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CREEPING FLOW THROUGH AN ANNULAR STENOSIS IN A PIPE

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Abstract. The creeping flow disturbance of Poiseuille flow due to a disk can be determined by the use of a distribution of "ringlet" force singularities but the method does not readily adapt to the complementary problem involving an annular constriction. Here it is shown that a solvable Fredholm integral equation of the second kind with bounded kernel can be obtained for an Abel transform of the density function. The exponential decay associated with the biorthogonal eigenfunctions ensures that the flow adjusts to the presence of the constriction in at most a pipe length of half a radius on either side. Methods that depend on matching series at the plane of the constriction appear doomed to failure. The physical quantities of interest are the additional pressure drop and the maximum velocity. The lubricating effect of inlets is demonstrated by extending the analysis to a periodic array of constrictions.

1. Introduction. The effect of a constriction or stenosis on axisymmetric Poiseuille flow is of interest over a wide range of Reynolds numbers, from high speed exhaust gases through arterial blood flow to the transport of crude oil. Likely shapes for the stenosis are a differentiable axisymmetric surface with only one local minimum of the pipe radius, a nonzero length of narrower concentric pipe or a thin annular disk whose outer radius equals that of the pipe. The last two of these shapes allow the use of cylindrical polar coordinates but are likely to yield mixed boundary value problems. In the creeping flow approximation, the rapid exponential decay of the pipe eigenfunctions ensures that disturbances to the flow are confined within a pipe length of half a radius both upstream and downstream.

Though presented as a mixed boundary value problem, the solution given by Shail and Norton [1] for the creeping flow disturbance due to a disk is equivalent to that constructed by using "ringlet" force singularities, modified to take account of the pipe wall [2]. In [1], the authors remark that their method cannot handle the annular constriction; this is due to several integrals not having closed form evaluations. Related difficulties arise when the "ringlet" distribution technique is applied to an annular disk [3] and are manifested in the integral equation by the appearance in the kernel of a term that is singular at the inner rim. After transfer to the outer rim, this singularity is seen in Sec. 3 to be eliminated when the outer rim coincides

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with the pipe wall, as is the case for the annular stenosis. After solving the integral equation of the second kind by means of Chebyshev approximation, the additional pressure drop due to the constriction and the maximum flow velocity are computed for radii ratios in the range 0.2 to 0.9. However, an accurate estimate of the regions of separated flow, on either side of the annular constriction, is unavailable because the series expansion is in terms of pipe eigenfunctions with exponential decay away from instead of towards the axisymmetric corner.

The pipe eigenfunctions of the repeated Stokes operator are described in Sec. 5 but do not furnish a method of solution because the linear systems obtained are ill-conditioned. The difficulty is due to the mixed conditions. In using the biorthogonality of the tangential velocity and pressure eigenfunctions, it is necessary to introduce the unknown pressure jump across the constriction. The normal velocity and vorticity are continuous in the hole, by construction, and so an equation for the unknown function is obtained by requiring the total normal velocity at the annular constriction to be zero. The situation is similar when the stenosis is a nonzero length of narrower concentric pipe. In this case the biorthogonality of the normal velocity and vorticity in the narrower pipe must also be used before obtaining the governing equation, as above. Thus it seems imperative, as in this paper, to avoid the use of pipe eigenfunctions in the construction of the velocity field for flows of this type. Such a strategy was employed by Dagan, Weinbaum, and Pfeffer [4] in their infinite series solution for the creeping flow through an orifice of finite length. Their method appears to be amenable to adaptation for the flow through the narrower pipe stenosis. A related two-dimensional problem, creeping flow through a sudden contraction, was solved with a finite-difference scheme by Vrentas and Duda [5] and with eigenfunction expansions, matched to corner solutions for greater accuracy, by Phillips [6]. Both papers show flow patterns with separation well within the region where the eigenfunctions are significant. Phillips arranges to separate the velocities from the stresses in using the biorthogonality properties but does not establish the convergence of his truncation process which involves several inversions of linear systems. The third and fourth derivatives of the stream function, used in the stress matching, have nonintegrable singularities at the reentrant corner.

The pipe eigenfunctions do, however, play a necessary role in extending the analysis to a periodic array of constrictions in order to demonstrate the lubricating advantage of inlets.

2. Basic formulation. The pressure driven flow of incompressible viscous fluid within a circular pipe of unit radius is constricted by a symmetrically located thin rigid annular disk of radii a and 1. Cylindrical polar coordinates (ρ, θ, z) are chosen so that the cylindrical boundary is at $\rho = 1$ $(-\infty < z < \infty, -\pi < \theta \leq \pi)$ and the annular constriction is at $z = 0$ $(a \leq \rho \leq 1, -\pi < \theta \leq \pi)$. The total velocity field consists of a steady parabolic Poiseuille flow with axial velocity V and a disturbance flow $\mathbf{v}$ that vanishes as $|z| \to \infty$ and for which the Reynolds number is assumed to be small enough for application of the creeping flow equations

$$\mu \nabla^2 \mathbf{v} = \operatorname{grad} p, \tag{2.1}$$

$$\operatorname{div} \mathbf{v} = 0, \tag{2.2}$$

where μ is the viscosity and p the dynamic pressure associated with $\mathbf{v}$. On writing the total velocity field as

