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A NEW METHOD FOR SOLVING DYNAMICALLY ACCELERATING CRACK PROBLEMS: PART 1. THE CASE OF A SEMI-INFINITE MODE III CRACK IN ELASTIC MATERIAL REVISITED

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Abstract. Presented here is a new method for constructing solutions to dynamically accelerating, semi-infinite crack problems. The problem of a semi-infinite, anti-plane shear (mode III) crack accelerating dynamically in an infinite, linear, homogeneous, and isotropic elastic body has been solved previously by Freund and Kostrov. However, their methods are based upon the construction of a certain Green's function for the ordinary two-dimensional wave equation and do not generalize to either the opening mode problem in elastic material or viscoelastic material. What is presented here is a new approach based upon integral transforms and complex variable techniques that does, in principle, generalize to both the opening modes of deformation and viscoelastic material. Moreover, the method presented here produces directly, for the mode III crack, a simple closed form expression for the crack-face profile for arbitrary applied crack-face tractions. Generalizations to opening modes of deformation and viscoelastic material produce integral equations for the crack-face displacement profile that in some cases admit closed form solutions and otherwise can be solved numerically. In contrast, the method of Freund and Kostrov yields, in mode III, an expression for the stress in front of the crack, but for opening modes provides only the stress intensity factor.

1. Introduction. Dynamic crack growth has been an important topic in fracture mechanics for more than thirty years. Many researchers have made tremendous efforts to construct analytical solutions to a variety of canonical boundary value problems serving as idealized mathematical models of dynamically propagating cracks. The excellent book by Freund [2] gives an up-to-date account of the development of the subject. One of the milestones in this effort was the solution presented by Kostrov [4] for a dynamically accelerating, semi-infinite, anti-plane shear crack in an infinite, homogeneous, isotopic elastic body. Kostrov's method depended upon the construction of a Green's function for the wave equation in a half-plane satisfying certain mixed boundary conditions. As shown by Freund [2], one can in a rather straightfor-

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ward fashion apply the Green's function method to construct both the stress ahead of the moving crack and the crack-face displacement. Freund first considers the case of a point load applied to the crack-faces and then shows how the solution for more general loadings follows by a superposition argument.

For all intents and purposes, this appears to solve the problem completely and further consideration would hardly seem warranted. However, the Green's function method has somewhat limited applicability. In particular, it does not seem to generalize to viscoelastic material. With the exception of one approximate analysis by Goleniewski [3], there exist no solutions for dynamically accelerating cracks in viscoelastic material. The purpose of this paper is to present a new derivation of the solution to the elastic problem described in the first paragraph. The method utilizes integral transform and complex variable techniques that are directly generalizable to viscoelastic material and to opening modes of deformation. It is interesting to note that it has been widely believed that these techniques could not be applied to accelerating crack problems since when formulated in terms of a coordinate system moving with the crack-tip, the governing wave equations have time-dependent coefficients whereas when formulated in a fixed coordinate system in which the crack-tip is moving, one has incomplete time histories for the boundary values at all points in front of the initial crack-tip position (c.f. [2], pg. 368]). It is shown here how these apparent impediments can be surmounted. In contrast to Kostrov's method for elastic material, the method presented here produces directly an elegant closed form expression for the crack-face displacement for general crack-face loading. Point loads then become a trivial special case. The generalizations to viscoelastic material and opening modes of deformation will be addressed in future papers.

After submitting this paper for publication, the authors became aware of two other alternative approaches to deriving elastodynamic crack solutions due to Burridge [1] and Willis [5]. Through clever utilization of results concerning homogeneous solutions for the elastic wave equations, Burridge treats the corresponding opening mode problem while Willis considers both the opening mode and anti-plane shear problems. However, their methods, like those of Freund and Kostrov, do not seem to generalize to the viscoelastic wave equations. It is interesting to note that both here and in Burridge's paper, which present very different approaches to solving accelerating crack problems, certain similar fractional integral and differential operators play crucial roles in the analysis.

2. Problem formulation and reduction to a Riemann-Hilbert problem. The problem to be considered is that of an infinite, linear, homogeneous, isotropic, elastic body containing a semi-infinite mode III crack, initially lying along $x_1 < 0$, $x_2 = 0$, that begins to propagate to the right at $t = 0$ due to the application of time-dependent crack-face tractions. For $t > 0$, the crack-tip is located at the point $a(t)$ where $a(t)$ is a nonnegative, nondecreasing function of time subject to the restriction $\dot{a}(t) < c$ where c denotes the shear wave speed of the material. Due to anti-plane symmetry one need only consider a single wave equation for the out-of-plane displacement,

$u_3(x_1, x_2, t)$ in the half-space $x_2 > 0$,

$$\ddot{u}_3 = c^2 \Delta u_3, \tag{2.1}$$

where Δ denotes the two-dimensional Laplacian. The revelant constitutive equation is

$$\sigma_{23}(x_1, x_2, t) = \mu \frac{\partial u_3}{\partial x_2} \tag{2.2}$$

with μ being the shear modulus.

The relevant initial-boundary value problem for the wave equations (2.1) that must be solved has initial conditions

$$u_3(x_1, x_2, 0) = \dot{u}_3(x_1, x_2, 0) = 0$$

and boundary conditions

$$\begin{gathered} u_3(x_1, 0, t) = 0 \quad \text{for } x_1 > a(t), \\ \sigma_{23}(x_1, 0, t) = \sigma_e(x_1, t) \quad \text{for } x_1 < a(t), \\ u_3(x_1, x_2, t) \to 0 \quad \text{as } x_2 \to \infty. \end{gathered} \tag{2.3}$$

For much of the ensuing analysis it will prove convenient to switch from the reference coordinate system (x_1, x_2) to a coordinate system (x, y) moving with the crack-tip. To this end, the following notation is adopted:

$$\begin{gathered} x = x_1 - a(t), \qquad y = x_2, \\ w(x, y, t) = w(x_1 - a(t), x_2, t) = u_3(x_1, x_2, t), \\ \sigma(x, y, t) = \sigma(x_1 - a(t), x_2, t) = \sigma_{23}(x_1, x_2, t). \end{gathered} \tag{2.4}$$

The initial-boundary value problem (2.1)–(2.3) will be solved by applying the Fourier transform in x_1 defined by

$$\hat{f}(p, x_2, t) = \int_{-\infty}^{\infty} e^{ipx_1} f(x_1, x_2, t)\, dx_1,$$

and then by applying the Laplace transform in t given by

$$\hat{\bar{f}}(p, x_2, s) = \int_0^{\infty} e^{-ts} \hat{f}(p, x_2, t)\, dt.$$

Transforming (2.1) and (2.2), one obtains

$$s^2 \hat{\bar{u}}_3(p, x_2, s) = c^2 \left[\frac{\partial^2}{\partial x_2^2} - p^2 \right] \hat{\bar{u}}_3(p, x_2, s), \tag{2.5}$$

$$\hat{\bar{\sigma}}_{23}(p, x_2, s) = \mu \frac{\partial}{\partial x_2} \hat{\bar{u}}_3(p, x_2, s), \tag{2.6}$$

where c denotes the shear wave speed defined by $c^2 = \mu/\rho$.

