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THE GRAVITY WAVES CREATED BY A MOVING SOURCE IN A FLUID OF FINITE DEPTH

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Abstract. The linear gravity waves created by a moving oscillatory source are considered in a fluid of finite uniform depth bounded on one side by a vertical cliff. The unsteady asymptotic waves are determined by Bliestien's asymptotic expansion to the exact Fourier transform solution. Several physically interesting results are obtained. The fronts of each of the waves generated by the source are determined for all times and for all values of the parameters. The physical cause of the singularity of solution for certain values of the parameters is explained.

1. Introduction. We study the linear two-dimensional gravity wave problem generated by a source which moves with a constant velocity V and oscillates with a frequency ω. The fluid is incompressible inviscid and of finite uniform depth h and is bounded on one side by a vertical cliff.

This problem without the cliff has been considered by Debnath and Rosenblat [3] and there are similar problems considered by many other authors. The problem with surface tension has been investigated by Pramanik and Majumdar [4]. The linear case and also some nonlinear considerations have been studied by Akylas [1]. A similar problem on a sloping beach has been considered by Sarkar [5]. Generally the long time behaviour of the resulting far field waves are determined by the asymptotic estimate of the integrals in the Fourier transform solution and the asymptotic method in most cases has been the method of stationary phase. However, this method fails when a stationary point of certain integral coincides with its pole. Methods are available to overcome this difficulty. The method of Bliestien [2] is particularly suitable for the present case.

Mathematically the coincidence of a stationary point with the pole represents some functional relationship between three parameters a, b, and c, where a represents the dimensionless velocity V, b the dimensionless frequency ω, and c the dimensionless ratio between the distance and the time. Treating these relations as some surfaces in the (a, b, c) space, these surfaces are determined for all possible values of the parameters. Then it is ultimately shown that these surfaces physically represent the positions of the wave fronts for various values of the parameters V, ω,

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and h and each front moves with the group velocity of the corresponding wave. As found by Debnath and Rosenblat, it is also found here that four progressive waves are generated by the source. Referring to a fixed set of axes, three waves and their fronts move in the direction of motion of the source. Among them one front remains in the upstream side and two in the downstream side. The remaining wave and its front move in the opposite direction of motion of the source and on reaching the cliff are reflected back and this reflection occurs for all possible values of the parameters, though for large values of ω the front of the reflected wave remains very near to the cliff. The physical reason for the occurrence of the singularity of solution for certain values of the parameters is explained. It is observed that for those values of the parameters the two wave fronts, one in the upstream side and the other in the downstream side, coincide at the source. As a result the energy created by the source cannot come out of it.

2. Formulation. We consider the two-dimensional problem of wave generation on the surface of incompressible inviscid fluid of finite uniform depth bounded on one side by a vertical cliff. The wave generating mechanism is the applied free surface pressure distribution $p(x, t)$ which is switched on at the initial moment $t = 0$ and at the same time moves with a uniform velocity V away from the cliff, where the x-axis is along the undisturbed free surface and the y-axis is vertically upwards. Then in the fixed coordinate system with the origin at the cliff we have the following initial-boundary value problem:

$$\varphi_{xx} + \varphi_{yy} = 0 \quad \text{in } 0 < x < \infty, \ -h < y < 0, \ t \geq 0, \tag{1}$$

$$\left.\begin{aligned} &\varphi_t + g\eta + \frac{1}{\rho}p(x, t) = 0 \qquad (2)\\ &\eta_t = \varphi_y \qquad (3) \end{aligned}\right] \quad \text{at } y = 0.$$

If we suppose that the pressure distribution initially occupies the interval $(0, l)$ in Eq. (2), then $p(x, t)$ is nonzero only over the interval $(Vt, l + Vt)$.

$$\varphi_x = 0 \quad \text{at } x = 0, \qquad \varphi_y = 0 \quad \text{at } y = -h, \tag{4}$$

$$\varphi(x, y, 0) = 0, \qquad \eta(x, 0) = 0, \tag{5}$$

where $\varphi(x, y, t)$ is the velocity potential, $\eta(x, t)$ is the free surface elevation, and h is the depth of the fluid. The functions here are assumed defined as generalized functions as was done in [3].

To solve the problem we write φ and p in the following integral form:

$$\varphi = \sqrt{\frac{2}{\pi}} \int_0^\infty A(k, t) \cosh k(y + h) \cos kx \, dk, \tag{6a}$$

$$p(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \overline{p}(k, t) \cos kx \, dk. \tag{6b}$$

Substitution of (6) into (2) and (3) gives

$$\frac{\partial^2 A}{\partial t^2} + \sigma^2 A = -\frac{1}{\rho} \operatorname{sech} kh \frac{\partial \overline{p}}{\partial t},$$

where $\sigma^2 = gk \tanh kh$.