$$-V(1-\rho^2)\hat{\mathbf{z}} + \mathbf{v}(\rho, z),$$

where $\mathbf{v} = u\hat{\boldsymbol{\rho}} + w\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\rho}}$ and $\hat{\mathbf{z}}$ denote unit vectors in the radial and axial directions, the total pressure field is given by

$$4\mu V z + p(\rho, z).$$

The quantities of interest here are then

$$\Delta P = \lim_{z\to\infty}[p(\rho, z) - p(\rho, -z)] \quad \text{and} \quad V - w(0, 0),$$

which are respectively the additional pressure drop due to the constriction and the maximum velocity in the flow which evidently occurs at the center of the constriction. The plane of symmetry $z = 0$ ensures that u and w are respectively odd and even functions of z and allows the arbitrary constant in p to be chosen so that $p(\rho, -z) = -p(\rho, z)$ for all $z > 0$. The boundary conditions on the radial and axial velocity components are

$$u = 0 = w \quad \text{at } \rho = 1 \ (-\infty < z < \infty), \tag{2.3}$$

$$w = V(1-\rho^2) \quad \text{at } z = 0 \ (a \le \rho \le 1). \tag{2.4}$$

The symmetry ensures the vanishing of u and the continuity of the tangential stresses at $z = 0$ and so it remains to achieve a continuous pressure field by imposing the condition

$$p = 0 \quad \text{at } z = 0 \ (0 \le \rho < a). \tag{2.5}$$

The velocity field can be constructed by use of an axisymmetric distribution of point force singularities over the annular constriction. Davis [2] showed that the application of this method to the complementary problem of a disk in Poiseuille flow is equivalent to the analysis given by Shail and Norton [1]. However, neither calculation can be readily adapted to the annular constriction because several integrals can no longer be evaluated in closed form and, as described by Davis [3], the replacement of the disk by an annular disk adds to the kernel of the integral equation a term which must be singular at either the inner or the outer rim. It is expected that a singularity at $\rho = 1$ is likely to be immaterial because the density function for the "ringlet" force singularities must surely tend to zero at the pipe wall.

The axial velocity due to the point force $8\pi\mu\hat{\mathbf{z}}$ at the origin is $(1 - z\partial/\partial z) \times (\rho^2 + z^2)^{-1/2}$. So the axial velocity that occurs when the same total force is applied uniformly over the ring $z = 0$, $\rho = \rho_0$ is equal to

$$\left(1 - z\frac{\partial}{\partial z}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\phi}{(\rho^2 + \rho_0^2 - 2\rho\rho_0\cos\phi + z^2)^{1/2}}$$
$$= \left(1 - z\frac{\partial}{\partial z}\right)\int_0^\infty e^{-k|z|}J_0(k\rho)J_0(k\rho_0)\,dk.$$

The corresponding radial velocity is, from (2.2), equal to

$$z\int_0^\infty ke^{-k|z|}J_1(k\rho)J_0(k\rho_0)\,dk\,.$$

When solutions of the creeping flow equations (2.1) and (2.2) are added to these velocities and conditions (2.3) applied, it may be shown that the velocity field

$$U(\rho\,,\,z:\rho_0)\hat{\boldsymbol{\rho}}+W(\rho\,,\,z:\rho_0)\hat{\mathbf{z}}$$

due to a "ringlet" force singularity of strength $8\pi\mu\hat{\mathbf{z}}$ placed at $z=0\,,\ \rho=\rho_0$ inside a rigid cylinder of unit radius is given by

$$\begin{aligned}W(\rho\,,\,z\,;\,\rho_0)=\left(1-z\frac{\partial}{\partial z}\right)&\left\{\int_0^\infty e^{-k|z|}J_0(k\rho)J_0(k\rho_0)\,dk\right.\\&\left.-\frac{2}{\pi}\int_0^\infty\frac{K_0}{I_0}I_0(\rho_0\lambda)I_0(\rho\lambda)\cos\lambda z\,d\lambda\right\}\\&-\frac{2}{\pi}\int_0^\infty\left[\rho_0I_1(\rho_0\lambda)-\frac{I_1}{I_0}I_0(\rho_0\lambda)\right]\left[\rho I_1(\rho\lambda)-\frac{I_1}{I_0}I_0(\rho\lambda)\right]\frac{\cos\lambda z\,d\lambda}{I_1^2-I_0I_2}\,,\end{aligned}\tag{2.6}$$

$$\begin{aligned}U(\rho\,,\,z\,;\,\rho_0)=z&\left\{\int_0^\infty ke^{-k|z|}J_1(k\rho)J_0(k\rho_0)\,dk\right.\\&\left.+\frac{2}{\pi}\int_0^\infty\frac{K_0}{I_0}\lambda I_1(\rho\lambda)I_0(\rho_0\lambda)\cos\lambda z\,d\lambda\right\}\\&-\frac{2}{\pi}\int_0^\infty\left[\rho_0I_1(\rho_0\lambda)-\frac{I_1}{I_0}I_0(\rho_0\lambda)\right]\left[\rho I_2(\rho\lambda)-\frac{I_1}{I_0}I_1(\rho\lambda)\right]\frac{\sin\lambda z\,d\lambda}{I_1^2-I_0I_2}\,,\end{aligned}$$

where the modified Bessel functions are evaluated, both here and below, at λ unless otherwise stated. The property $W(1\,,\,z:\rho_0)=0$ is easily evident from (2.6) and similarly $U(1\,,\,z:\rho_0)$ may be reduced to the integral of an exact derivative, i.e.,

$$U(1\,,\,z:\rho_0)=\frac{2}{\pi}\left[\frac{I_0(\rho_0\lambda)}{I_0}\sin\lambda z\right]_0^\infty=0\,.$$