The general solution of the ordinary differential equation (2.5) satisfying (2.3_3) is

$$\hat{\bar{u}}_3(p, x_2, s) = A(p, s) e^{-\beta(p, s) x_2} \tag{2.7}$$

with $A(p, s) = \hat{\bar{u}}_3(p, 0, s)$ as yet unknown and $\beta(p, s)$ given by

$$\beta(p, s) = \sqrt{p^2 + s^2/c^2}. \tag{2.8}$$

In (2.7) the square root must be chosen so that $\mathrm{Re}(\beta(p, s)) > 0$. Combining (2.6)–(2.8) and letting $x_2 \to 0$ result in the basic boundary relation

$$\hat{\bar{\sigma}}_{23}(p, 0, s) = -\mu\beta(p, s)\hat{\bar{u}}_3(p, 0, s). \tag{2.9}$$

It now proves convenient to reformulate (2.9) in the moving coordinate system (2.4). In particular, it is easily seen that (2.9) becomes

$$\eta(p, s) = -\mu\beta(p, s)\xi^-(p, s), \tag{2.10}$$

where $\eta(p, s)$ and $\xi^-(p, s)$ are defined by

$$\begin{aligned} \eta(p, s) &= \int_0^\infty e^{-ts} e^{ipa(t)} \hat{\sigma}(p, 0, t)\, dt, \\ \xi^-(p, s) &= \int_0^\infty e^{-ts} e^{ipa(t)} \hat{w}(p, 0, t)\, dt. \end{aligned} \tag{2.11}$$

The boundary relation can now be viewed as the linear jump condition of a Riemann-Hilbert (hereafter abbreviated R-H) boundary value problem. More specifically, if $\sigma^\pm(x, 0, t) = \sigma(x, 0, t)$ for $x \gtrless 0$, respectively, then (2.10) may be rearranged as

$$\eta^+(p, s) = -\mu\beta(p, s)\xi^-(p, s) - \eta^-(p, s) \tag{2.12}$$

in which $\eta^\pm(p, s)$ are given by

$$\begin{aligned} \eta^+(p, s) &= \int_0^\infty e^{-ts} e^{ipa(t)} \hat{\sigma}^+(p, 0, t)\, dt \\ &= \int_0^\infty e^{-ts}\, dt \int_{a(t)}^\infty e^{ipx} \sigma^+(x - a(t), 0, t)\, dx \end{aligned} \tag{2.13}$$

$$\eta^-(p, s) = \int_0^\infty e^{-ts}\, dt \int_{-\infty}^{a(t)} e^{ipx} \sigma^-(x - a(t), 0, t)\, dx. \tag{2.14}$$

Thus, in particular, $\eta^-(p, s)$ is calculated from the known crack-face traction $\sigma^-(x, t)$. If $\eta^+(p, s)$ and $\xi^-(p, s)$ had natural analytic extensions for $\mathrm{Im}(z) \gtrless 0$, respectively, with algebraic asymptotic behavior as $\mathrm{Im}(|z|) \to \infty$, then (2.12) could be viewed as a regular R-H boundary value problem and solved by standard methods. However, such is obviously not the case here since, in particular, the natural analytic extension of $\xi^-(p, s)$ to the lower complex half-plane, $\mathrm{Im}(z) < 0$, grows exponentially as $\mathrm{Im}(z) \to -\infty$.

To circumvent this difficulty it proves convenient to decompose $\xi^-(p, s)$ as

$$\begin{aligned} \xi^-(p, s) &= \int_0^\infty e^{-ts}\, dt \int_{-\infty}^{a(t)} e^{ipx} w(x - a(t), 0, t)\, dx \\ &= \hat{\bar{d}}_L(p, s) + \hat{\bar{d}}_R(p, s) \end{aligned} \tag{2.15}$$

with

$$d_L(x, t) = \begin{cases} w(x - a(t), 0, t) & \text{if } -\infty < x < 0, \\ 0 & \text{if } 0 < x, \end{cases}$$
$$d_R(x, t) = \begin{cases} 0 & \text{if } -\infty < x < 0, a(t) < x, \\ w(x - a(t), 0, t) & \text{if } 0 < x < a(t) \end{cases} \tag{2.16}$$

and

$$\hat{\bar{d}}_L(p, s) = \int_0^\infty e^{-ts} dt \int_{-\infty}^0 e^{ipx} w(x - a(t), t)\, dx, \tag{2.17}$$

$$\hat{\bar{d}}_R(p, s) = \int_0^\infty e^{-ts} dt \int_0^{a(t)} e^{ipx} w(x - a(t), t)\, dx. \tag{2.18}$$

From (2.16) one sees immediately that $d_L(x, t)$ is the current crack-face displacement, but only up to the initial location of the crack-tip, whereas $d_R(x, t)$ is the current crack-face displacement from the initial location of the crack-tip up to its current position. Moreover, from (2.17) it is clear that $\hat{\bar{d}}_L(p, s)$ has a natural analytic extension to the lower half of the complex plane that vanishes at infinity. Consequently, one can pose the following regular R-H problem: Find $F^{\pm}(z, s)$ analytic for $\mathrm{Im}(z) \gtrless 0$, respectively, and such that

$$F^+(p, s) = -\mu\beta(p, s)F^-(p, s) + g(p, s),$$
$$\lim_{\mathrm{Im}(z)\to\pm\infty} F^{\pm}(z, s) = 0, \tag{2.19}$$

where

$$F^+(p, s) = \eta^+(p, s), \qquad F^-(p, s) = \hat{\bar{d}}_L(p, s),$$
$$g(p, s) = -\eta^-(p, s) - \mu\beta(p, s)\hat{\bar{d}}_R(p, s). \tag{2.20}$$