A solution of this equation using the initial conditions (4) gives

$$A = -\frac{1}{\rho}\operatorname{sech} kh \int_0^t \overline{p}(k, \tau)\cos\sigma(t-\tau)\, d\tau .$$

Then by Eq. (2) we obtain

$$\eta = \sqrt{\frac{2}{\pi}}\frac{1}{g\rho}\int_0^\infty \sigma\cos kx\, dk \int_0^t \overline{p}(k, \tau)\sin\sigma(t-\tau)\, d\tau . \tag{7}$$

Now, since $p(x, t)$ is nonzero only over the interval $(Vt, l+Vt)$, the inversion of (6b) gives

$$\begin{aligned}\overline{p}(k, t) &= \sqrt{\frac{2}{\pi}}\int_{Vt}^{l+Vt} p(x, t)\cos kx\, dx \\ &= \sqrt{\frac{2}{\pi}}\int_0^l p(x+Vt, t)\cos k(x+Vt)\, dx .\end{aligned}$$

Again we suppose that $p(x, t) = f(x)e^{i\omega t}$ and that the value of the function $f(x)$ depends only on the distance from the end point of the pressure region. Substitution of all these into (7) finally gives

$$\eta = \frac{2}{\pi g\rho}\int_0^\infty \sigma\cos kx\, dk \int_0^t \sin\sigma(t-\tau)e^{i\omega\tau}\, d\tau \int_0^l f(\alpha)\cos k(\alpha+V\tau)\, d\alpha .$$

After completing the integration over τ, the integral for η can be broken up into a number of integrals, each containing one or two poles. The existence of these integrals is assured by the assumption of generalized function. Thus, we get

$$4\pi\rho g\eta = \sum_{m=1}^{4}\sum_{n=1}^{4} I_{mn}, \tag{8}$$

where

$$I_{11} = \int_0^\infty F_1(k)e^{i[k(x+Vt)+\omega t]}\, dk , \qquad I_{12} = \int_0^\infty F_1(k)e^{i[-k(x-Vt)+\omega t]}\, dk ,$$

$$I_{13} = -\int_0^\infty F_1(k)e^{i(kx+\sigma t)}\, dk , \qquad I_{14} = -\int_0^\infty F_1(k)e^{i(-kx+\sigma t)}\, dk ,$$

$$I_{21} = -\int_0^\infty F_2(k)e^{i[k(x+Vt)+\omega t]}\, dk , \qquad I_{22} = -\int_0^\infty F_2(k)e^{i[-k(x-Vt)+\omega t]}\, dk ,$$

$$I_{23} = \int_0^\infty F_2(k)e^{i(kx-\sigma t)}\, dk , \qquad I_{24} = \int_0^\infty F_2(k)e^{-i(kx+\sigma t)}\, dk ,$$

$$I_{31} = -\int_0^\infty F_3(k)e^{i[-k(x+Vt)+\omega t]}\, dk , \qquad I_{32} = -\int_0^\infty F_3(k)e^{i[k(x-Vt)+\omega t]}\, dk ,$$

$$I_{33} = -\int_0^\infty F_3(k)e^{i(kx+\sigma t)}\, dk , \qquad I_{34} = \int_0^\infty F_3(k)e^{i(-kx+\sigma t)}\, dk ,$$

$$I_{41} = \int_0^\infty F_4(k)e^{i[-k(x+Vt)+\omega t]}\, dk , \qquad I_{42} = \int_0^\infty F_4(k)e^{i[k(x-Vt)+\omega t]}\, dk ,$$

$$I_{43} = -\int_0^\infty F_4(k)e^{i(kx-\sigma t)}\, dk , \qquad I_{44} = -\int_0^\infty F_4(k)e^{-i(kx+\sigma t)}\, dk ,$$

and

$$F_1(k) = \frac{\sigma f_1(k)}{kV - \sigma + \omega}, \qquad F_2(k) = \frac{\sigma f_1(k)}{kV + \sigma + \omega},$$
$$F_3(k) = \frac{\sigma f_2(k)}{kV + \sigma - \omega}, \qquad F_4(k) = \frac{\sigma f_2(k)}{kV - \sigma - \omega},$$
$$f_1(k) = \int_0^l f(x)e^{ikx}\,dx, \qquad f_2(k) = \int_0^l f(x)e^{-ikx}\,dx,$$
$$\sigma = [gk \tanh kh]^{1/2}.$$

3. Asymptotic unsteady waves. We are interested in the asymptotic waves for large values of t and x such that x/t remains finite. The dominant contributions to this asymptotic value come from the stationary point and the poles of the integrals. We note the poles are the roots of the following equations:

$$\left.\begin{aligned} \sigma &= kV + \omega, \\ \sigma &= -kV + \omega, \\ \sigma &= kV - \omega. \end{aligned}\right] \tag{9}$$