Thus a solution for $\mathbf{v}$ that satisfies (2.1), (2.2), (2.3), and (2.5) may be written in the form

$$\mathbf{v}=V\int_a^1[U(\rho\,,\,z\,;\,\rho_0)\hat{\boldsymbol{\rho}}+W(\rho\,,\,z\,;\,\rho_0)\hat{\mathbf{z}}]\gamma(\rho_0)\,d\rho_0\tag{2.7}$$

with the density function $\gamma(\rho_0)$ to be determined by condition (2.4) which yields

$$\int_a^1W(\rho\,,0;\,\rho_0)\gamma(\rho_0)\,d\rho_0=1-\rho^2\qquad(a\le\rho\le1)\,.\tag{2.8}$$

3. The governing integral equation. In the consideration of various disk problems, Davis [2] found that equations of type (2.8) could be more amenable to the application of suitable Abel type operators than to solution as successive Abel integral equations. Thus, after defining

$$\begin{aligned}\frac{d}{dt}\int_t^1\frac{\rho J_0(k\rho)\,d\rho}{(\rho^2-t^2)^{1/2}}&=-\frac{t}{(1-t^2)^{1/2}}M(k\,,\,t)\,,\\\frac{d}{dt}\int_t^1\frac{\rho^2J_1(k\rho)\,d\rho}{(\rho^2-t^2)^{1/2}}&=-\frac{t}{(1-t^2)^{1/2}}N(k\,,\,t)\,,\end{aligned}\tag{3.1}$$

where

$$M(k, t) = J_0(k) + k \int_t^1 \left(\frac{1 - t^2}{\rho^2 - t^2} \right)^{1/2} J_1(k\rho)\, d\rho,$$
$$N(k, t) = J_1(k) - k \int_t^1 \left(\frac{1 - t^2}{\rho^2 - t^2} \right)^{1/2} \rho J_0(k\rho)\, d\rho = -\frac{\partial M}{\partial k}, \tag{3.2}$$

the application of the operator

$$\frac{d}{dt} \int_t^1 \frac{\rho\, d\rho}{(\rho^2 - t^2)^{1/2}}$$

to the integral equation (2.8) yields, with the kernel given by setting $z = 0$ in (2.6),

$$\frac{t}{(1 - t^2)^{1/2}} \int_a^1 \left\{ \int_0^\infty M(k, t) J_0(\rho_0 k)\, dk - \frac{2}{\pi} \int_0^\infty \frac{K_0}{I_0} I_0(\rho_0 \lambda) M(i\lambda, t)\, d\lambda \right.$$
$$- \frac{2}{\pi} \int_0^\infty \left[\rho_0 I_1(\rho_0 \lambda) - \frac{I_1}{I_0} I_0(\rho_0 \lambda) \right]$$
$$\left. \times \left[i^{-1} N(i\lambda, t) - \frac{I_1}{I_0} M(i\lambda, t) \right] \frac{d\lambda}{I_1^2 - I_0 I_2} \right\}$$
$$\times \gamma(\rho_0)\, d\rho_0 = 2t(1 - t^2)^{1/2} \qquad (a \le t \le 1). \tag{3.3}$$

The reason for choosing the limits $(t, 1)$ in (3.1) becomes apparent after defining

$$H(u) = \int_a^u \frac{\gamma(\rho_0)\, d\rho_0}{(u^2 - \rho_0^2)^{1/2}},$$

i.e.,

$$\gamma(\rho_0) = \frac{2}{\pi} \frac{d}{d\rho_0} \int_a^{\rho_0} \frac{H(u) u}{(\rho_0^2 - u^2)^{1/2}}\, du, \tag{3.4}$$

with limits chosen to avoid singularities at the rim $\rho = a$ in the subsequent integral equation. An integration by parts then shows that

$$\int_a^1 J_0(\rho_0 k) \gamma(\rho_0)\, d\rho_0 = \frac{2}{\pi} \int_a^1 M(k, u) \frac{H(u) u}{(1 - u^2)^{1/2}}\, du,$$

where M is given by (3.2), and hence substitution in (3.3) yields

$$\frac{t}{(1 - t^2)^{1/2}} \int_a^1 \left\{ \frac{2}{\pi} \int_0^\infty M(k, t) M(k, u)\, dk - \frac{4}{\pi^2} \int_0^\infty \frac{K_0}{I_0} M(i\lambda, u) M(i\lambda, t)\, d\lambda \right.$$
$$- \frac{4}{\pi^2} \int_0^\infty \left[i^{-1} N(i\lambda, u) - \frac{I_1}{I_0} M(i\lambda, u) \right]$$
$$\left. \times \left[i^{-1} N(i\lambda, t) - \frac{I_1}{I_0} M(i\lambda, t) \right] \frac{d\lambda}{I_1^2 - I_0 I_2} \right\}$$
$$\times \frac{H(u) u}{(1 - u^2)^{1/2}}\, du = 2t(1 - t^2)^{1/2} \qquad (a \le t \le 1). \tag{3.5}$$

Thus, in transforming from (2.8) to (3.5), one Fredholm integral equation of the first kind with symmetric kernel has been replaced by another. However, the key to further progress is to now obtain an equation of the second kind, a task readily achieved for the disk problems where M is a cosine function. Next it will be shown that the first integral in the kernel of (3.5) has a value that includes a multiple of $\delta(u-t)$.