Since both $d_R(x, t)$ and $d_L(x, t)$ are unknown, the inhomogeneous term $g(p, s)$ as well as $F^+(p, s)$ and $F^-(p, s)$ in the R-H problem (2.19) are all unknown. Consequently, (2.19) cannot be solved directly for $F^+(p, s)$ and $F^-(p, s)$. However, by formally solving (2.19), an integral relation expressing $d_L(x, t)$ as an integral operator of $d_R(x, t)$ and a separate integral equation from which $d_R(x, t)$ can be determined will both be derived. In this way, the displacement along the entire crack will be constructed. It will then be straightforward to derive expressions for the stress ahead of the crack and the Stress Intensity Factor (SIF).

3. Derivation of the governing integral equations. The R-H problem (2.19) and (2.20) can be formally solved by first considering the corresponding homogeneous R-H problem

$$X^+(p, s) = -\mu\beta(p, s)X^-(p, s). \tag{3.1}$$

For most of the following, s will be assumed to be real and positive. When necessary the generalization of specific formulas to complex s will follow by analytic continuation arguments. From (2.8) one can easily solve (3.1) by inspection. Convenient forms for $X^{\pm}(z, s)$ are given by

$$X^+(z, s) = \sqrt{z + is/c}, \qquad X^-(z, s) = \frac{-1}{\mu\sqrt{z - is/c}} \tag{3.2}$$

in which the branch cuts for $X^+(z, s)$ and $X^-(z, s)$ lie along the negative and positive imaginary axes, respectively. The unique solution to the inhomogeneous R-H problem (2.19) is then given by

$$\begin{aligned} F^{\pm}(z, s) &= X^{\pm}(z, s)\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(\tau, s)}{X^+(\tau, s)}\frac{d\tau}{(\tau - z)} \\ &= -X^{\pm}(z, s)\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\eta^-(\tau, s)}{X^+(\tau, s)}\frac{d\tau}{(\tau - z)} \\ &\quad + X^{\pm}(z, s)\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\hat{\bar{d}}_R(\tau, s)}{X^-(\tau, s)}\frac{d\tau}{(\tau - z)}. \end{aligned} \tag{3.3}$$

Applying the Plemelj formula to (3.3) along with (2.20) and (3.2) yields

$$\begin{aligned} \hat{\bar{d}}_L(p, s) &= F^-(p, s) \\ &= \hat{\bar{d}}_0(p, s) \\ &\quad - \frac{1}{2}\hat{\bar{d}}_R(p, s) + \frac{1}{\sqrt{p - is/c}}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\hat{\bar{d}}_R(\tau, s)\sqrt{\tau - is/c}\frac{d\tau}{(\tau - p)}, \end{aligned} \tag{3.4}$$

in which

$$\hat{\bar{d}}_0(p, s) = \frac{-\eta^-(p, s)}{2\mu\sqrt{p^2 + s^2/c^2}} + \frac{1}{\mu\sqrt{p - is/c}}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\eta^-(\tau, s)}{\sqrt{\tau + is/c}}\frac{d\tau}{(\tau - p)}, \tag{3.5}$$

and

$$\begin{aligned} \eta^+(p, s) &= F^+(p, s) \\ &= \hat{\bar{t}}_0(p, s) - \frac{\mu}{2}\sqrt{p^2 + s^2/c^2}\,\hat{\bar{d}}_R(p, s) \\ &\quad - \mu\sqrt{p + is/c}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\hat{\bar{d}}_R(\tau, s)\sqrt{\tau - is/c}\frac{d\tau}{(\tau - p)} \end{aligned} \tag{3.6}$$

with

$$\hat{\bar{t}}_0(p, s) = -\frac{1}{2}\eta^-(p, s) - \sqrt{p + is/c}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\eta^-(\tau, s)}{\sqrt{\tau + is/c}}\frac{d\tau}{(\tau - p)}. \tag{3.7}$$

The right-hand sides of (3.5) and (3.7) are identical in form to the expressions for the double Fourier and Laplace transforms of the crack-face displacement and corresponding stress (in front of the crack) that one would obtain for a stationary crack subject to a loading that, in general, is a function of $a(t)$. An interesting special case is that for which the crack-face loading is stationary relative to the initial crack-tip position. More specifically, for crack-face loads $\sigma_e(x_1, t)$ of the form

$$\sigma_e(x_1, t) = H(-x_1)f(x_1, t), \tag{3.8}$$

one has from (2.4) and (2.14) that

$$\eta^-(p, s) = \int_0^{\infty} e^{-ts}\,dt\int_{-\infty}^{0} e^{ipx}f(x, t)\,dx. \tag{3.9}$$

Substitution of (3.9) into (3.5) followed by Fourier and Laplace inversion yields a function $d_0(x, t)$ that is identical to the crack-face displacement for a stationary crack (ignoring the fact that x is in actuality a Galilean variable) subjected to the loading form (3.8) with x substituted for x_1.

The desired equations for $d_R(x, t)$ and $d_L(x, t)$ are obtained by applying both the inverse Fourier and Laplace transforms to (3.4)–(3.7). It is convenient to first calculate $\sqrt{p - is/c}\,\hat{\bar{d}}_R(p, s)$. To this end, it is shown in the appendix that if

$$\hat{\bar{\delta}}_L^{-1/2}(p, s) \equiv \sqrt{s + ipc}\,\hat{\bar{d}}_R(p, s) \tag{3.10}$$

then

$$\delta_L^{-1/2}(x, t) = \left[\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right]\frac{1}{\sqrt{\pi}}\int_{\max\{0, b_1^{-1}(t+x/c)\}}^{\min\{t, t+x/c\}} d_R(x + c(t-q), q)\frac{dq}{\sqrt{t-q}}, \tag{3.11}$$

where $b_1^{-1}(q)$ denotes the inverse function to

$$b_1(q) = q + a(q)/c. \tag{3.12}$$

We will also use, below, the function $b_0(q)$ defined by

$$b_0(q) = q - a(q)/c, \tag{3.13}$$

which by virtue of the restriction $\dot{a}(t) < c$, is invertible on $0 < q < \infty$. An interesting observation from (3.11) is that $\delta_R^{-1/2}(r, t)$ is a fractional derivative of $d_R(r, t)$ of order $\frac{1}{2}$ along the left propagating characteristic direction.

