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THE ANISOTROPIC ELASTIC SEMI-INFINITE STRIP

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Abstract. It is known that the stresses of an isotropic elastic semi-infinite strip decay exponentially at large distance x_1 from the end $x_1 = 0$ if the sides $x_2 = \pm 1$ are traction free and the loading at $x_1 = 0$ is in self-equilibrium. We study the associated problem for a general anisotropic elastic strip. Eight different side conditions at $x_2 = \pm 1$ and eight different end conditions at $x_1 = 0$ are considered. With the Stroh formalism, all these different side and end conditions are encompassed in one simple formulation. It is shown that, for certain side conditions, the loading at $x_1 = 0$ need not be in self-equilibrium. The decay factor for the strip of monoclinic materials with the plane of symmetry at $x_3 = 0$ and with the sides $x_2 = \pm 1$ being traction free is derived, and it has a remarkably simple expression. Numerical calculations of the smallest decay factor are presented.

1. Introduction. In a Cartesian coordinate system (x_1, x_2), let an isotropic elastic semi-infinite strip of width two units occupy the region

$$0 \leq x_1 < +\infty, \qquad -1 \leq x_2 \leq +1. \tag{1.1}$$

The sides of the strip $x_2 = \pm 1$ are traction free, that is,

$$\sigma_{21} = \sigma_{22} = 0 \quad \text{at } x_2 = \pm 1. \tag{1.2}$$

Under the assumption of plane stress or plane strain, if all stress and displacement components approach zero as x_1 becomes large, the classical solution for the stress is of an exponential decay form,

$$\sigma_{ij} = e^{-\lambda x_1} f_{ij}(x_2) \qquad (i, j = 1, 2), \tag{1.3}$$

where the decay exponent λ is determined by the eigenequation

$$\sin 2\lambda = \pm 2\lambda. \tag{1.4}$$

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Related problems for the isotropic elastic semi-infinite strip have been considered by Papkovich [1], Fadle [2, 3] and Johnson and Little [4]. Extensions to certain anisotropic elastic strips have been carried out by Horgan and his coworkers [5–9].

In this paper we consider the associated problem for the semi-infinite strip of general anisotropic materials. For anisotropic materials the in-plane and anti-plane displacements are in general coupled and we have to consider all three displacements. We shall use the three-dimensional Cartesian coordinate system (x_1, x_2, x_3) or (x, y, z), and assume that the displacement vector (u_1, u_2, u_3) depends on x_1 and x_2 only. The side conditions are, in addition to (1.2),

$$\sigma_{23} = 0 \quad \text{at } x_2 = \pm 1 . \tag{1.5}$$

In fact we will consider eight different side conditions, one of which is (1.2) and (1.5). Coupled with eight end conditions prescribed at $x = 0$, there are a total of 64 different boundary conditions. As we will see in the paper a simple formulation based on the Stroh formalism [10–20] encompasses all 64 boundary conditions with little additional effort.

2. Basic equations. Let u_i, σ_{ij} be the displacement and the stress, respectively, of a homogeneous anisotropic elastic body. The equations governing the equilibrium and the stress-strain laws can be written as

$$\sigma_{ij,j} = 0, \tag{2.1}$$

$$\sigma_{ij} = C_{ijks} u_{k,s} . \tag{2.2}$$

In the above, a comma stands for differentiation, repeated indices imply summation and C_{ijks} are the elasticity constants which are assumed to be fully symmetric and positive definite so that the strain energy is positive.

If the displacement u_k depends on x_1 and x_2 only, so does the stress. To satisfy (2.1) the stress function $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ may be introduced such that

$$\sigma_{1i} = -\phi_{i,2} \quad \sigma_{2i} = \phi_{i,1} \qquad (i = 1, 2, 3). \tag{2.3}$$

Substituting (2.2) into (2.3) yields

$$\begin{cases} \mathbf{Q}\mathbf{u}_{,1} + \mathbf{R}\mathbf{u}_{,2} = -\boldsymbol{\phi}_{,2}, \\ \mathbf{R}^{\mathrm{T}}\mathbf{u}_{,1} + \mathbf{T}\mathbf{u}_{,2} = \boldsymbol{\phi}_{,1}, \end{cases} \tag{2.4}$$

where $\mathbf{u}$ is the displacement vector whose elements are (u_1, u_2, u_3) and the components of the 3×3 matrices $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$ are

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2} . \tag{2.5}$$

We see that $\mathbf{Q}$ and $\mathbf{T}$ are symmetric and positive definite.

