nu) \\ \xi_2(t, \nu) \end{pmatrix}. \quad (5.09)$$

Comparing with (5.01), the damping terms agree just in case

$$-2\gamma a^2\rho^{-1}(1-\hat{h}(\nu)) = lp_2(\nu) + \alpha q_2(\nu). \quad (5.10)$$

Using (1.11) and (1.20) along with the definition of the constant a and the fact (cf. (4.20)) that $p_2(\nu) = \mathscr{O}(\nu^{-2})$, $\nu^2 \to \infty$, we conclude that (5.10) requires

$$2\gamma = \rho EI/4\sigma^2, \quad (5.11)$$

which establishes a simple relationship between γ of the spatial hysteresis model and the parameter σ of the shear diffusion process. Equation (5.10) is equivalent to

$$\hat{h}(\nu) = 1 + (\theta K/2\gamma)p_2(\nu) + (4\sigma^2/\rho)q_2(\nu). \quad (5.12)$$

Since $p_2(\nu)$ and $q_2(\nu)$ are even, negative for all ν, and of orders ν^2 and ν^4, respectively, as $\nu \to 0$, $\hat{h}(\nu)$ is even and less than 1 for all $\nu \neq 0$, with $\hat{h}(0) = 1$.

Using (5.11) with Proposition 4.1 we see that

$$\hat{h}(\nu) = (4\sigma^2/\rho)p_2(\nu) + \Delta\mathscr{O}(\nu^{-2}) + \mathscr{O}(\nu^{-4}) = \mathscr{O}(\nu^{-2}), \qquad \nu^2 \to \infty. \quad (5.13)$$

From this it follows that $h(\eta)$ and $h'(\eta)$ are square integrable on the interval $-\infty < \eta < \infty$. To show that $h(\eta)$ is integrable on that interval it is enough to prove $\hat{h}(\nu)$ and $\hat{h}'(\nu)$ square integrable on $-\infty < \nu < \infty$. The first is evident from (5.13). From (4.25) and (4.26), along with the immediately preceding material, $\hat{p}_2(\nu)$ and $\hat{q}_2(\nu)$ satisfy equations

$$(1 + lr)\hat{p}_2'(\nu) + \frac{d}{d\nu}(\nu^{-2} f(\hat{p}_2(\nu), \hat{q}_2(\nu), \nu)),$$

$$q_2'(\nu) - lr(r - s\alpha)p_2'(\nu) + \frac{d}{d\nu}(\nu^{-2} g(\hat{p}_2(\nu), q_2'(\nu), \nu)),$$

where f and g have continuous and bounded derivatives with respect to all arguments as $\nu^2 \to \infty$. From this we obtain the boundedness of $p_2'(\nu)$ and $q_2'(\nu)$ as $\nu^2 \to \infty$ and then the square integrability of $\hat{h}'(\nu)$ follows from (5.12) and the formulae

$$p_2'(\nu) = \nu^{-2}\hat{p}_2'(\nu) - 2\nu^{-3}\hat{p}_2(\nu), \qquad q_2'(\nu) = \nu^{-2}\hat{q}_2'(\nu) - 2\nu^{-3}\hat{q}_2(\nu).$$

From the Plancherel theorem the Euler-Bernoulli energy expression for the system (5.07) is a positive multiple of

$$\int_{-\infty}^{\infty} \left| \frac{d}{dt}\hat{w}(\nu, t) \right|^2 + \nu^4 |\hat{w}(\nu, t)|^2 \, d\nu.$$

The standard computation with (5.08) shows that this is strictly decreasing with increasing t if $1 - \hat{h}(\nu) > 0$, $\nu \neq 0$, which is ensured by the result $\hat{h}(\nu) < 1$, $\nu \neq 0$, already obtained. This completes the proof of the theorem.

REMARK. We cannot show $h(\eta) > 0$, $-\infty < \eta < \infty$, at this writing.

6. Concluding remarks; the modified spatial hysteresis model. Comparison of (5.09) with (5.02) and (5.03) reveals that we continue to have some loose ends matching the two systems in the infinite interval case which is the subject of Theorem 5.2. The additional terms added to $-\nu^4$ in (5.02), (5.03) have no counterpart in (5.09). The originally postulated spatial hysteresis model [9] made no provision for modification of the elasticity term $EI\frac{\partial^4 w}{\partial x^4}$, but the thermoelastic/shear diffusion model does result in such modification in addition to introducing the damping terms which we have already studied.