Substitution of (3.10) into (3.4) yields

$$\hat{\bar{d}}_L(p, s) = \hat{\bar{d}}_0(p, s) - \frac{1}{2\sqrt{s + ipc}}\left[\hat{\bar{\delta}}_L^{-1/2}(p, s) - \frac{1}{\pi i}\int_{-\infty}^{\infty}\hat{\bar{\delta}}_L^{-1/2}(\tau, s)\frac{d\tau}{(\tau - p)}\right]. \tag{3.14}$$

It is useful now to recall the well-known formula relating the Fourier and Hilbert transforms,

$$\mathscr{F}[i\,\mathrm{sgn}(\xi)\mathscr{F}^{-1}(f)(\xi)](x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{dt}{(t-x)}, \tag{3.15}$$

in which $\mathscr{F}$ and $\mathscr{F}^{-1}$ denote the Fourier transform and its inverse, respectively. Applying (3.15) to the Hilbert transform in (3.14), one has

$$\begin{aligned}\frac{1}{\pi i}\int_{-\infty}^{\infty}\hat{\bar{\delta}}_L^{-1/2}(\tau, s)\frac{d\tau}{(\tau - p)} &= \mathscr{F}\{\mathrm{sgn}(r)\bar{\delta}_L^{-1/2}(r, s)\}\\ &= -\hat{\bar{\delta}}_L^{-1/2}(p, s) + 2\int_0^{\infty} e^{ixp}\bar{\delta}_L^{-1/2}(x, s)\,dx \qquad (3.16)\\ &= \hat{\bar{\delta}}_L^{-1/2}(p, s) - 2\int_{-\infty}^{0} e^{ixp}\bar{\delta}_L^{-1/2}(x, s)\,dx. \qquad (3.17)\end{aligned}$$

Substitution of (3.17) into (3.14) yields

$$\hat{\bar{d}}_L(p, s) = \hat{\bar{d}}_0(p, s) - \frac{1}{\sqrt{s + icp}}\int_{-\infty}^{0} e^{ixp}\bar{\delta}_L^{-1/2}(x, s)\,dx. \tag{3.18}$$

In a manner similar to the derivation of (3.11) in the appendix, it can be shown that

$$\begin{aligned}&\frac{1}{\sqrt{s + ipc}}\int_{-\infty}^{0} e^{ixp}\bar{\delta}_L^{-1/2}(x, s)\,dx \qquad (3.19)\\ &\quad = \int_0^{\infty} e^{-ts}\,dt\int_{-\infty}^{0} e^{ixp}\,dx\frac{1}{\sqrt{\pi}}\int_{\max\{0, t+x/c\}}^{t}\delta_L^{-1/2}(x + c(t-r), r)\frac{dr}{\sqrt{t-r}}.\end{aligned}$$

Substitution of (3.19) into (3.18) followed by double Fourier and Laplace inversions yields for $x < 0$

$$d_L(x\,,\,t) = d_0(x\,,\,t) - \frac{1}{\sqrt{\pi}}\int_{\max\{0\,,\,t+x/c\}}^{t} \delta_L^{-1/2}(x + c(t-r)\,,\,r)\frac{dr}{\sqrt{t-r}}. \qquad (3.20)$$

Further substitution of (3.11) into (3.20) yields finally the desired equation relating $d_L(x\,,\,t)$ and $d_R(x\,,\,t)$ for $x < 0$

$$d_L(x\,,\,t) = d_0(x\,,\,t) - \frac{1}{\pi}\int_{\max\{0\,,\,t+x/c\}}^{t} \frac{dr}{\sqrt{t-r}}\frac{\partial}{\partial r}\int_{\max\{0\,,\,b_1^{-1}(t+x/c)\}}^{t+x/c} d_R(x + c(t-q)\,,\,q)\frac{dq}{\sqrt{r-q}}. \qquad (3.21)$$

An integral equation will now be derived from which $d_R(x\,,\,t)$ can be determined. Subsequent substitution into (3.21) gives $d_L(x\,,\,t)$, thereby completing the calculation of the crack-face displacement. To this end, the double Fourier-Laplace inversion of (3.6) must be carried out in a manner similar to the derivation of (3.21). Substituting (3.10) into (3.6) and making use of (3.16), one has

$$\hat{\bar{f}}_R(p\,,\,s) = \hat{\bar{t}}_0(p\,,\,s) - \frac{\mu}{c}\sqrt{s - ipc}\int_0^\infty e^{irp}\overline{\delta}_L^{-1/2}(r\,,\,s)\,dr \qquad (3.22)$$

with

$$f_R(x\,,\,t) = \begin{cases} 0 & \text{if } 0 < x < a(t), \\ \sigma^+(x - a(t)\,,\,0\,,\,t) & \text{if } a(t) < x. \end{cases} \qquad (3.23)$$

Analogous to (3.19), it can be shown that

$$\begin{aligned} &\sqrt{s-ipc}\int_0^\infty e^{irp}\overline{\delta}_L^{-1/2}(r\,,\,s)\,dr \\ &= \sqrt{s-ipc}\int_0^\infty e^{-ts}\,dt\int_0^\infty e^{ipx}\delta_L^{-1/2}(x\,,\,t)\,dx \\ &= \int_0^\infty e^{-ts}\,dt\int_0^\infty e^{ipx}\,dx\frac{1}{\sqrt{\pi}}\left[\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right] \\ &\quad\times\int_{\max\{0\,,\,t-x/c\}}^{t} \delta_L^{-1/2}(x - c(t-r)\,,\,r)\frac{dr}{\sqrt{t-r}}. \end{aligned} \qquad (3.24)$$

Substitution of (3.24) into (3.22) followed by double Laplace-Fourier inversion produces for $x > 0$

$$f_R(x\,,\,t) = t_0(x\,,\,t) - \frac{\mu}{c\sqrt{\pi}}\left[\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right]\int_{\max\{0\,,\,t-x/c\}}^{t} \delta_L^{-1/2}(x - c(t-r)\,,\,r)\frac{dr}{\sqrt{t-r}}. \qquad (3.25)$$

Appealing to (3.23), one arrives finally at the desired integral equation for $d_R(x\,,\,t)$ by restricting x and t to the set $\{(x\,,\,t)\colon 0 < x < a(t)\}$

$$t_0(x\,,\,t) = \frac{\mu}{c\sqrt{\pi}}\left[\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right]\int_{t-x/c}^{t} \delta_L^{-1/2}(x - c(t-r)\,,\,r)\frac{dr}{\sqrt{t-r}} \qquad (3.26)$$

with $\delta_L^{-1/2}(x\,,\,t)$ given by (3.11).