Equations (2.4) can be rewritten as [15]

$$\frac{\partial \mathbf{w}}{\partial y} = \mathbf{N}\frac{\partial \mathbf{w}}{\partial x}, \tag{2.6}$$

where

$$\mathbf{w} = \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\phi} \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^{\mathrm{T}} \end{bmatrix}, \tag{2.7}$$

$$\begin{cases} \mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}, & \mathbf{N}_2 = \mathbf{T}^{-1} = \mathbf{N}_2^{\mathrm{T}}, \\ \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} - \mathbf{Q} = \mathbf{N}_3^{\mathrm{T}}. \end{cases} \tag{2.8}$$

The 6×6 matrix $\mathbf{N}$ is not symmetric but the matrix $\mathbf{JN}$ is, i.e.,

$$(\mathbf{JN})^{\mathrm{T}} = \mathbf{JN}, \tag{2.9}$$

or

$$\mathbf{N}^{\mathrm{T}}\mathbf{J} = \mathbf{JN} \tag{2.10}$$

where $\mathbf{J}$ is the 6×6 matrix

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{2.11}$$

and $\mathbf{I}$ is the 3×3 unit matrix.

Consider the following eigenrelation

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}. \tag{2.12}$$

There are six eigenvalues p_a and six eigenvectors $\boldsymbol{\xi}_\alpha$. Since p cannot be real if the strain energy is positive [10], we let

$$\mathrm{Im}(p_\alpha) > 0, \quad p_{\alpha+3} = \overline{p}_\alpha, \quad \boldsymbol{\xi}_{\alpha+3} = \overline{\boldsymbol{\xi}}_\alpha \qquad (\alpha = 1, 2, 3), \tag{2.13}$$

where Im denotes the imaginary part and the overbar stands for the complex conjugate. We assume that $\mathbf{N}$ is simple or semisimple so that the $\boldsymbol{\xi}_\alpha$ span a six-dimensional space. A modified solution for nonsemisimple $\mathbf{N}$ can be found in [16, 18]. Writing the 6-vector $\boldsymbol{\xi}_\alpha$ as two 3-vectors $\mathbf{a}_\alpha$, $\mathbf{b}_\alpha$:

$$\boldsymbol{\xi}_\alpha = \begin{bmatrix} \mathbf{a}_\alpha \\ \mathbf{b}_\alpha \end{bmatrix} \qquad (\alpha = 1, 2, 3), \tag{2.14}$$

the eigenrelation (2.12) for the six eigenvalues and the six eigenvectors is

$$\mathbf{N}\begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix}\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{P}} \end{bmatrix}, \tag{2.15}$$

where

$$\begin{cases} \mathbf{P} = \mathrm{diag}(p_1, p_2, p_3), \\ \mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \qquad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]. \end{cases} \tag{2.16}$$

Using the following orthogonality relation [14, 15, 17],

$$\begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix}\begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \overline{\mathbf{B}}^{\mathrm{T}} & \overline{\mathbf{A}}^{\mathrm{T}} \end{bmatrix} = \mathbf{I} = \begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \overline{\mathbf{B}}^{\mathrm{T}} & \overline{\mathbf{A}}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix}, \tag{2.17}$$