Consider the triple integral equations

$$\int_0^\infty A(k)J_0(k\rho)\,dk = f(\rho) \qquad (a<\rho<1), \tag{3.6}$$

$$\int_0^\infty kA(k)J_0(k\rho)\,dk = 0 \qquad (0<\rho<a,\ \rho>1). \tag{3.7}$$

With $M(k, t)$ defined by (3.1), Eq. (3.6) shows that

$$\frac{t}{(1-t^2)^{1/2}}\int_0^\infty M(k, t)A(k)\,dk = \frac{d}{dt}\int_t^1 \frac{f(\rho)\rho}{(\rho^2-t^2)^{1/2}}\,d\rho \qquad (a<t<1). \tag{3.8}$$

But, on writing

$$\int_0^\infty kA(k)J_0(k\rho)\,dk = X(\rho) \qquad (a<\rho<1),$$

(3.7) completes a Hankel transform which, on inversion, gives

$$A(k) = \int_a^1 X(\rho)J_0(k\rho)\rho\,d\rho. \tag{3.9}$$

Then substitution in (3.6) and a standard calculation described by Sneddon [7] yields the Abel integral equation

$$\int_\rho^1 \frac{S(t)\,dt}{(t^2-\rho^2)^{1/2}} = f(\rho) - \frac{2}{\pi}\int_1^\infty \frac{dt}{(t^2-\rho^2)^{1/2}}\int_a^1 \frac{X(u)u}{(t^2-u^2)^{1/2}}\,du$$

whose solution yields the integral equation

$$\begin{aligned} S(t) = &-\frac{d}{dt}\int_t^1 \frac{\rho f(\rho)\,d\rho}{(\rho^2-t^2)^{1/2}} \\ &- \frac{4}{\pi^2}\frac{t}{(1-t^2)^{1/2}}\int_a^1\int_1^\infty \frac{(v^2-1)\,dv}{(v^2-t^2)(v^2-u^2)}\frac{uS(u)}{(1-u^2)^{1/2}}\,du \qquad (a<t<1) \end{aligned} \tag{3.10}$$

for the function $S(t)$ defined for $a \le t \le 1$ by

$$S(t) = \int_a^t \frac{X(u)u}{(t^2-u^2)^{1/2}}\,du, \quad \text{i.e., } X(\rho) = \frac{2}{\pi\rho}\frac{d}{d\rho}\int_a^\rho \frac{S(t)t}{(\rho^2-t^2)^{1/2}}\,dt.$$

But substitution for X in (3.9) yields

$$A(k) = \frac{2}{\pi}\int_a^1 M(k, u)\frac{S(u)u}{(1-u^2)^{1/2}}\,du,$$

and then comparison of (3.8) and (3.10) shows that

$$\frac{2}{\pi}\int_0^\infty M(k,t)M(k,u)\,dk = \frac{(1-t^2)^{1/2}(1-u^2)}{tu}\delta(t-u) + \frac{4}{\pi^2}K^*(t,u) \qquad (0<t,u<1), \tag{3.11}$$

where

$$\begin{aligned} K^*(t,u) &= \int_1^\infty \frac{(v^2-1)\,dv}{(v^2-t^2)(v^2-u^2)} \\ &= \frac{1}{t^2-u^2}\left[\frac{1-u^2}{2u}\ln\left(\frac{1+u}{1-u}\right) - \frac{1-t^2}{2t}\ln\left(\frac{1+t}{1-t}\right)\right] \qquad (0<t\neq u<1), \\ K^*(u,u) &= \frac{1}{2u^2}\left[\frac{u^2+1}{2u}\ln\left(\frac{1+u}{1-u}\right)-1\right] \qquad (0<u<1). \end{aligned} \tag{3.12}$$

It may be shown directly that

$$\int_0^\infty M(k,t)J_0(k)\,dk = \frac{2}{\pi}K^*(t,1) \qquad (0<t<1), \tag{3.13}$$

i.e., (3.11) is valid if either $t=1$ or $u=1$.

The substitution of (3.11) into (3.5) now yields the integral equation

$$H(t) + \frac{4}{\pi^2}\frac{t}{(1-t^2)^{1/2}}\int_a^1 \mathscr{K}(t,u)H(u)\frac{u\,du}{(1-u^2)^{1/2}} = 2t(1-t^2)^{1/2} \qquad (a\le t<1), \tag{3.14}$$

where

$$\begin{aligned} \mathscr{K}(t,u) = K^*(t,u) &- \int_0^\infty \frac{K_0}{I_0}M(i\lambda,t)M(i\lambda,u)\,d\lambda \\ &- \int_0^\infty \left[i^{-1}N(i\lambda,t)-\frac{I_1}{I_0}M(i\lambda,t)\right]\left[i^{-1}N(i\lambda,u)-\frac{I_1}{I_0}M(i\lambda,u)\right]\frac{d\lambda}{I_1^2-I_0I_2}. \end{aligned} \tag{3.15}$$