4. Solution of the governing integral equations. Equation (3.26) can be solved by recognizing that the right-hand side is a fractional derivative of $d_R(x, t)$ along each of the two characteristic directions. More specifically, we first make the change of variables $\xi = t - x/c$, $\eta = t + x/c$ and note that since $0 < x < a(t)$, η and ξ must satisfy

$$\xi < \eta < 2b_0^{-1}(\xi) - \xi, \tag{4.1}$$

or equivalently,

$$2b_1^{-1}(\eta) - \eta < \xi < \eta, \tag{4.2}$$

where $b_0^{-1}(\xi)$ and $b_1^{-1}(\eta)$ denote the inverse functions to $b_0(q)$ and $b_1(q)$ given by (3.12) and (3.13). Making this change of variable in (3.26) one concludes that

$$t_0\left(\frac{c(\eta-\xi)}{2}, \frac{\eta+\xi}{2}\right) = \frac{\mu}{c}\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial\eta}\int_\xi^\eta \delta_L^{-1/2}\left(\frac{c(r-\xi)}{2}, \frac{r+\xi}{2}\right)\frac{dr}{\sqrt{\eta-r}}. \tag{4.3}$$

For fixed ξ (i.e., along a fixed characteristic), equation (4.3) is an Abel integral equation on the interval $\xi < \eta < 2b_0^{-1}(\xi) - \xi$, which is easily inverted to yield

$$\delta_L^{-1/2}(x, t) = \delta_L^{-1/2}\left(\frac{c(\eta-\xi)}{2}, \frac{\eta+\xi}{2}\right) = \frac{c}{\mu\sqrt{2\pi}}\int_\xi^\eta t_0\left(\frac{c(q-\xi)}{2}, \frac{q+\xi}{2}\right)\frac{dq}{\sqrt{\eta-q}}. \tag{4.4}$$

Substitution of (3.11) into (4.4) followed by restriction to the constant $\eta = t + x/c$ characteristic yields an Abel integral equation on the ξ interval, $2b_1^{-1}(\eta) - \eta < \xi < \eta$, whose solution is

$$\begin{aligned} d_R(x, t) &= d_R\left(\frac{c(\eta-\xi)}{2}, \frac{\eta+\xi}{2}\right) \\ &= \frac{c}{2\mu\pi}\int_{2b_1^{-1}(\eta)-\eta}^{\xi}\frac{dr}{\sqrt{\xi-r}}\int_r^\eta t_0\left(\frac{c(q-r)}{2}, \frac{q+r}{2}\right)\frac{dq}{\sqrt{\eta-q}} \qquad (4.5)\\ &= \frac{c}{\mu\pi}\int_{b_1^{-1}(t+x/c)}^{t}\frac{dr}{\sqrt{t-r}}\int_r^{t+x/c} t_0\left(c(q-r), q+r-t-\frac{x}{c}\right)\frac{dq}{\sqrt{t+x/c-q}}. \qquad (4.6)\end{aligned}$$

The functions $d_0(x, t)$ and $t_0(x, t)$ in (3.5) and (3.7) can be calculated by methods similar to those employed above to derive the integral equations (3.21) and (3.25). In particular, one can show that for $x < 0$, $d_0(x, t)$ is given by

$$d_0(x, t) = \frac{-c}{\pi\mu}\int_{\max\{0, t+x/c\}}^{t}\frac{dq}{\sqrt{t-q}}\int_0^q \sigma^-(x + c(-2q+t+b_0(r)), 0, r)\frac{dr}{\sqrt{q-r}}, \tag{4.7}$$

whereas for $x > 0$, $t_0(x, t)$ is given by

$$\begin{aligned} t_0(x, t) = &-\left[\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right]\frac{1}{\pi}\int_{\max\{0, t-x/c\}}^{t}\frac{dq}{\sqrt{t-q}} \\ &\times\int_0^{\min\{q, b_0^{-1}(t-x/c)\}} \sigma^-(x - ct + cb_0(r), 0, r)\frac{dr}{\sqrt{q-r}}. \end{aligned} \tag{4.8}$$

From (4.7) one readily sees that $d_0(x, t)$ is the crack-face displacement one would obtain for a stationary crack subjected to the crack-face loading

$$\sigma_e(x, t) = \sigma^-(x - a(t), t) \quad \text{for } x < 0. \tag{4.9}$$

Substituting (4.8) into (4.6), it is shown in the appendix that for $0 < x < a(t)$

$$d_R(x, t) = \frac{-c}{\mu\pi} \int_{b_1^{-1}(t+x/c)}^{t} \frac{dq}{\sqrt{t-q}} \int_0^q \sigma^-(x + c(-2q + t + b_0(r)), 0, r) \frac{dr}{\sqrt{q-r}}. \tag{4.10}$$

Finally appealing to (2.16), one obtains for the crack-face displacement on the interval $-a(t) < x < 0$

$$\begin{aligned} w(x, 0, t) = \frac{-c}{\mu\pi} &\int_{b_1^{-1}(b_1(t)+x/c)}^{t} \frac{dq}{\sqrt{t-q}} \\ &\times \int_0^q \sigma^-(x + c(-2q + b_1(t) + b_0(r)), 0, r) \frac{dr}{\sqrt{q-r}}. \end{aligned} \tag{4.11}$$

As for $d_L(x, t)$ on the interval $-\infty < x < 0$, integrating (3.21) by parts shows that for $t + x/c > 0$

$$\begin{aligned} d_L(x, t) &= d_0(x, t) + d_R\left(0+, t + \frac{x}{c}\right) \\ &- \frac{1}{\pi} \int_{t+x/c}^{t} \frac{dr}{\sqrt{t-r}} \int_{b_1^{-1}(t+x/c)}^{t+x/c} \frac{d}{dq} d_R(x + c(t-q), q) \frac{dq}{\sqrt{r-q}}. \end{aligned} \tag{4.12}$$