(2.15) leads to

$$\mathbf{N} = \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix}\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{P}} \end{bmatrix}\begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \overline{\mathbf{B}}^{\mathrm{T}} & \overline{\mathbf{A}}^{\mathrm{T}} \end{bmatrix}$$

and

$$e^{\lambda \mathbf{N}} = \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix}\begin{bmatrix} \langle e^{\lambda p} \rangle & \mathbf{0} \\ \mathbf{0} & \langle e^{\lambda \overline{p}} \rangle \end{bmatrix}\begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \overline{\mathbf{B}}^{\mathrm{T}} & \overline{\mathbf{A}}^{\mathrm{T}} \end{bmatrix}, \tag{2.18}$$

where

$$\begin{cases} \langle e^{\lambda p} \rangle = \mathrm{diag}(e^{\lambda p_1}, e^{\lambda p_2}, e^{\lambda p_3}), \\ \langle e^{\lambda \overline{p}} \rangle = \mathrm{diag}(e^{\lambda \overline{p}_1}, e^{\lambda \overline{p}_2}, e^{\lambda \overline{p}_3}). \end{cases} \tag{2.19}$$

If the 6×6 matrix $e^{\lambda \mathbf{N}}$ is partitioned into four 3×3 matrices as

$$e^{\lambda \mathbf{N}} = \begin{bmatrix} (e^{\lambda \mathbf{N}})_1 & (e^{\lambda \mathbf{N}})_2 \\ (e^{\lambda \mathbf{N}})_3 & (e^{\lambda \mathbf{N}})_1^{\mathrm{T}} \end{bmatrix}, \tag{2.20}$$

it is clear that

$$\begin{cases} (e^{\lambda \mathbf{N}})_1 = \mathbf{A}\langle e^{\lambda p}\rangle \mathbf{B}^{\mathrm{T}} + \overline{\mathbf{A}}\langle e^{\lambda \bar{p}}\rangle \overline{\mathbf{B}}^{\mathrm{T}}, \\ (e^{\lambda \mathbf{N}})_2 = \mathbf{A}\langle e^{\lambda p}\rangle \mathbf{A}^{\mathrm{T}} + \overline{\mathbf{A}}\langle e^{\lambda \bar{p}}\rangle \overline{\mathbf{A}}^{\mathrm{T}}, \\ (e^{\lambda \mathbf{N}})_3 = \mathbf{B}\langle e^{\lambda p}\rangle \mathbf{B}^{\mathrm{T}} + \overline{\mathbf{B}}\langle e^{\lambda \bar{p}}\rangle \overline{\mathbf{B}}^{\mathrm{T}}. \end{cases} \tag{2.21}$$

3. The eigenfunctions $\mathbf{w}^{(k)}$. For the semi-infinite strip subject to a self-equilibrated loading at $x = 0$ the stress is of negligible magnitude at distances x which are large compared with the width of the strip according to Saint-Venant's principle [5–9]. We therefore choose the solutions of (2.6) in the following form:

$$\mathbf{w}(x, y) = \sum_{k=1}^{\infty} C_k e^{-\lambda_k x} \mathbf{w}^{(k)}(y), \tag{3.1}$$

where C_k and λ_k, with $\mathrm{Re}\,\lambda_k > 0$ and Re denoting the real part, are complex constants to be determined as are the eigenfunctions $\mathbf{w}^{(k)}$. Substituting the series (3.1) into (2.6) leads to the following ordinary differential equation for the eigenfunctions $\mathbf{w}^{(k)}$:

$$\frac{d}{dy}\mathbf{w}^{(k)} = -\lambda_k \mathbf{N}\mathbf{w}^{(k)}. \tag{3.2}$$

A general solution of (3.2) is

$$\mathbf{w}^{(k)}(y) = e^{-\lambda_k p y} \boldsymbol{\xi}, \tag{3.3}$$

where p and $\boldsymbol{\xi}$ are, respectively, an eigenvalue and eigenvector of $\mathbf{N}$ in (2.12).