While the second and the third equations each always have one root, say α_3 and α_4, respectively, the first may have two roots, say α_1 and α_2 $(\alpha_1 < \alpha_2)$. For certain values of the parameters V, ω, and h the two roots may coincide, thereby forming the double pole. We shall elaborate on this point below. Meanwhile we denote this case by writing $V = V^*$ and note that for $V < V^*$ both α_1 and α_2 exist, and for $V > V^*$ none exist. There may be one stationary point to various integrals. This stationary point $k = \alpha_0$ is the solution of the equation

$$\sigma' = x/t. \tag{10}$$

We note that α_0 exists only when $\sqrt{gh} > x/t$. The asymptotic value of an integral containing either a pole or a stationary point can be obtained by well-known standard methods. For an integral containing both a stationary point and a pole, Bliestien's method may be conveniently applied. We consider the integral I_{14}. For $V^* > V$ and $(gh)^{1/2} > x/t$, we break it up as follows:

$$I_{14} = \frac{1}{(\alpha_1 - \alpha_2)}(J_1 - J_2), \tag{11}$$

where

$$J_1 = \int_0^\infty \frac{\sigma f_1(k)}{q(k)} \frac{e^{it(\sigma - xk/t)}}{k - \alpha_1}\,dk$$

and

$$J_2 = \int_0^\infty \frac{\sigma f_1(k)}{q(k)} \frac{e^{it(\sigma - xk/t)}}{k - \alpha_2}\,dk;$$

here

$$q(\alpha_n) = (-1)^n \frac{\sigma'(\alpha_n) - V}{\alpha_2 - \alpha_1}, \qquad n = 1, 2.$$

To evaluate J_1 the following transformation is made:

$$\sigma - \frac{x}{t}k - \left(\sigma_1 - \frac{x}{t}\alpha_1\right) = -\left(\frac{1}{2}u^2 + a_1 u\right), \tag{12}$$

where we introduce the notation

$$\sigma_n = \sigma(\alpha_n), \qquad n = 0, 1, 2, 3, 4. \tag{13}$$

Then it follows that the pole $k = \alpha_1$ and the stationary point $k = \alpha_0$ correspond respectively to $u = 0$ and $u = -a_1$. Also we get the relations

$$\frac{1}{2}a_1^2 = (\sigma_0 - \sigma_1) - \frac{x}{t}(\alpha_0 - \alpha_1), \tag{14}$$

$$\operatorname{sgn} a_1 = \operatorname{sgn}(\alpha_1 - \alpha_0). \tag{15}$$

These two relations determine the unknown parameter a_1. Then, using Bliestien's expansion, J_1 is approximately reduced to

$$\begin{aligned} J_1 &= e^{i[t(\sigma_1 - a_1^2/2) - \alpha_1 x]}\left[A_1 \int_{-\infty}^{\infty} \frac{1}{u - a_1} e^{-itu^2/2}\, du + B_1 \int_{-\infty}^{\infty} e^{-itu^2/2}\, du\right] \\ &= -\pi(1+i)A_1 \operatorname{sgn} a_1 \operatorname{cis}\left(\frac{1}{2}ta_1^2\right) e^{i(\sigma_1 t - \alpha_1 x)} \\ &\quad + B_1\left(\frac{\pi}{t}\right)^{1/2}(1-i)e^{i[t(\sigma_1 + a_1^2/2) - \alpha_1 x]}, \end{aligned}$$

where

$$A_1 = \frac{\sigma_1 f_1(\alpha_1)}{\sigma'(\alpha_1) - V}, \qquad B_1 = \frac{\sigma_0 f_1(\alpha_0)}{(\sigma_0 - \alpha_0 V - \omega)}\left\{-\frac{1}{\sigma''(\alpha_0)}\right\}^{1/2} + \frac{A_1}{a_1},$$

$$\operatorname{cis}(x) = c(x) + is(x),$$

$$[c(x), s(x)] = \frac{1}{\sqrt{2\pi}} \int_0^X [\cos x, \sin x]\frac{1}{\sqrt{x}}\, dx.$$