K^* gives the kernel appropriate to the annular disk in infinite fluid; the additional terms, due to the pipe wall, are such as to ensure, as expected, bounded behavior as $t\to 1$. This is because, by successive use of (3.2), (3.1), and (3.13),

$$\begin{aligned} \mathscr{K}(t,1) &= K^*(t,1) - \int_0^\infty K_0M(i\lambda,t)\,d\lambda \\ &= K^*(t,1) - \frac{\pi}{2}\int_0^\infty J_0(k)M(k,t)\,dk = 0 \end{aligned}$$

and moreover $\mathscr{K}(t,u) = O[(1-u)\ln(1-u)]$ as $u\to 1$ with $t\neq 1$. Hence the symmetric kernel in (3.14) is such that

$$\frac{t}{(1-t^2)^{1/2}}\mathscr{K}(t,u)\frac{u}{(1-u^2)^{1/2}} \to 0 \quad \text{as either } t \text{ or } u \to 1$$

and so $H(t) \to 0$ as $t \to 1$. Thus the integral equation (3.14) has a bounded kernel for the range of interest $a \le t < 1$.

The numerical solution of (3.14) has been achieved by using El-Gendi's method [8] based on the Clenshaw-Curtis quadrature scheme to express the integral of a typical smooth function $g(u)$, defined in $(0, 1)$ in terms of the set $\{g_j\,;\, j = 0, 1, \ldots, 2N\}$ of approximate values of $g(u)$ at $u = \frac{1}{2}[1 - \cos(j\pi/2N)]$. Thus

$$\int_0^1 g(u)\,du \simeq \sum_{j=0}^{N}{}'' b_j(g_j + g_{2N-j}),$$

where

$$b_j = \frac{1}{N}\sum_{r=0}^{N}{}'' \frac{1}{1-4r^2} T_{2r}[\cos(j\pi/2N)] \qquad (0 \le j \le N),$$

T_m denotes the mth Chebyshev polynomial, and the attachment of double primes to a summation symbol indicates that the first and last terms are to be halved. In this way (3.14) is converted, since $H(1) = 0$, to a set of $2N$ simultaneous equations for which an IMSL routine is available. $N = 3$ was found to be sufficient throughout but accuracy with the λ-integrals was more difficult to achieve.

4. The pressure drop and maximum velocity. From (2.1) and (2.6), the pressure field $P(\rho, z : \rho_0)$ due to the "ringlet" singularity is given by

$$\begin{aligned}
&\mu^{-1}P(\rho, z : \rho_0) \\
&\quad = -2\frac{\partial}{\partial z}\left\{\int_0^\infty e^{-k|z|}J_0(k\rho)J_0(k\rho_0)\,dk - \frac{2}{\pi}\int_0^\infty \frac{K_0}{I_0}I_0(\rho_0\lambda)I_0(\rho\lambda)\cos\lambda z\,d\lambda\right\} \\
&\quad - \frac{4}{\pi}\int_0^\infty \left[\rho_0 I_1(\rho_0\lambda) - \frac{I_1}{I_0}I_0(\rho_0\lambda)\right] I_0(\rho\lambda)\frac{\sin\lambda z\,d\lambda}{I_1^2 - I_0 I_2}.
\end{aligned}$$

In particular, by Dirichlet's lemma,

$$\begin{aligned}
P(\rho, \pm\infty; \rho_0) &= \mp 2\mu \lim_{\lambda\to 0}\left\{\left[\lambda\frac{K_0}{I_0}I_0(\rho_0\lambda) + \frac{\rho_0 I_1(\rho_0\lambda) - \frac{I_1}{I_0}I_0(\rho_0\lambda)}{I_1^2 - I_0 I_2}\right]\lambda I_0(\rho\lambda)\right\} \\
&= \pm 8\mu(1 - \rho_0^2).
\end{aligned}$$

Consequently the additional pressure drop ΔP due to the constriction is given, from (2.7), by

$$\Delta P = 16\mu V\int_a^1 (1 - \rho_0^2)\gamma(\rho_0)\,d\rho_0 = \frac{64}{\pi}\mu V\int_a^1 H(t)t(1-t^2)^{1/2}\,dt \tag{4.1}$$

after substitution of (3.4).

The maximum velocity can be similarly shown, after substitution of (2.6) and use of (3.2), to be given by

$$\begin{aligned} v_{\max} &= V - V\int_a^1 W(0,0;\rho_0)\gamma(\rho_0)\,d\rho_0 \\ &= V + \frac{2}{\pi}V\int_a^1 \frac{H(t)t}{(1-t^2)^{1/2}}\left\{\frac{2}{\pi}\int_0^\infty \frac{K_0}{I_0}[M(i\lambda, t) - I_0]\,d\lambda - \frac{1-t^2}{t^2}\right. \\ &\qquad \left. -\frac{2}{\pi}\int_0^\infty \left[i^{-1}N(i\lambda, t) - \frac{I_1}{I_0}M(i\lambda, t)\right]\frac{I_1}{I_0}\frac{d\lambda}{I_1^2 - I_0 I_2}\right\}dt \end{aligned} \tag{4.2}$$

in which the expression in curly brackets has the factor $(1-t^2)$. Table 1 displays, for various a, values of the dimensionless pressure drop and maximum velocity, given by (4.1) and (4.2) respectively after solving the integral equation (3.14) for $H(t)$.