Using the notation of (2.16) and (2.19), the general solution obtained by a linear combination of six solutions of (3.3) associated with six p's can be written as

$$\begin{bmatrix} \mathbf{u}^{(k)} \\ \phi^{(k)} \end{bmatrix} = \mathbf{w}^{(k)} = \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \langle e^{-\lambda_k p y}\rangle & \mathbf{0} \\ \mathbf{0} & \langle e^{-\lambda_k \bar{p} y}\rangle \end{bmatrix} \begin{bmatrix} \mathbf{q}_k \\ \mathbf{h}_k \end{bmatrix} \tag{3.4}$$

in which $\mathbf{q}_k$ and $\mathbf{h}_k$ are 3×1 constant matrices to be determined by the side conditions at $y = \pm 1$.

For the side conditions at $y = \pm 1$ consider one of the following eight conditions:

$$\sigma_{21} = 0, \quad \sigma_{22} = 0, \quad \sigma_{23} = 0. \tag{3.5a}$$

$$\sigma_{21} = 0, \quad \sigma_{22} = 0, \quad u_3 = 0. \tag{3.5b}$$

$$\sigma_{21} = 0, \quad u_2 = 0, \quad \sigma_{23} = 0. \tag{3.5c}$$

$$\sigma_{21} = 0, \quad u_2 = 0, \quad u_3 = 0. \tag{3.5d}$$

$$u_1 = 0, \quad \sigma_{22} = 0, \quad \sigma_{23} = 0. \tag{3.5e}$$

$$u_1 = 0, \quad \sigma_{22} = 0, \quad u_3 = 0. \tag{3.5f}$$

$$u_1 = 0, \quad u_2 = 0, \quad \sigma_{23} = 0. \tag{3.5g}$$

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0. \tag{3.5h}$$

Conditions (3.5a) are identical to (1.2) and (1.5). From $(2.3)_2$, $\sigma_{2i} = 0$ at $x_2 = \pm 1$ implies that $\phi_i = \text{const}$ at $x_2 = \pm 1$. Let T_i be the total traction at $x_1 = 0$, i.e.,

$$T_i = \int_{-1}^{+1} \sigma_{1i}(0, x_2)\, dx_2 . \tag{3.6}$$

If T_i vanishes, it follows from $(2.3)_1$ that

$$\phi_i(0, 1) - \phi_i(0, -1) = 0 .$$

Since $\phi_i = \text{const}$ at $x_2 = \pm 1$, without loss in generality we set

$$\phi_i = 0 \quad \text{at } x_2 = \pm 1 ,$$

whenever $\sigma_{2i} = 0$ at $x_2 = \pm 1$. Thus σ_{21}, σ_{22}, σ_{23} in (3.5) can be replaced, respectively, by ϕ_1, ϕ_2, ϕ_3 and the eight different side conditions in (3.5) are rewritten as

$$u_1\phi_1 = 0, \quad u_2\phi_2 = 0, \quad u_3\phi_3 = 0 \qquad \text{at } y = \pm 1 . \tag{3.7}$$

In terms of the vectors $\mathbf{u}$ and $\boldsymbol{\phi}$, we may write (3.7) as

$$\mathbf{I}_u\mathbf{u} + \mathbf{I}_\phi\boldsymbol{\phi} = \mathbf{0} \quad \text{at } \; y = \pm 1 , \tag{3.8}$$

where

$$\mathbf{I}_u + \mathbf{I}_\phi = \mathbf{I} \tag{3.9}$$

and $\mathbf{I}_u$, $\mathbf{I}_\phi$ are 3×3 diagonal matrices whose diagonal elements are either zero or one. The special cases (3.5a) and (3.5h) correspond, respectively, to $\mathbf{I}_u = \mathbf{0}$, $\mathbf{I}_\phi = \mathbf{I}$ and $\mathbf{I}_\phi = \mathbf{0}$, $\mathbf{I}_u = \mathbf{I}$. It is readily shown that

$$\mathbf{I}_u\mathbf{I}_u = \mathbf{I}_u , \quad \mathbf{I}_\phi\mathbf{I}_\phi = \mathbf{I}_\phi , \quad \mathbf{I}_u\mathbf{I}_\phi = \mathbf{0} . \tag{3.10}$$