The integral J_2 can be similarly evaluated. The complete asymptotic values of the integral I_{14} are as follows:

$$\begin{aligned} I_{14} &= -\pi^{1/2}(1+i)\sum_{n=1}^{2}\left[\pi^{1/2}A_n \operatorname{sgn} a_n \operatorname{cis}\left(\frac{1}{2}a_n^2 t\right) e^{i(\sigma_n t - \alpha_n x)}\right. \\ &\qquad \left. + it^{-1/2}B_n e^{i\{t(\sigma_n + a_n^2/2 - \alpha_n x)\}}\right] \\ &\qquad\qquad (\text{for } \sqrt{gh} > x/t,\ V^* > V) \\ &= \pi i \sum_{n=1}^{2} A_n e^{i(\sigma_n t - \alpha_n x)} \quad (\text{for } \sqrt{gh} < x/t,\ V^* > V) \\ &= \left[\frac{2\pi}{t|\sigma_0''|}\right]^{1/2} \frac{\sigma_0 f_1(\alpha_0)}{\sigma_0 - \alpha_0 V - \omega} e^{i(\sigma_0 t - \alpha_0 x - \pi/4)} \quad (\text{for } \sqrt{gh} > x/t,\ V^* < V). \end{aligned}$$

The quantities a_2, A_2, and B_2 are defined similarly to the corresponding quantities with suffix l. All other integrals in (8) can be similarly evaluated. The far field asymptotic wave is the combination of all those asymptotic values. In the following we write down this asymptotic wave. We incorporate the Heaviside step function H to include all possible cases.

$$\begin{aligned}
4\pi\rho g\eta = &-\pi(1+i)H\left(\sqrt{gh}-\frac{x}{t}\right)\\
&\times\left[H(V^*-V)A_1\operatorname{sgn}a_1\operatorname{cis}\left(\frac{1}{2}a_1^2t\right)e^{i(\sigma_1t-\alpha_1x)}\right.\\
&\quad+H(V^*-V)A_2\operatorname{sgn}a_2\operatorname{cis}\left(\frac{1}{2}a_2^2t\right)e^{i(\sigma_2t-\alpha_2x)}\\
&\quad+A_3\operatorname{sgn}a_3\operatorname{cis}\left(\frac{1}{2}a_3^2t\right)e^{i(\sigma_3t-\alpha_3x)}\\
&\quad\left.-iA_4\operatorname{sgn}a_4\overline{\operatorname{cis}}\left(\frac{1}{2}a_4^2t\right)e^{-i(\sigma_4t-\alpha_4x)}\right]\\
&-\pi iH\left(\frac{x}{t}-\sqrt{gh}\right)[H(V^*-V)A_1e^{i(\sigma_1t-\alpha_1x)}\\
&\quad+H(V^*-V)A_2e^{i(\sigma_2t-\alpha_2x)}+A_3e^{i(\sigma_3t-\alpha_3x)}\\
&\quad-A_4e^{-i(\sigma_4t-\alpha_4x)}]\\
&+\pi i\operatorname{sgn}(x-Vt)-[H(V^*-V)A_1e^{i(\sigma_1t-\alpha_1x)}+H(V^*-V)A_2e^{i(\sigma_2t-\alpha_2x)}\\
&\quad-A_3e^{i(\sigma_3t+\alpha_3x)}-A_4e^{-i(\sigma_4t-\alpha_4x)}]\\
&+\pi iA_3[e^{i(\sigma_3t+\alpha_3x)}+e^{i(\sigma_3t-\alpha_3x)}]\\
&+H\left(\sqrt{gh}-\frac{x}{t}\right)\left(\frac{\pi}{t}\right)^{1/2}(1-i)\\
&\times\left[H(V^*-V)B_1e^{i(\sigma_1t-\alpha_1x+a_1^2t/2)}+B_2H(V^*-V)e^{i(\sigma_2t-\alpha_2x+a_2^2t/2)}\right.\\
&\quad\left.+B_3e^{i(\sigma_3t-\alpha_3x+a_3^2t/2)}+iB_4e^{-i(\sigma_4t-\alpha_4x+a_4^2t/2)}\right.\\
&+H\left(\sqrt{gh}-\frac{x}{t}\right)\left[\frac{2\pi}{t|\sigma''(\alpha_0)|}\right]^{-1/2}\left[\frac{H(V-V^*)\sigma_0f_1(\alpha_0)}{\sigma_0-\alpha_0V-\omega}\right]e^{i(\sigma_0t-\alpha_0x-\pi/4)}\\
&\quad\left.+\frac{\sigma_0f_1(\alpha_0)}{\sigma_0+\alpha_0V+\omega}e^{i(\sigma_0t-\alpha_0x-\pi/4)}\right].\qquad(16)
\end{aligned}$$

Here the quantities a_3 and a_4 are related respectively to the roots α_3 and α_4 in the same way as a_1 is related to α_1 and

$$A_3=\frac{\sigma_3f_2(\alpha_3)}{\sigma'(\alpha_3)+V},\qquad B_3=\frac{A_3}{a_3}+\frac{\sigma_0f_2(\alpha_0)}{\sigma_0+\alpha_0V-\omega}\left\{-\frac{1}{\sigma''(\alpha_0)}\right\}^{1/2},$$

$$A_4=\frac{\sigma_4f_2(\alpha_4)}{\sigma'(\alpha_4)-V},\qquad B_4=\frac{A_4}{a_4}+\frac{\sigma_0f_2(\alpha_0)}{\sigma_0-\alpha_0V+\omega}\left\{-\frac{1}{\sigma''(\alpha_0)}\right\}^{1/2}.$$