Since $d_0(0-, t) = 0$ and the integral in (4.12) also vanishes as $x \to 0-$, one sees immediately from (4.12) that $d_L(0-, t) = d_R(0+, t)$. It is shown in the appendix that if (4.7) and (4.10) are substituted into (3.21) that for $x < 0$

$$d_L(x, t) = \frac{-c}{\mu\pi} \int_{\max\{0, b_1^{-1}(t+x/c)\}}^{t} \frac{dq}{\sqrt{t-q}} \int_0^q \sigma^-(x + c(-2q - t + b_0(r)), 0, r) \frac{dr}{\sqrt{q-r}}. \tag{4.13}$$

Thus, it is seen that (4.13) extends (4.11) to the case $x < -a(t)$. Furthermore, it should be noted in (4.13) that for $t + x/c < 0$, $d_L(x, t) = d_0(x, t)$. Thus it is seen that the crack-face displacement is simply that for a "stationary" crack until sufficient time has elapsed for the disturbance due to the moving crack-tip to reach the location $x < 0$.

For completeness, it will now be shown how to construct an expression for the stress, $\sigma^+(x, t)$, in front of the crack-tip, and in particular how to extract the stress intensity factor from the formulas generated above. To this end, one sees from (3.23) that for $x > 0$,

$$\sigma^+(x, t) = f_R(x + a(t), t) \tag{4.14}$$

with $f_R(x, t)$ given by (3.25). Evaluating (3.25) requires the consideration of two cases: (i) $\xi = t - x/c < 0$ and (ii) $\xi = t - x/c > 0$. The calculations are similar in both cases, and since our primary interest is in constructing the stress intensity factor, only case (ii) will be presented. Therefore, in (3.25) it will be assumed that

$a(t) < x < ct$. Since $t_0(x, t)$ is given by (4.8), it remains only to calculate the second term in (3.25)

$$\frac{-2\mu}{c\sqrt{\pi}}\frac{\partial}{\partial\eta}\int_{\xi}^{t}\delta_L^{-1/2}(c(r-\xi), r)\frac{dr}{\sqrt{t-r}} \tag{4.15}$$

in which use is again made of the characteristic coordinates η and ξ. It is easily verified that in (4.15), one need only evaluate $\delta_L^{-1/2}(x, t)$ for $t > 0$ and $0 < x < a(t)$. Under these restrictions, substitution of (4.10) into (3.11) followed by an interchange in the order of integration yields

$$\delta_L^{-1/2}(x, t) = \frac{-c}{\mu\sqrt{\pi}}\int_0^t \sigma^-(c(b_0(r)-\xi), 0, r)\frac{dr}{\sqrt{t-r}}. \tag{4.16}$$

A crucial observation from (3.11) is that $\delta_L^{-1/2}(x, t) = 0$ if $x > a(t)$ from which it follows that $\delta_L^{-1/2}(c(r-\xi), r) = 0$ for $r > b_0^{-1}(\xi)$. Substitution of (4.16) into (4.15) now gives for the second term in (3.25)

$$\frac{2}{\pi}\frac{\partial}{\partial\eta}\int_{\xi}^{b_0^{-1}(\xi)}\frac{dr}{\sqrt{t-r}}\int_0^r \sigma^-(c(b_0(q)-\xi), 0, q)\frac{dq}{\sqrt{r-q}}. \tag{4.17}$$

Combining (3.25), (4.8), and (4.17) one has for $a(t) < x < ct$

$$f_R(x, t) = \int_0^{b_0^{-1}(\xi)} \sigma^-(c(b_0(q)-\xi), 0, q)h(\xi, t, q)\,dq \tag{4.18}$$

in which

$$h(\xi, t, q) = \frac{-1}{2\pi}\int_q^{b_0^{-1}(\xi)}(t-r)^{-3/2}(r-q)^{-1/2}\,dr. \tag{4.19}$$

Finally, substitution of (4.18) into (4.14) yields for $x > 0$

$$\sigma^+(x, t) = \int_0^{b_0^{-1}(b_0(t)-x/c)} \sigma^-(c(b_0(q)-b_0(t))+x, 0, q)h\left(b_0(t)-\frac{x}{c}, t, q\right)dq. \tag{4.20}$$

The stress intensity factor is easily extracted from (4.20) by rewriting $h(\xi, t, q)$ in (4.19) as

$$h(\xi, t, q) = \frac{-1/\pi}{\sqrt{t-b_0^{-1}(\xi)}\sqrt{b_0^{-1}(\xi)-q}} + h_1(\xi, t, q) \tag{4.21}$$

with $h_1(\xi, t, q)$ given by

$$h_1(\xi, t, q) = \frac{1}{2\pi}\int_{b_0^{-1}(\xi)}^{t}(t-r)^{-1/2}(r-q)^{-3/2}\,dr.$$

It is easily seen that $h_1(\xi, t, q)$ produces no contribution to the crack-tip stress singularity. Thus, near the crack-tip $\sigma^+(x, t)$ is asymptotically represented by

$$\sigma^+(x, t) \sim \frac{1}{\sqrt{t-b_0^{-1}(b_0(t)-x/c)}}X(x, t) \tag{4.22}$$

with

$$X(x, t) = \frac{-1}{\pi} \int_0^{b_0^{-1}(b_0(t)-x/c)} \sigma^- \left(c\left(b_0(q) - b_0(t) + \frac{x}{c}\right), 0, q \right) \times \frac{dq}{\sqrt{b_0^{-1}(b_0(t) - x/c) - q}}. \tag{4.23}$$

Using the formula $\dot{b}_0(t) = 1 - \dot{a}(t)/c$ one easily computes the limit

$$\lim_{x \to 0+} \sqrt{\frac{x}{t - b_0^{-1}(b_0(t) - x/c)}} = \lim_{x \to 0+} \sqrt{\dot{b}_0(b_0^{-1}(b_0(t) - x/c))} = \sqrt{c - \dot{a}(t)}$$

from which it follows that

$$\sigma(x, t) \sim \frac{K(t)}{\sqrt{x}} \quad \text{as } x \to 0+,$$

where the stress intensity factor, $K(t)$, is given by

$$K(t) = -\sqrt{c - \dot{a}(t)} \frac{1}{\pi} \int_0^t \sigma^-(c(b_0(q) - b_0(t)), 0, q) \frac{dq}{\sqrt{t - q}}. \tag{4.24}$$

5. Conclusion. Two noteworthy features of the above analysis are the remarkably simple form for the crack-face displacement even for general loadings given by (4.11) and the fact that transform and complex variable methods can be adapted to handle accelerating crack problems. This latter fact suggests that the methods employed here can be applied to opening mode problems for which no expressions for the crack-face displacement are available in the literature and viscoelastic material models for which no solutions have yet been obtained. These latter problems are the subject of future papers.