Substituting (3.4) into (3.8) yields

$$\begin{cases} \mathbf{K}\langle e^{-\lambda_k p}\rangle \mathbf{q}_k + \overline{\mathbf{K}}\langle e^{-\lambda_k \bar{p}}\rangle \mathbf{h}_k = \mathbf{0} , \\ \mathbf{K}\langle e^{\lambda_k p}\rangle \mathbf{q}_k + \overline{\mathbf{K}}\langle e^{\lambda_k \bar{p}}\rangle \mathbf{h}_k = \mathbf{0} , \end{cases} \tag{3.11}$$

where

$$\mathbf{K} = \mathbf{I}_u\mathbf{A} + \mathbf{I}_\phi\mathbf{B} . \tag{3.12}$$

For the special cases (3.5a) and (3.5h), $\mathbf{K}$ reduces to $\mathbf{B}$ and $\mathbf{A}$, respectively. It is shown in the Appendix that $\mathbf{K}$ is nonsingular and hence $\mathbf{K}^{-1}$ exists. Eliminating $\mathbf{q}_k$ in (3.11) leads to

$$\{\langle e^{2\lambda_k p}\rangle \mathbf{K}^{-1}\overline{\mathbf{K}} - \mathbf{K}^{-1}\overline{\mathbf{K}}\langle e^{2\lambda_k \bar{p}}\rangle\}\langle e^{-2\lambda_k \bar{p}}\rangle \mathbf{h}_k = \mathbf{0} .$$

Hence λ_k is a root of the determinant

$$\|\langle e^{2\lambda p}\rangle K^{-1}\overline{\mathbf{K}} - \mathbf{K}^{-1}\overline{\mathbf{K}}\langle e^{2\lambda \bar{p}}\rangle\| = 0 \tag{3.13}$$

and $\mathbf{q}_k$, $\mathbf{h}_k$ are determined from (3.11).

Rewriting (3.13) as

$$\|\mathbf{K}\langle e^{2\lambda p}\rangle \mathbf{K}^{-1} - \overline{\mathbf{K}}\langle e^{2\lambda \bar{p}}\rangle \overline{\mathbf{K}}^{-1}\| = 0 ,$$

or, since $\mathbf{K}\mathbf{K}^{\mathrm{T}}$ is purely imaginary (see the Appendix),

$$\|(\mathbf{K}\langle e^{2\lambda p}\rangle\mathbf{K}^{\mathrm{T}} + \overline{\mathbf{K}}\langle e^{2\lambda\overline{p}}\rangle\overline{\mathbf{K}}^{\mathrm{T}})(\mathbf{K}\mathbf{K}^{\mathrm{T}})^{-1}\| = 0,$$

we obtain from (2.21) the results that λ_k is a root of

$$\|(e^{2\lambda\mathbf{N}})_3\| = 0 \quad \text{and} \quad \|(e^{2\lambda\mathbf{N}})_2\| = 0$$

for the special cases (3.5a) and (3.5h), respectively.

Equations (3.11) remain the same if λ_k is replaced by $-\lambda_k$. Thus if λ_k is a root so is $-\lambda_k$. Denoting by $\mathbf{w}^{(-k)}$ the eigenfunction associated with $-\lambda_k$, we have from (3.2) and (3.4),

$$\frac{d}{dy}\mathbf{w}^{(-k)} = \lambda_k\mathbf{N}\mathbf{w}^{(-k)}, \tag{3.14}$$

$$\begin{bmatrix}\mathbf{u}^{(-k)}\\ \phi^{(-k)}\end{bmatrix} = \mathbf{w}^{(-k)} = \begin{bmatrix}\mathbf{A} & \overline{\mathbf{A}}\\ \mathbf{B} & \overline{\mathbf{B}}\end{bmatrix}\begin{bmatrix}\langle e^{\lambda_k p y}\rangle & \mathbf{0}\\ \mathbf{0} & \langle e^{\lambda_k \overline{p} y}\rangle\end{bmatrix}\begin{bmatrix}\mathbf{q}_k\\ \mathbf{h}_k\end{bmatrix}. \tag{3.15}$$