TABLE 1.
Values of the dimensionless pressure drop and maximum velocity, computed from the numerical solution of the governing integral equation.

a	$\Delta P/32\mu V$	$v_{\max}/V$
0.9	0.0185	1.067
0.8	0.0777	1.251
0.7	0.198	1.568
0.6	0.427	2.079
0.5	0.884	2.921
0.45	1.279	3.546
0.4	1.876	4.381
0.35	2.806	5.508
0.3	4.303	7.031
0.25	6.75	9.022
0.2	10.69	11.31

5. Pipe eigenfunctions and separation. An alternative form for the disturbance velocity $\mathbf{v}$ given by (2.7) is

$$\mathbf{v} = V\,\mathrm{curl}\left\{\rho^{-1}\hat{\boldsymbol{\theta}}\sum_{n\neq 0} c_n\lambda_n^{-1}\psi_n^{(1)}(\rho)e^{-\lambda_n|z|}\right\}, \tag{5.1}$$

where the pipe eigenfunctions are defined by

$$\psi_n^{(1)}(\rho) = \lambda_n \left[\frac{\rho^2 J_2(\lambda_n \rho)}{J_1(\lambda_n)} - \frac{\rho J_1(\lambda_n \rho)}{J_0(\lambda_n)} \right] \tag{5.2}$$

and vanish with their first derivatives at $\rho = 1$ because $J_1^2 = J_0 J_2$ determines the complex eigenvalues $\{\lambda_n, \lambda_{-n} = \bar{\lambda}_n; n \geq 1\}$ in the first and fourth quadrants, arranged in order of ascending real part. Thus $\lambda_1 = 4.463 + 1.468i$, $\lambda_2 = 7.693 + 1.727i$, etc. (Dorrepaal et al. [9]). The no-slip conditions at the pipe wall are satisfied by (5.1) and the complex conjugate coefficients $\{c_n, c_{-n} = \bar{c}_n; n \geq 1\}$ are determined by the flow requirements at $z = 0$, namely, the radial velocity is everywhere zero, the pressure is continuous in the hole $\rho < a$, and the total normal velocity must vanish at the constriction $a \leq \rho \leq 1$. Thus

$$\sum_{n \neq 0} c_n \psi_n^{(1)}(\rho) = 0, \qquad 0 \leq \rho \leq 1, \tag{5.3}$$

$$\sum_{n \neq 0} c_n \Psi_n^{(1)}(\rho) = 1 - \rho^2, \qquad a \leq \rho \leq 1, \tag{5.4}$$

$$\sum_{n \neq 0} c_n \lambda_n \Psi_n^{(2)}(\rho) = \frac{\Delta P}{2\mu V}, \qquad 0 \leq \rho < a, \tag{5.5}$$

where

$$\Psi_n^{(1)}(\rho) = (\rho\lambda_n)^{-1} \frac{d\psi_n^{(1)}}{d\rho} = \lambda_n \left[\frac{\rho J_1(\lambda_n \rho)}{J_1(\lambda_n)} - \frac{J_0(\lambda_n \rho)}{J_0(\lambda_n)} \right], \tag{5.6}$$

$$\Psi_n^{(2)}(\rho) = \lambda_n^{-2} \rho^{-1} \frac{d}{d\rho} \left[\rho \frac{d\Psi_n^{(1)}}{d\rho} \right] + \Psi_n^{(1)} = 2 \frac{J_0(\lambda_n \rho)}{J_1(\lambda_n)},$$

and, as in (4.1), ΔP is the additional pressure drop due to the constriction. The biorthogonality property

$$\int_0^1 [\Psi_m^{(2)} \Psi_n^{(1)} + \Psi_m^{(1)} \Psi_n^{(2)}] \rho \, d\rho = -2\delta_{mn},$$

which can be readily established from the differential equations (see, for example, Yoo and Joseph [10]), can now be used by considering the ρ-derivative of (5.3) and an extended form of (5.5). Thus, from

$$\sum_{n \neq 0} c_n \lambda_n \Psi_n^{(1)}(\rho) = 0 \qquad (0 \leq \rho \leq 1),$$

$$\sum_{n \neq 0} c_n \lambda_n \Psi_n^{(2)}(\rho) = \frac{\Delta P}{2\mu V} - \begin{cases} 0 & (0 \leq \rho < a), \\ 2\gamma(\rho)/\rho & (\rho > a), \end{cases}$$

it follows that

$$-2\lambda_m c_m = \frac{\Delta P}{2\mu V} \int_0^1 \Psi_m^{(1)}(\rho) \rho \, d\rho - 2 \int_a^1 \Psi_m^{(1)}(\rho) \gamma(\rho) \, d\rho.$$

Here the first integral vanishes by virtue of (5.6) and the function $\gamma(\rho)$ has been chosen to have the same meaning as in Sec. 2, namely, the density of ringlet singularities on the axisymmetric constriction. Substitution for $\{c_n\}$ in the remaining velocity condition (5.4) then yields

$$\int_a^1 \sum_{n\neq 0} \lambda_n^{-1}\Psi_n^{(1)}(\rho)\Psi_n^{(1)}(\rho_0)\gamma(\rho_0)\,d\rho_0 = 1-\rho^2 \qquad (a\le\rho\le 1). \tag{5.7}$$

Thus the kernel $W(\rho\,,0:\rho_0)$ in (2.8) has symmetric expansion

$$W(\rho\,,0:\rho_0) = \sum_{n\neq 0} \lambda_n^{-1}\Psi_n^{(1)}(\rho)\Psi_n^{(1)}(\rho_0) \tag{5.8}$$

in terms of nonorthogonal functions given by (5.6). Attempts to solve a truncated version of (5.7) for the coefficients $\{c_n\}$ have been thwarted by ill-conditioned equations. An integral equation similar to (5.7) was established by Ross [11] for both the annular stenosis and the corresponding constriction in parallel plate flow.