The change from $-\nu^4$ in the lower left-hand corner of the matrix in (5.09) to the asymptotic, as $\nu^2 \to \infty$, term $-(1 + \theta K^2/EI)\nu^4$ in (5.03) may be described as *thermal stiffening*—a familiar phenomenon in thermoelasticity. Its effect is stronger at high temporal (and hence spatial) frequencies because less time is available from one part of a vibration cycle to another in which the temperature differential can be dissipated by lateral conduction. Perhaps more disconcerting are the positive terms $((\rho EI)^{1/2}/\tau)\nu^6$ and $(\tau(\rho EI)^{1/2}/4\sigma^2)\nu^4$ in (5.02) and (5.03), respectively. These terms, which correspond to reduction of the bending stiffness of the beam due to coupling with the shear deformation process, can be explained by noting that for a given value of $\|w\|^2_{L^2[0,L]}$ the minimum value of

$$\tau\|\beta\|^2_{L^2[0,L]} + EI\left\|\left(\frac{\partial\beta}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)\right\|^2_{L^2[0,L]}$$

(cf. (2.14)) is smaller than the minimum value of $\|EI(\frac{\partial^2 w}{\partial x^2})\|^2_{L^2[0,L]}$; the addition degree of freedom represented by the shear displacement allows a given degree of bending to be achieved with a smaller applied force (or resultant restoring force, however one chooses to regard it).

The elasticity modifications just discussed in connection with the thermoelastic/shear diffusion model can be matched, admittedly in a rather ad hoc manner, in the spatial hysteresis framework by a fairly natural modification of the latter. In place of (5.06) we use the integro-partial differential equation

$$\rho\frac{\partial^2 w}{\partial t^2} - 2\gamma\frac{\partial}{\partial x}\int_{-\infty}^{\infty} h(x-\xi)\left(\frac{\partial^2 w}{\partial t\partial x}(x,t) - \frac{\partial^2 w}{\partial t\partial x}(\xi,t)\right)d\xi$$
$$+ (EI + \theta K^2)\frac{\partial^4 w}{\partial x^4} + \delta\frac{\partial}{\partial x}\int_{-\infty}^{\infty} f(x-\xi)\left(\frac{\partial w}{\partial x}(x,t) - \frac{\partial w}{\partial x}(\xi,t)\right)d\xi = 0, \tag{6.01}$$

where f, the *elastic interaction kernel*, has many of the same properties as does the earlier *dissipation interaction kernel* h. We thereby introduce spatial nonlocality into the elastic term as well as the dissipation term. Equation (5.07) is then similarly modified and $-\nu^4$ in the lower left-hand corner of the matrix in (5.09) becomes

$$-\left(1 + \frac{\theta K^2}{EI}\right)\nu^4 + \frac{\delta a^2\nu^2}{\rho}(1 - \hat{f}(\nu)), \tag{6.02}$$

where $\hat{f}$ is the transform of the elastic interaction kernel f in the same sense as $\hat{h}$ is the transform of the dissipation interaction kernel h. Comparing with (5.01), (5.02), (5.03) we must have

$$\begin{aligned} -\frac{\theta K^2}{EI}\nu^4 + \frac{\delta a^2\nu^2}{\rho}(1-\hat{f}(\nu)) &= l\nu^2 p_1(\nu) + \alpha\nu^2 q_1(\nu) \\ &= \begin{cases} \mathscr{O}(\nu^6), & \nu \to 0, \\ -\theta K^2/EI\nu^4 + \tau(\rho EI)^{1/2}/4\sigma^2 + \mathscr{O}(1), & \nu^2 \to \infty. \end{cases} \end{aligned} \tag{6.03}$$

The required integrability of f necessitates

$$\frac{\delta a^2}{\rho} = \frac{\tau(\rho EI)^{1/2}}{4\sigma^2}, \quad \text{i.e., } \delta = \frac{\tau\rho^{3/2}EI^{1/2}}{4\sigma^2 a^2},$$

from which it is then clear that

$$\begin{aligned} \hat{f}(\nu) &= 1 - \frac{\rho}{\delta a^2}\left(\frac{\theta K^2}{EI}\nu^2 + lp_1(\nu) + \alpha q_1(\nu)\right) \\ &= \begin{cases} 1 - 4\theta\sigma^2K^2\nu^2/\tau\rho^{1/2}EI^{3/2} + \mathscr{O}(\nu^4), & \nu \to 0, \\ \mathscr{O}(\nu^{-2}), & \nu^2 \to \infty. \end{cases} \end{aligned}$$