By taking the complex conjugate of (3.11), it is easily shown that if λ_k is a complex root so is its complex conjugate $\overline{\lambda}_k$. Moreover, $\mathbf{q}_k$, $\mathbf{h}_k$ for $\overline{\lambda}_k$ are identical to $\overline{\mathbf{h}}_k$, $\overline{\mathbf{q}}_k$ for λ_k, respectively. In employing the series solutions of (3.1), $\mathbf{w}^{(k)}$ associated with λ_k and $\overline{\lambda}_k$ should be considered as two independent solutions. The two solutions are complex conjugates of each other, assuring us that $\mathbf{w}(x, y)$ is real.

4. Orthogonality of eigenfunctions $\mathbf{w}^{(k)}$. From (3.2), (3.14), and (2.10),

$$\begin{aligned}\frac{d}{dy}\{(\mathbf{w}^{(-m)})^{\mathrm{T}}\mathbf{J}\mathbf{w}^{(k)}\} &= \left(\frac{d}{dy}\mathbf{w}^{(-m)}\right)^{\mathrm{T}}\mathbf{J}\mathbf{w}^{(k)} + (\mathbf{w}^{(-m)})^{\mathrm{T}}\mathbf{J}\left(\frac{d}{dy}\mathbf{w}^{(k)}\right)\\ &= (\lambda_m - \lambda_k)(\mathbf{w}^{(-m)})^{\mathrm{T}}\mathbf{J}\mathbf{N}\mathbf{w}^{(k)}.\end{aligned}$$

When both sides are integrated from $y = -1$ to $y = 1$,

$$\{\mathbf{u}^{(-m)^{\mathrm{T}}}\phi^{(k)} + \phi^{(-m)^{\mathrm{T}}}\mathbf{u}^{(k)}\}_{-1}^{+1} = (\lambda_m - \lambda_k)\int_{-1}^{1}(\mathbf{w}^{(-m)})^{\mathrm{T}}\mathbf{J}\mathbf{N}\mathbf{w}^{(k)}\,dy.$$

The left-hand side vanishes due to (3.7) and we have

$$\int_{-1}^{1}(\mathbf{w}^{(-m)})^{\mathrm{T}}\mathbf{J}\mathbf{N}\mathbf{w}^{(k)}\,dy = \begin{cases}0 & \text{if } \lambda_k \neq \lambda_m,\\ J_k & \text{if } \lambda_k = \lambda_m.\end{cases} \tag{4.1}$$

This is the orthogonality relation for the eigenfunctions $\mathbf{w}^{(k)}$.

The value of J_k can be determined as follows. From (3.4), (3.15), (2.15), (2.17), and the fact that the product of two diagonal matrices commutes, the integrand on the left of (4.1) when $k = m$ can be shown to be

$$(\mathbf{w}^{(k)})^{\mathrm{T}}\mathbf{J}\mathbf{N}\mathbf{w}^{(k)} = \mathbf{q}_k^{\mathrm{T}}\mathbf{P}\mathbf{q}_k + \mathbf{h}_k^{\mathrm{T}}\overline{\mathbf{P}}\mathbf{h}_k.$$

This is independent of y and hence

$$J_k = 2\{\mathbf{q}_k^{\mathrm{T}}\mathbf{P}\mathbf{q}_k + \mathbf{h}_k^{\mathrm{T}}\overline{\mathbf{P}}\mathbf{h}_k\}. \tag{4.2}$$

Since $(\mathbf{q}_k, \mathbf{h}_k)$ is unique up to an arbitrary multiplicative constant, we may choose the constant such that $J_k = 1$.