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Appendix. The following illustrates the techniques needed to derive several of the results appearing in the preceding text.

First the derivation of (3.11) will be shown. If $\hat{\bar{\delta}}_L^{-1/2}(p, s) \equiv \sqrt{s + ipc}\, \hat{\bar{d}}_R(p, s)$ then

$$\hat{\bar{\delta}}_L^{-1/2}(p, s) = \sqrt{s + ipc} \int_0^\infty e^{-ts}\, dt \int_{-\infty}^\infty e^{ipx} \bar{d}_R(x, t)\, dx. \tag{A.1}$$

Making the change of variables $s_1 = s + ipc$, $x_1 = x + ct$ in (A.1) it is found that

$$\hat{\bar{\delta}}_L^{-1/2}(p, s) = \frac{s_1}{\sqrt{s_1}} \int_0^\infty e^{-ts_1}\, dt \int_{-\infty}^\infty e^{ipx_1} \bar{d}_R(x_1 - ct, t)\, dx_1. \tag{A.2}$$

Recalling the formula $\mathscr{L}(t^{-1/2}) = \sqrt{\pi}/\sqrt{s}$, one easily sees that

$$\begin{aligned} \hat{\bar{\delta}}_L^{-1/2}(p, s) &= s_1 \int_0^\infty e^{-ts_1}\, dt \int_{-\infty}^\infty e^{ipx_1}\, dx_1 \frac{1}{\sqrt{\pi}} \int_0^t d_R(x_1 - cq, q) \frac{dq}{\sqrt{t - q}} \\ &= \int_0^\infty e^{-ts_1}\, dt \int_{-\infty}^\infty e^{ipx_1}\, dx_1 \frac{\partial}{\partial t} \frac{1}{\sqrt{\pi}} \int_0^t d_R(x_1 - cq, q) \frac{dq}{\sqrt{t - q}}. \end{aligned} \tag{A.3}$$

Note that the partial with respect to t in (A.3) is made assuming a constant x_1 value. Furthermore, from the definition of $d_R(x, t)$ in (2.16) it can be seen that

$$d_R(x_1 - cq, q) = 0 \quad \text{if } q > x_1/c \text{ or } q < b_1^{-1}(x_1/c).$$

Thus

$$\hat{\bar{\delta}}_L^{-1/2}(p, s) = \int_0^\infty e^{-ts_1}\, dt \int_{-\infty}^\infty e^{ipx_1}\, dx_1 \frac{\partial}{\partial t} \frac{1}{\sqrt{\pi}} \int_{\max\{0, b_1^{-1}(x_1/c)\}}^{\min\{t, x_1/c\}} d_R(x_1 - cq, q) \frac{dq}{\sqrt{t-q}}. \tag{A.4}$$

Equation (3.11) follows from changing x_1 and s_1 in (A.4) back to x and s.

We shall now consider the derivation of (4.10). If (4.8) is rewritten in the characteristic coordinates $\eta = t + x/c$ and $\xi = t - x/c$, then it is found that for $\xi < \eta < 2b_0^{-1}(\xi) - \xi$, i.e., $0 < x < a(t)$

$$t_0\left(\frac{c(\eta - \xi)}{2}, \frac{\eta + \xi}{2}\right) = \frac{-2}{\pi} \frac{\partial}{\partial \eta} \int_\xi^{(\xi+\eta)/2} \frac{dq}{\sqrt{(\xi+\eta)/2 - q}} \times \int_0^q \sigma^-(-c\xi + cb_0(r), 0, r) \frac{dr}{\sqrt{q - r}}. \tag{A.5}$$

Substituting (A.5) into (4.5), one obtains

$$d_R(c(\eta - \xi)/2, (\eta + \xi)/2) = \frac{-c}{\mu\pi^2} \int_{2b_1^{-1}(\eta)-\eta}^{\xi} \frac{dr}{\sqrt{\xi - r}} \int_r^\eta \frac{dq}{\sqrt{\eta - q}} \frac{\partial}{\partial q} \int_r^{(q+r)/2} \frac{dv}{\sqrt{(q+r)/2 - v}} \times \int_0^v \sigma^-(-cr + cb_0(u), 0, u) \frac{du}{\sqrt{v - u}}. \tag{A.6}$$

In (A.6) let

$$F(v, r) = \int_0^v \sigma^-(-cr + cb_0(u), 0, u) \frac{du}{\sqrt{v - u}}$$

and

$$G(r, q) = \int_r^{(q+r)/2} F(v, r) \frac{dv}{\sqrt{(q+r)/2 - v}}.$$

From the observation that

$$\frac{d}{dt} \int_a^t \frac{g(r)}{\sqrt{t - r}}\, dr = \frac{g(a)}{\sqrt{t - a}} + \int_a^t \frac{g'(r)}{\sqrt{t - r}}\, dr, \tag{A.7}$$

it is found that

$$\int_r^\eta \frac{\partial}{\partial q} G(r, q) \frac{dq}{\sqrt{\eta - q}} = -\frac{G(r, r)}{\sqrt{\eta - r}} + \frac{\partial}{\partial \eta} \int_r^\eta G(r, q) \frac{dq}{\sqrt{\eta - q}}. \tag{A.8}$$

If the prescribed stress on the crack-faces, $\sigma^-(x, 0, t)$, is assumed bounded then $G(r, r) = 0$. The term $\int_r^\eta G(r, q) \frac{dq}{\sqrt{\eta - q}}$ in (A.8) may be simplified by reversing the variables of integration to obtain

$$\int_r^\eta G(r, q) \frac{dq}{\sqrt{\eta - q}} = \int_r^{(\eta+r)/2} F(v, r)\, dv \int_{2v-r}^\eta \frac{\sqrt{2}\, dq}{\sqrt{\eta - q}\sqrt{q - (2v - r)}}. \tag{A.9}$$