4. Steady states waves. For $t \to \infty$, the above system reduces to a steady wave system with constant amplitude and the expression for η takes a simple form. To write down this expression for η in a convenient way we introduce the following notation:

$$\eta_1 = \frac{iA_1}{2\rho g} e^{i(\sigma_1 t - \alpha_1 x)}, \qquad \eta_2 = \frac{-iA_2}{2\rho g} e^{i(\sigma_2 t - \alpha_2 x)},$$
$$\eta_3 = \frac{iA_3}{2\rho g} e^{i(\sigma_3 t + \alpha_3 x)}, \qquad \eta_3' = \frac{iA_3}{2\rho g} e^{i(\sigma_3 t - \alpha_3 x)}, \tag{17}$$
$$\eta_4 = \frac{iA_4}{2\rho g} e^{-i(\sigma_4 t - \alpha_4 x)}.$$

Then we have the following expression for the steady state value of η:

$$\begin{aligned} 2\eta = {} & -H\left(\sqrt{gh} - \frac{x}{t}\right)[H(V^* - V)\,\mathrm{sgn}(\alpha_1 - \alpha_0)\eta_1 - H(V^* - V)\,\mathrm{sgn}(\alpha_2 - \alpha_0)\eta_2 \\ & \qquad + \mathrm{sgn}(\alpha_3 - \alpha_0)\eta_3' - \mathrm{sgn}(\alpha_4 - \alpha_0)\eta_4] \\ & - H\left(\frac{x}{t} - \sqrt{gh}\right)[H(V^* - V)\eta_1 - H(V^* - V)\eta_2 + \eta_3' - \eta_4)] \\ & + \mathrm{sgn}\left(\frac{x}{t} - V\right)[H(V^* - V)\eta_1 - H(V^* - V)\eta_2 - \eta_3 - \eta_4] + (\eta_3 + \eta_3'). \end{aligned} \tag{18}$$

From the above expression it follows that the nature of the ultimate wave pattern depends upon the relative magnitude between the elements of the pairs (V, V^*), $(x/t, V)$, $(x/t, \sqrt{gh})$, and (α_n, α_0), $n = 1, \ldots, 4$. This implies that the existence and position of each of five waves represented in (17) depend upon the values of the parameters V, ω, and h and also on the ratio x/t. For given values of these parameters the distributions of the waves can be determined in the following way. First we investigate the case when the elements of each of the pairs coincide. Let us consider the case $\alpha_0 = \alpha_1$, $\alpha_0 = \alpha_2$. The condition for this occurrence can be written as

$$\sigma = kV + \omega, \qquad \sigma' = x/t. \tag{19}$$

We make the transformation $kh = \lambda$. Then (19) reduces to

$$\sigma = a\lambda + b, \qquad \sigma' = c, \tag{20}$$

where

$$a = \frac{V}{\sqrt{gh}}, \quad b = \frac{\omega\sqrt{h}}{\sqrt{g}}, \quad c = \frac{x}{t}\frac{1}{\sqrt{gh}}, \quad \sigma = (\lambda \tanh \lambda)^{1/2}.$$

Now (20) can be regarded as a surface in the (a, b, c) space where λ is a variable parameter. In fact this represents two surfaces, S_1 representing the case $\alpha_0 = \alpha_1$ and S_2 representing the case $\alpha_0 = \alpha_2$. The range of values of the parameter λ is determined by the fact that all the quantities a, b, c are positive. It is also clear that branches of the surfaces exist only in the region $c < 1$ and $a < 1$. The last condition arises from the fact that for $a > 1$, the poles α_1 and α_2 do not exist. Now we consider a section of these surfaces by a plane $a =$ constant which is shown in Fig. 1. The section of a surface S_n by a plane $a =$ constant will be denoted by the curve c_n, $n = 0, 1, 2, 3, 4$.