5. The series solution. Equation (3.1) for $x = 0$ is

$$\mathbf{w}(0, y) = \sum_{k=1}^{\infty} C_k \mathbf{w}^{(k)}(y). \tag{5.1}$$

At the end $x = 0$, only one function each from the following three pairs,

$$(u_1(0, y), \sigma_{11}(0, y)), \quad (u_2(0, y), \sigma_{12}(0, y)), \quad (u_3(0, y), \sigma_{13}(0, y)), \tag{5.2}$$

may be prescribed. Since

$$\phi_i = -\int_{-1}^{y} \sigma_{1i}(0, \eta)\, d\eta,$$

we may replace $\sigma_{11}(0, y)$, $\sigma_{12}(0, y)$, $\sigma_{13}(0, y)$ in (5.2) by $\phi_1(0, y)$, $\phi_2(0, y)$, $\phi_3(0, y)$, respectively. Thus the three prescribed end conditions can be one of the following eight possibilities:

$$\phi_1(0, y), \quad \phi_2(0, y), \quad \phi_3(0, y). \tag{5.2a}$$
$$\phi_1(0, y), \quad \phi_2(0, y), \quad u_3(0, y). \tag{5.2b}$$
$$\phi_1(0, y), \quad u_2(0, y), \quad \phi_3(0, y). \tag{5.2c}$$
$$\phi_1(0, y), \quad u_2(0, y), \quad u_3(0, y). \tag{5.2d}$$
$$u_1(0, y), \quad \phi_2(0, y), \quad \phi_3(0, y). \tag{5.2e}$$
$$u_1(0, y), \quad \phi_2(0, y), \quad u_3(0, y). \tag{5.2f}$$
$$u_1(0, y), \quad u_2(0, y), \quad \phi_3(0, y). \tag{5.2g}$$
$$u_1(0, y), \quad u_2(0, y), \quad u_3(0, y). \tag{5.2h}$$

Let $\mathbf{I}_u^o$, $\mathbf{I}_\phi^o$ be 3×3 diagonal matrices whose diagonal elements are either zero or one and satisfy (3.9), i.e.,

$$\mathbf{I}_u^o + \mathbf{I}_\phi^o = \mathbf{I}. \tag{5.3}$$

By properly choosing $\mathbf{I}_u^o$, $\mathbf{I}_\phi^o$, any of the eight prescribed end conditions can be given by

$$\mathbf{I}_u^o \mathbf{u}(0, y) \quad \text{and} \quad \mathbf{I}_\phi^o \boldsymbol{\phi}(0, y).$$

The unknown end conditions are

$$\mathbf{I}_\phi^o \mathbf{u}(0, y) \quad \text{and} \quad \mathbf{I}_u^o \boldsymbol{\phi}(0, y) \tag{5.4}$$

which can be replaced in terms of the eigenfunctions $\mathbf{w}^{(k)}(y)$. Introducing the 6×6 matrix

$$\mathbf{I}^o = \begin{bmatrix} \mathbf{I}_u^o & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_\phi^o \end{bmatrix}$$

we have

$$\begin{aligned} \mathbf{w}(0, y) &= \mathbf{I}^o \mathbf{w}(0, y) + (\mathbf{I} - \mathbf{I}^o)\mathbf{w}(0, y) \\ &= \mathbf{I}^o \mathbf{w}(0, y) + \sum_{k=1}^{\infty} C_k \{\mathbf{w}^{(k)}(y) - \mathbf{I}^o \mathbf{w}^{(k)}(y)\}. \end{aligned} \tag{5.5}$$

From the orthogonality relation (4.1), (5.1) leads to

$$\int_{-1}^{1} (\mathbf{w}^{(-m)}(y))^{\mathrm{T}} \mathbf{J}\mathbf{N}\mathbf{w}(0, y)\, dy = C_m J_m.$$

Substitution of (5.5) into the above yields the following equations for C_m:

$$\sum_{k=1}^{\infty} \gamma_{mk} C_k = \Delta_m \qquad (m = 1, 2, \ldots) \tag{5.6}$$

where

$$\begin{cases} \gamma_{mk} = \int_{-1}^{1} (\mathbf{w}^{(-m)}(y))^{\mathrm{T}} \mathbf{J N I}^{o} \mathbf{w}^{(k)}(y)\, dy, \\ \Delta_m = \int_{-1}^{1} (\mathbf{w}^{(-m)}(y))^{\mathrm{T}} \mathbf{J N I}^{o} \mathbf{w}(0, y)\, dy. \end{cases} \tag{5.7}$$

Equations (5.6) are an infinite system of simultaneous equations for C_m. Approximate solutions may be obtained by truncating (5.6) to a finite number of equations and C_m.