The integral, $\int_{2v-r}^{\eta} \frac{dq}{\sqrt{\eta-q}\sqrt{q-(2v-r)}}$ is recognized to be the beta function, $B(z, w)$, for $z = w = \frac{1}{2}$ where $B(\frac{1}{2}, \frac{1}{2}) = \pi$. Thus

$$\frac{\partial}{\partial \eta}\int_r^{\eta} G(r, q)\frac{dq}{\sqrt{\eta - q}} = \sqrt{2}\pi\frac{\partial}{\partial \eta}\int_r^{(\eta+r)/2} F(v, r)\, dv = \frac{\pi}{\sqrt{2}}F\left(\frac{\eta+r}{2}, r\right). \quad (A.10)$$

Substituting (A.10) into (A.6), it is found that

$$d_R\left(\frac{c(\eta-\xi)}{2}, \frac{\eta+\xi}{2}\right) = \frac{-c}{\mu\pi\sqrt{2}}\int_{2b_1^{-1}(\eta)-\eta}^{\xi} F\left(\frac{\eta+r}{2}, r\right)\frac{dr}{\sqrt{\xi - r}}, \quad (A.11)$$

which after the change of variable $u = (\eta + r)/2$ becomes (4.10) when the characteristic coordinates ξ and η are switched back to x and t.

Finally, we shall present the derivation of (4.13) for $t + x/c > 0$. If (4.10) is substituted into (3.21) and the result and (4.7) are rewritten using the characteristic coordinates ξ and η, it is found that

$$d_L\left(\frac{c(\eta-\xi)}{2}, \frac{\eta+\xi}{2}\right) = d_0\left(\frac{c(\eta-\xi)}{2}, \frac{\eta+\xi}{2}\right) + \frac{c}{\mu\pi^2}\int_{\eta}^{(\xi+\eta)/2}\frac{dr}{\sqrt{(\xi+\eta)/2 - r}}\frac{\partial}{\partial r}\int_{b_1^{-1}(\eta)}^{\eta} I(q, \eta)\frac{dq}{\sqrt{r-q}} \quad (A.12)$$

$$d_0\left(\frac{c(\eta-\xi)}{2}, \frac{\eta+\xi}{2}\right) = \frac{-c}{\mu\pi}\int_{\eta}^{(\xi+\eta)/2} H(q, \eta)\frac{dq}{\sqrt{(\xi+\eta)/2 - q}}, \quad (A.13)$$

where $H(v, \eta) = \int_0^v \sigma^-(-2cv + c\eta + cb_0(u), 0, u)\frac{du}{\sqrt{v-u}}$ and $I(q, \eta) = \int_{b_1^{-1}(\eta)}^{q} H(v, \eta)\frac{dv}{\sqrt{q-v}}$. Note that

$$\int_{b_1^{-1}(\eta)}^{\eta} I(q, \eta)\frac{dq}{\sqrt{r-q}} = \int_{b_1^{-1}(\eta)}^{r} I(q, \eta)\frac{dq}{\sqrt{r-q}} - \int_{\eta}^{r} I(q, \eta)\frac{dq}{\sqrt{r-q}}. \quad (A.14)$$

In a manner similar to the derivation of (A.10), it can be shown that

$$\frac{\partial}{\partial r}\int_{b_1^{-1}(\eta)}^{\eta} I(q, \eta)\frac{dq}{\sqrt{r-q}} = \pi H(r, \eta) - \frac{\partial}{\partial r}\int_{\eta}^{r} I(q, \eta)\frac{dq}{\sqrt{r-q}}. \quad (A.15)$$

Employing (A.7), (A.15) becomes

$$\frac{\partial}{\partial r}\int_{b_1^{-1}(\eta)}^{\eta} I(q, \eta)\frac{dq}{\sqrt{r-q}} = \pi H(r, \eta) - \frac{I(\eta, \eta)}{\sqrt{r-\eta}} - \int_{\eta}^{r}\frac{\partial}{\partial q}I(q, \eta)\frac{dq}{\sqrt{r-q}}. \quad (A.16)$$

It can then be shown that

$$\begin{aligned}\int_{\eta}^{(\xi+\eta)/2} & \frac{dr}{\sqrt{(\xi+\eta)/2 - r}}\frac{\partial}{\partial r}\int_{b_1^{-1}(\eta)}^{\eta} I(q, \eta)\frac{dq}{\sqrt{r-q}} \\ &= \pi\int_{\eta}^{(\xi+\eta)/2} H(r, \eta)\frac{dr}{\sqrt{(\xi+\eta)/2 - r}} \\ &\quad - I(\eta, \eta)\int_{\eta}^{(\xi+\eta)/2}\frac{dr}{\sqrt{r-\eta}\sqrt{(\xi+\eta)/2 - r}} \\ &\quad - \int_{\eta}^{(\xi+\eta)/2}\frac{dr}{\sqrt{(\xi+\eta)/2 - r}}\int_{\eta}^{r}\frac{\partial}{\partial q}I(q, \eta)\frac{dq}{\sqrt{r-q}}. \end{aligned} \quad (A.17)$$

If the variables of integration q and r are reversed in the last integral of (A.17) and the resulting beta integral is replaced by π, then (A.17) becomes

$$\int_{\eta}^{(\xi+\eta)/2} \frac{dr}{\sqrt{(\xi+\eta)/2-r}} \frac{\partial}{\partial r} \int_{b_1^{-1}(\eta)}^{\eta} I(q,\eta) \frac{dq}{\sqrt{r-q}} = \pi \int_{\eta}^{(\xi+\eta)/2} H(r,\eta) \frac{dr}{\sqrt{(\xi+\eta)/2-r}} - \pi I\left(\frac{\xi+\eta}{2}, \eta\right). \tag{A.18}$$

Substituting (A.18) and (A.13) into (A.12), it is found that

$$d_L\left(\frac{c(\eta-\xi)}{2}, \frac{\eta+\xi}{2}\right) = \frac{-c}{\mu\pi} \int_{b_1^{-1}(\eta)}^{(\xi+\eta)/2} H(r,\eta) \frac{dr}{\sqrt{(\xi+\eta)/2-r}}, \tag{A.19}$$

which becomes (4.13) when written as a function of x and t.

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