Separation of the flow at the cylinder wall $\rho = 1$ will occur wherever there is a change of sign of the tangential stress associated with the total flow $\mathbf{v} - V(1-\rho^2)\hat{\mathbf{z}}$. From (2.7) the equation for separation is therefore

$$2 + \int_a^1 \frac{\partial W}{\partial\rho}(1\,,z\,;\rho_0)\gamma(\rho_0)\,d\rho_0 = 0. \tag{5.9}$$

Now, by use of the identity

$$\int_0^\infty e^{-k|z|}J_0(k\rho)J_0(k\rho_0)\,dk = \frac{2}{\pi}\int_0^\infty K_0(\rho\lambda)I_0(\rho_0\lambda)\cos\lambda z\,d\lambda \qquad (\rho>\rho_0),$$

the ρ-derivative at $\rho = 1$ of the expression in curly brackets in (2.6) is equal to

$$-\frac{2}{\pi}\int_0^\infty \frac{I_0(\rho_0\lambda)}{I_0}\cos\lambda z\,d\lambda.$$

Also, an integration by parts shows that

$$z\frac{\partial}{\partial z}\int_0^\infty \frac{I_0(\rho_0\lambda)}{I_0}\cos\lambda z\,d\lambda$$
$$= -\int_0^\infty \left\{\frac{I_0(\rho_0\lambda)}{I_0} + \frac{\rho_0\lambda}{I_0}I_1(\rho_0\lambda) - \frac{\lambda I_1}{I_0^2}I_0(\rho_0\lambda)\right\}\cos\lambda z\,d\lambda$$

and hence the ρ-derivative of (2.6) at $\rho = 1$ may be written

$$\frac{\partial W}{\partial\rho}(1\,,z:\rho_0) = -\frac{4}{\pi}\int_0^\infty \frac{\rho_0 I_1(\rho_0\lambda)I_1 - I_0(\rho_0\lambda)I_2}{I_1^2 - I_0I_2}\cos\lambda z\,d\lambda.$$

Substitution into (5.9) then yields

$$\frac{2}{\pi}\int_0^\infty\int_a^1 [\rho_0I_1(\rho_0\lambda)I_1 - I_0(\rho_0\lambda)I_2]\gamma(\rho_0)\,d\rho_0\frac{\cos\lambda z}{I_1^2 - I_0I_2}\,d\lambda = 1, \tag{5.10}$$

which, by contour integration, can be written

$$2\,\mathrm{Re}\sum_{n=1}^\infty \lambda_n e^{-\lambda_n|z|}\int_a^1\left[\frac{\rho_0J_1(\rho_0\lambda_n)}{J_1(\lambda_n)} - \frac{J_0(\rho_0\lambda_n)}{J_0(\lambda_n)}\right]\gamma(\rho_0)\,d\rho_0 = 1. \tag{5.11}$$

This is identical to the separation condition

$$\sum_{n\neq 0} c_n\lambda_n e^{-\lambda_n|z|} = 1$$

which may be derived directly from (5.1).

Evidently the theory of Moffatt vortices implies that the flow pattern must have a nested sequence of toroidal vortices in the cylindrical corner $\rho = 1$, $z = 0$. However, the structure of $\mathbf{v}$ in (5.1) shows that the disturbance of the Poiseuille flow due to the constriction is essentially confined to $|z| < \frac{1}{2}$. Hence the outer vortex on either side of the constriction extends less than a half radius along the pipe, as in the computed streamlines displayed by Vrentas and Duda [5] and Phillips [6] for two-dimensional flow through a contraction. So (5.11) does not furnish an efficient means for determining the position of even the largest vortex because the terms have rapid exponential decay away from instead of towards the corner. Further, because the velocities are exponentially small in the vortices, there is insufficient accuracy available to solve the separation condition in the form

$$\frac{4}{\pi^2}\int_0^\infty \int_a^1 \frac{H(t)t}{(1-t^2)^{1/2}}[i^{-1}N(i\lambda, t)I_1 - M(i\lambda, t)I_2]\,dt\frac{\cos\lambda z}{I_1^2 - I_0 I_2}\,d\lambda = 1,$$

obtained from (5.10) by use of (3.2) and (3.4) as in (4.2).