The portion AB represents the case $\alpha_0 = \alpha_1$ and BC represents the case $\alpha_0 = \alpha_2$. The point B separates the two cases so that $\alpha_0 = \alpha_1 = \alpha_2$ at B. In a similar way

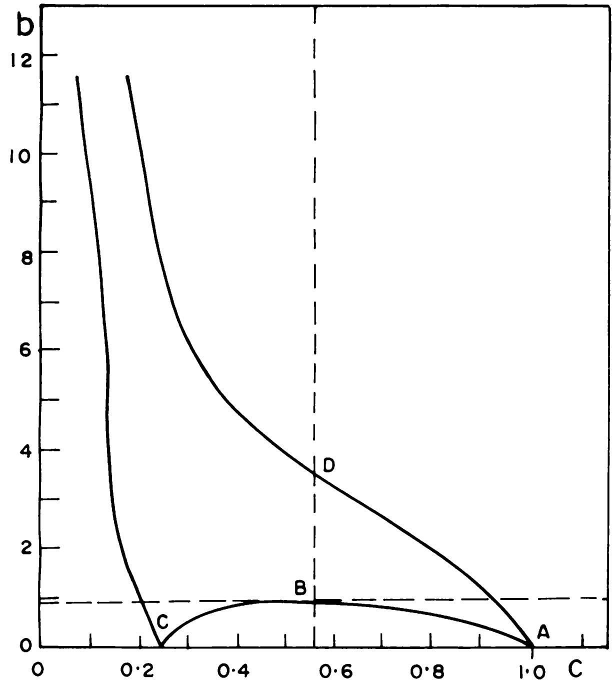

FIG. 1. Position of wave fronts for $a = 0.5$.

the conditions $\alpha_0 = \alpha_3$ and $\alpha_0 = \alpha_4$ represent the surfaces S_3 and S_4 , respectively, and their equations are

$$\sigma = -a\lambda + b, \qquad \sigma' = c, \tag{21}$$

$$\sigma = a\lambda - b, \qquad \sigma' = c. \tag{22}$$

These two surfaces exist for all values of a . For $a > 1$, the sections of these two surfaces in the (b, c) plane are shown in Fig. 2.

Let us now consider the case $V = V^*$, for which the condition is

$$\sigma = a\lambda + b, \qquad \sigma' = a. \tag{23}$$

This is again a surface S_0 in the (a, b, c) space and in fact this is a cylindrical surface with generators parallel to the c-axis. The section of this surface by a plane $c =$ constant has the constant form shown in Fig. 3. This is the curve obtained by Debnath and Rosenblat [3] though the parameters a and b are defined by them in a slightly different way. The section of S_0 by the plane $a =$ constant < 1 is a straight line, shown in Fig. 1 by the dotted line. This straight line touches the curves c_1 and c_2 at B . Now in the different regions separated by these curves the relative magnitudes between the quantities mentioned above can easily be determined. We at first consider the case in the (b, c) plane for a fixed value of a . We note that $V > V^*$ for points (b, c) above the dotted line, while the reverse takes place for (b, c) below this line. To determine the relative magnitude between α_0 and α_n , $n = 1, 2, 3, 4$, we proceed as follows. We fix the value of the quantity b and increase c from zero. Then all α_n are fixed and α_0 decreases from infinity. In

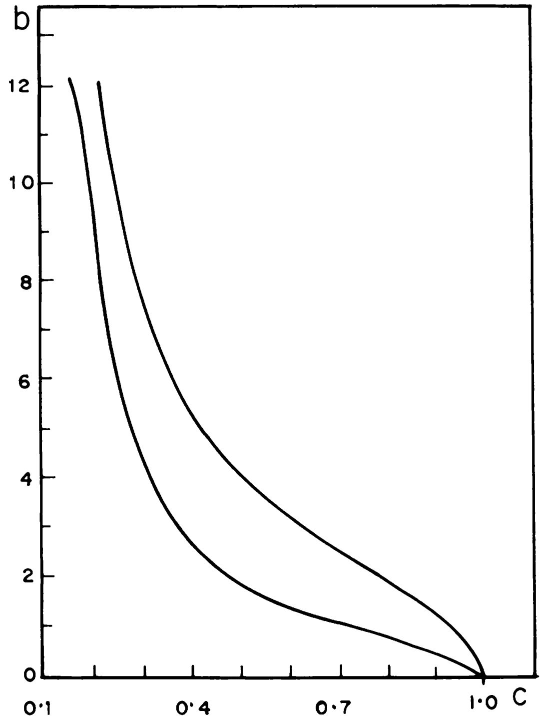

FIG. 2. Position of the wave fronts for $a = 1.5$.

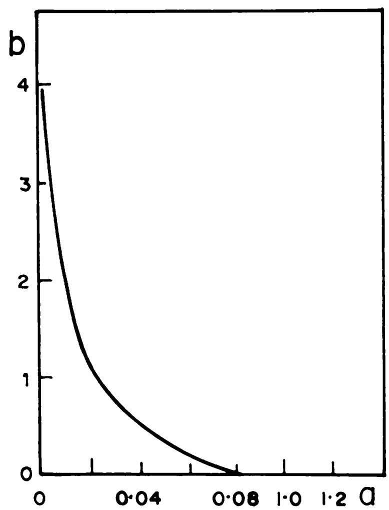

FIG. 3. The critical curve in the (a, b) plane.

this process in the (b, c) plane we first meet the curve c_4 where $\alpha_0 = \alpha_4$ and then in succession the curves c_2, c_1, and c_3 are intersected. It follows therefore that the inequality $\alpha_0 > \alpha_n$ takes place for points (b, c) to the left of the curve c_n, $n = 1, \ldots, 4$ and the reverse inequality occurs for (b, c) to the right of the corresponding curve.