From this formula and the properties of $p_1(\nu)$ and $q_1(\nu)$ as developed in Sec. 4 the square integrability of $f(\eta)$, $f'(\eta)$, $\hat{f}(\nu)$, and $\hat{f}'(\nu)$, and, in particular, the resulting integrability of $f(\eta)$ on $(-\infty, \infty)$ can be obtained by essentially the same arguments as were applied to $h(\eta)$, $h'(\eta)$, $\hat{h}(\nu)$, and $\hat{h}'(\nu)$ in Sec. 5.

From the form (6.03) of the terms added to the transformed elasticity operator, the definition of Δ following (4.22) and the formulae (4.29)–(4.32) for $p_1(\nu)$ and $q_1(\nu)$ it is clear that if Δ is sufficiently small and if the thermoelastic coupling parameter K is also sufficiently small (in fact, this is needed to ensure Δ small; cf. Sec. 4) then the added terms will be dominated by the original $EI\nu^4$ so that (cf. (6.02)) the quantity

$$\left(1 + \frac{\theta K^2}{EI}\right)\nu^4 - \frac{\delta a^2 \nu^2}{\rho}(1 - \hat{f}(\nu))$$

will be positive for all $\nu \neq 0$. Thus the form (cf. (5.14))

$$\int_{-\infty}^{\infty} \left|\frac{d}{dt}\hat{w}(\nu, t)\right|^2 + \left(\left(1 + \frac{\theta K^2}{EI}\right)\nu^4 - \frac{\delta a^2 \nu^2}{\rho}(1 - \hat{f}(\nu))\right) |\hat{w}(\nu, t)|^2 \, d\nu$$

is positively proportional to an energy form comparable to (5.14) and the Euler-Bernoulli energy and represents an "energy" form which is monotone decreasing for the modified thermoelastic/shear diffusion model (6.01). However, we are not able to assign a physical meaning to this form at the present time.

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References

[1] J. D. Achenbach, *Wave Propagation in Elastic Solids*, North-Holland/American Elsevier, Amsterdam-New York-Oxford, 1973, pp. 391 ff.

[2] C. W. Bert, *Material damping: an introductory review of mathematical models, measures and experimental technique*, J. Sound. Vibration **29**, 272–292 (1973)

[3] G. Chen and D. L. Russell, *A mathematical model for linear elastic systems with structural damping*, Quart. Appl. Math. **39**, 433–454 (1982)

[4] S. W. Hansen, *Frequency-proportional damping models for the Euler-Bernoulli beam equation*, Thesis, University of Wisconsin-Madison, December, 1988

[5] F.-L. Huang, *Proof of the holomorphic semigroup property for an Euler-Bernoulli beam structural damping model*, to appear

[6] I. A. Kunin, *Elastic Media with Microstructure* I; *One-dimensional Models*, Springer Series in Solid State Sciences, vol. 26, Springer-Verlag, New York, 1982

[7] J. E. Potter, *Matrix quadratic solutions*, SIAM J. Appl. Math. **14**, 496–501 (1966)

[8] R. C. Rogers and D. L. Russell, to appear

[9] D. L. Russell, *Mathematical models for the elastic beam with frequency-proportional damping*, Frontiers in Applied Mathematics (H. T. Banks, ed.), SIAM, to appear

[10] D. L. Russell, *On the positive square root of the fourth derivative operator*, Quart. Appl. Math. **46**, 751–773 (1988)

[11] D. L. Russell, *Mathematics of Finite-Dimensional Control Systems: Theory and Design*, Lecture Notes in Pure and Appl. Math., vol. 43, Marcel Dekker, New York, 1979 (cf. p. 225, ff.)

[12] D. L. Russell, *A general framework for the study of indirect damping mechanisms in elastic systems*, submitted to J. Math. Anal. Appl.

[13] S. P. Timoshenko, *Vibration Problems in Engineering*, 2nd ed., Van Nostrand, Princeton, New Jersey, 1955