6. Periodic array of annular constrictions. Suppose that, in addition to the annular constriction already considered, there are similar stenoses at $z = \pm 2mL$ $(m \geq 1)$, $a \leq \rho \leq 1$. Then the flow is periodic, period $2L$, and symmetric about planes $z = kL$ for all integers k. By introducing additional "ringlet" force singularities of strength $8\pi\mu\hat{z}$ at $z = \pm 2mL$ $(m \geq 1)$, $\rho = \rho_0$, the appropriate modification of the integral equation (2.8) is found to be

$$\int_a^1 W(\rho, 0; \rho_0)\gamma(\rho_0)\,d\rho_0 + \sum_{m=1}^{\infty}\int_a^1 [W(\rho, 2mL; \rho_0) + W(\rho, -2mL; \rho_0)]\gamma(\rho_0)d\rho_0$$
$$= 1 - \rho^2 \qquad (a \leq \rho \leq 1). \tag{6.1}$$

But the eigenfunction expansion (5.8) implies that

$$W(\rho, z; \rho_0) = \sum_{n\neq 0}\lambda_n^{-1}\Psi_n^{(1)}(\rho)\Psi_n^{(1)}(\rho_0)e^{-\lambda_n|z|} \tag{6.2}$$

and hence the additional kernel in (6.1) has the expansion

$$\sum_{m=1}[W(\rho, 2mL; \rho_0) + W(\rho, -2mL; \rho_0)] = \sum_{n\neq 0}\frac{e^{-\lambda_n L}}{\lambda_n \sinh\lambda_n L}\Psi_n^{(1)}(\rho)\Psi_n^{(1)}(\rho_0)$$

in terms of nonorthogonal functions given by (5.6). Now (3.1) shows that

$$\frac{d}{dt}\int_t^1 \frac{\rho\Psi_n^{(1)}(\rho)}{(\rho^2 - t^2)^{1/2}}\,d\rho = -\frac{\lambda_n t}{(1-t^2)^{1/2}}\left[\frac{N(\lambda_n, t)}{J_1(\lambda_n)} - \frac{M(\lambda_n, t)}{J_0(\lambda_n)}\right]$$

and then substitution of (3.4) yields

$$\int_a^1 \Psi_n^{(1)}(\rho_0)\gamma(\rho_0)\,d\rho_0 = \frac{2\lambda_n}{\pi}\int_a^1 \left[\frac{N(\lambda_n, u)}{J_1(\lambda_n)} - \frac{M(\lambda_n, u)}{J_0(\lambda_n)}\right]\frac{H(u)u}{(1-u^2)^{1/2}}\,du.$$

Thus the modified form of the integral equation (3.14) for $H(t)$ is

$$H(t) + \frac{4}{\pi^2}\frac{t}{(1-t^2)^{1/2}}\int_a^1 \widehat{\mathscr{K}}(t, u)H(u)\frac{u\,du}{(1-u^2)^{1/2}} = 2t(1-t^2)^{1/2} \qquad (a \le t \le 1), \tag{6.3}$$

where

$$\widehat{\mathscr{K}}(t, u) = \mathscr{K}(t, u) - \frac{\pi}{2}\sum_{n\neq 0}\frac{\lambda_n e^{-\lambda_n L}}{\sinh\lambda_n L}\left[\frac{N(\lambda_n, t)}{J_1(\lambda_n)} - \frac{M(\lambda_n, t)}{J_0(\lambda_n)}\right]\left[\frac{N(\lambda_n, u)}{J_1(\lambda_n)} - \frac{M(\lambda_n, u)}{J_0(\lambda_n)}\right]$$

and $\mathscr{K}$ is defined by (3.15). Similarly, the additional contribution to the maximum velocity $v_{\max}$, given by (4.2), can be shown to be

$$\frac{2}{\pi}V\int_a^1 \frac{H(t)t}{(1-t^2)^{1/2}}\sum_{n\neq 0}\frac{\lambda_n e^{-\lambda_n L}}{\sinh\lambda_n L}\cdot\frac{1}{J_0(\lambda_n)}\left[\frac{N(\lambda_n, t)}{J_1(\lambda_n)} - \frac{M(\lambda_n, t)}{J_0(\lambda_n)}\right]dt.$$

Since the pressure drop along a pipe length $2L$ (one period) due to a periodic array of ringlets is equal to the pressure drop over the whole pipe due to a single ringlet, it follows that the mean total pressure gradient in the periodic flow is $4\mu V + \Delta P/2L$, where ΔP is given by (4.1). Hence the pressure gradient to flux ratio is $(8\mu + \Delta P/LV)/\pi$, which may be compared with the corresponding ratio $8\mu/\pi a^4$ for Poiseuille flow in a pipe of radius a. The fluid in the "inlets" between the constrictions therefore achieves a lubricating advantage over the uniform pipe whenever a flux gain occurs for given pressure gradient, i.e.,

$$1 + \left(\frac{\Delta P}{8\mu V}\right)\frac{1}{L} < a^{-4}. \tag{6.4}$$

On noting that the lowest eigenvalue λ_1 has real part 4.463, it is seen that the kernel $\widehat{\mathscr{K}}$ in (6.3) differs from the kernel $\mathscr{K}$ in (3.14) by terms that decay rapidly with L, in accordance with the earlier comment that the disturbance of the Poiseuille flow due to a constriction is essentially confined within half the pipe radius on either side. Thus, for $L > \frac{1}{2}$, the computed values of $\Delta P/32\mu V$, displayed in Table 1, provide good estimates for insertion in the inequality (6.4), as then the flow through each stenosis is essentially unaffected by the presence of the other constrictions. It is found that (6.4) is easily satisfied, showing that a lubricating advantage always exists for $L > \frac{1}{2}$. It may be anticipated that parity is monotonically achieved as $L \to 0$.

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