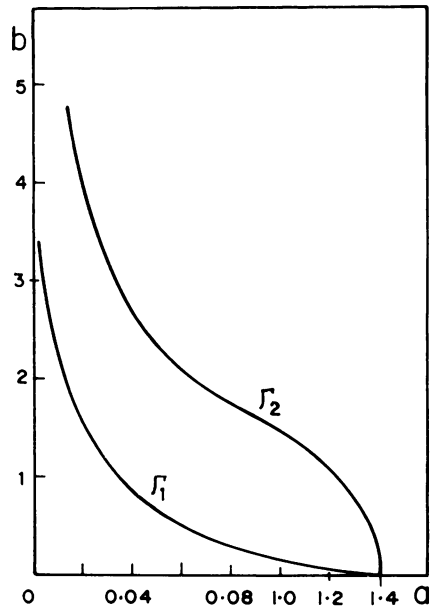

FIG. 4. The curves Γ_1 and Γ_2 in the $a = c$ plane.

Now since the above results hold for all possible values of a, the results are true in general in the (a, b, c) space. So in the above statements we are to replace "curves c_n" by "surfaces S_n." In addition to these surfaces there are two space curves which are important for our discussions. The first curve is the locus of the point B and the second is the locus of the point D for all possible values of a. Let these curves be respectively denoted by Γ_1 and Γ_2. Their equations may be written as

$$\Gamma_1: \left.\begin{aligned} a &= \sigma'(\lambda), \\ b &= \sigma(\lambda) - \lambda\sigma'(\lambda), \\ c &= \sigma'(\lambda), \end{aligned}\right] \qquad 0 \le \lambda < \infty, \tag{24}$$

$$\Gamma_2: \left.\begin{aligned} a &= \sigma'(\lambda), \\ b &= \sigma(\lambda) + \lambda\sigma'(\lambda), \\ c &= \sigma'(\lambda), \end{aligned}\right] \qquad 0 \le \lambda < \infty. \tag{25}$$

It is easy to see that Γ_1 is the curve along which the surface S_0 touches the surfaces S_1 and S_2 and both curves are in fact plane curves lying in the plane $a = c$. These curves are shown in Fig. 4 where $0A'$ is the line of intersection of the plane $a = c$ with the (a, c) plane. Now the plane $a = c$ has the physical significance that it is the plane generated by the positions of the source for all possible values of the source velocity a.

Thus the points to the left and to the right of this plane represent respectively the downstream and upstream sides. Thus the surface S_1 lies in the upstream side and the surface S_2 in the downstream side. For points (a, b) below the curve Γ_2, c_3 lies in the upstream side, while for (a, b) above Γ_2, c_3 is in the downstream side.

It is now easy to understand that the surfaces S_n physically represent four moving lines on the water surface. Let these lines be denoted by $x = x_n$, $n = 1, \ldots, 4$, and let $x = x_0$ represent the position of the source. Then the following inequalities at once follow:

$$x_4 < x_2 < x_0 < x_1 < x_3$$

for (a', b) below the curve Γ_1,

$$x_4 < x_0 < x_3$$

for (a', b) above the curve Γ_1 but below the curve Γ_2, and

$$x_4 < x_3 < x_0$$

for (a', b) above the curve Γ_2. Here a' is the distance along $0A'$.

The distribution of the five waves in the various regions in between the five lines is different. The following distributions can easily be determined from the general expression (18) with the help of the above discussions:

$$\begin{aligned} \eta &= \eta_3 + \eta_3' && \text{in } x < x_4, \\ &= \eta_3 + \eta_3' + \eta_4 && \text{in } x_4 < x < x_2, \\ &= \eta_2 + \eta_3 + \eta_3' + \eta_4 && \text{in } x_2 < x < x_0, \\ &= \eta_1 + \eta_3' && \text{in } x_0 < x < x_1, \\ &= \eta_3' && \text{in } x_1 < x < x_3, \\ &= 0 && \text{in } x_3 < x \end{aligned}$$

for (a', b) below the curve Γ_1,

$$\begin{aligned} \eta &= \eta_3 + \eta_3' && \text{in } x < x_4, \\ &= \eta_3 + \eta_3' + \eta_4 && \text{in } x_4 < x < x_0, \\ &= \eta_3' && \text{in } x_0 < x < x_3, \\ &= 0 && \text{in } x_3 < x \end{aligned}$$

for (a', b) above Γ_1 but below Γ_2, and

$$\begin{aligned} \eta &= \eta_3 + \eta_3' && \text{in } x < x_4, \\ &= \eta_3 + \eta_3' + \eta_4 && \text{in } x_4 < x < x_3, \\ &= \eta_3 + \eta_4 && \text{in } x_3 < x < x_0, \\ &= 0 && \text{in } x_0 < x \end{aligned}$$

for (a', b) above the curve Γ_2. A scrutiny of the above results at once leads us to the following conclusions. The waves η_1, η_2, η_3, and η_4 are the waves generated by the source and are the same waves found in horizontally unbounded fluid as found by Debnath and Rosenblat [3] and η_3' is the reflection of η_3 on the cliff. The surfaces S_n are nothing but the positions of four wave fronts for various values of the parameters a and b. Among them S_1, S_2, and S_4 are the wave fronts of the waves η_1, η_2, and η_4, respectively, and S_3 is the wave front of the reflected wave η_3'. Each wave front moves with the group velocity of the corresponding wave and moves in the direction of motion of the source. This is evident from the equations of representation of the surfaces S_n. There is no wave front of the wave η_3. It is the only wave that moves in the opposite direction of motion of the source and, since it has the greatest group velocity, is reflected before forming a front. Had there been no cliff, its front would be found on the other side of the cliff. For all values of the parameters the dimensionless group velocity of η_1 is greater than the dimensionless source velocity a. That is why this wave front is ahead of the source whenever η_1 exists. Similarly since the group velocity of η_2 and η_4 is always less than a, their fronts are behind the source. For points (a, b) below the curve Γ_2, the group velocity of η_3' is greater than a so that its front is ahead of the source. But for (a, b) above Γ_2, since the group velocity of η_3' is less than a, the front cannot reach the source.

It is noted that all the fronts always lie to the left of the plane $c = 1$. This follows from the fact that in a fluid of finite depth the maximum value of the dimensionless group velocity is 1 and the maximum distance covered by any front at any time from the starting point is given by $c = 1$. That is the reason for which some wave fronts exist in the upstream side for $a < 1$ while for $a > 1$ there is no front in the upstream side.

For a fixed value of $a < 1$ as b increases from zero, the group velocity η_1 decreases, remaining greater than a, and that of η_2 increases, remaining less than a, so that the two fronts gradually come close to each other. Ultimately, when b reaches the value such that the point (a, b) lies on the curve Γ_1, the group velocities of η_1 and η_2 become equal to a and the fronts coincide at the source. Then the waves η_1 and η_2, though generated by the source, cannot come out of the source. This is the cause of the singularity of the solution for these values of the parameters. As b still increases η_1 and η_2 do not exist and the group velocities of η_3 and η_4 continually decrease. For large values of b, the group velocities of η_3' and η_4 being small, their fronts remain always near the starting point at the cliff.

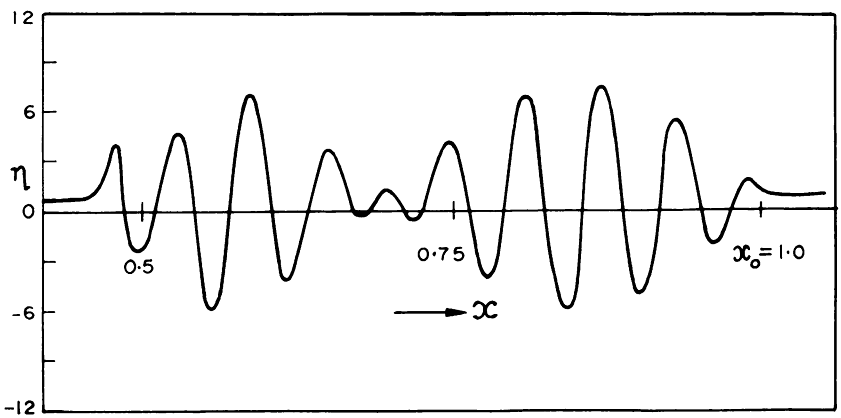

FIG. 5. Free surface elevation for $a = 0.1$, $b = 0.5$.

The free surface elevation is computed for the pressure distribution where $f(x) =$ constant $= p_0$ in $0 < x < l$. The results are expressed in dimensionless form. The wave number k is made dimensionless by the length scale h and η is made dimensionless by $p_0/(\rho g)$. Then x/h and $t(g/h)^{1/2}$ become respectively the dimensionless distance and time, and, in the expression for η, there appear the dimensionless quantities a, b, and l/h. Figure 5 shows the result for $a = 0.1$, $b = 0.5$, and $l/h = 1$. It follows from the figure that there is regular interference between the various components of the wave, and the main disturbance is confined downstream in the neighborhood of the source. In the upstream side and far downstream the disturbance is small.

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