6. Explicit solutions for C_m. The constants C_m in the series solution can be determined explicitly for certain anisotropic materials and end conditions which satisfy the relation

$$\mathbf{N I}^{o} + \mathbf{I}^{o} \mathbf{N} = \mathbf{N}. \tag{6.1}$$

When (6.1) holds, γ_{mk} of $(5.7)_1$ is

$$\gamma_{mk} = \int_{-1}^{1} (\mathbf{w}^{(-m)}(y))^{\mathrm{T}} \mathbf{J N w}^{(k)}(y)\, dy - \int_{-1}^{1} (\mathbf{w}^{(-m)}(y))^{\mathrm{T}} \mathbf{J I}^{o} \mathbf{N w}^{(k)}(y)\, dy.$$

With $\mathbf{Nw}^{(k)}$ in the second integral replaced in terms of $d\mathbf{w}^{(k)}/dy$ by (3.2), integration by parts, and use of (3.14) and (3.7) yields

$$\left(1 + \frac{\lambda_m}{\lambda_k}\right) \gamma_{mk} = \int_{-1}^{1} (\mathbf{w}^{(-m)}(y))^{\mathrm{T}} \mathbf{J N w}^{(k)}(y)\, dy.$$

In view of (4.1),

$$\gamma_{mk} = \begin{cases} 0 & \text{if } \lambda_k \neq \lambda_m, \\ \dfrac{J_k}{2} & \text{if } \lambda_k = \lambda_m, \end{cases} \tag{6.2}$$

and (5.6) has the explicit solution

$$C_m = 2 \frac{\Delta}{J_m}. \tag{6.3}$$

For the end conditions (5.2d) and (5.2e), it is readily shown that (6.1) is satisfied if $\mathbf{N}$ has the structure

$$\mathbf{N} = \begin{bmatrix} 0 & * & * & * & 0 & 0 \\ * & 0 & 0 & 0 & * & * \\ * & 0 & 0 & 0 & * & * \\ * & 0 & 0 & 0 & * & * \\ 0 & * & * & * & 0 & 0 \\ 0 & * & * & * & 0 & 0 \end{bmatrix}$$

where the $*$ denotes a possibly nonzero element. Monoclinic materials with the symmetry plane at $x = 0$ belong to this class.

For the end conditions (5.2c) and (5.2f), (6.1) is satisfied if

$$\mathbf{N} = \begin{bmatrix} 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 \\ 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 \\ 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 \end{bmatrix} .$$

Monoclinic materials with the symmetry plane at $y = 0$ belong to this class.

As to the remaining end conditions (5.2a), (5.2b), (5.2g), and (5.2h), it can be shown that no real materials satisfy (6.1). For these end conditions, (6.1) yields $N_{12} = 0$, which contradicts the fact that $N_{12} = -1$ for all materials [17].

The above discussion shows that explicit solutions for C_m and hence for the series solution (3.1) can be obtained for the end conditions (5.2d, e) or (5.2c, f) if the material has a symmetry plane at $x = 0$ or $y = 0$, respectively. Explicit solutions have not been found for the end conditions (5.2a, b) and (5.2g, h) for any materials, not even for isotropic materials [4].

7. Discussion of the end conditions. The discussion before and after (3.6) indicates that the total traction T_i at $x_1 = 0$ must vanish if $\sigma_{2i} = 0$ is prescribed at the sides $x_2 = \pm 1$. Thus for the side conditions (3.5a) all three tractions T_1, T_2, T_3 must vanish while for (3.5d) only T_1 is required to vanish. As to the side conditions (3.5h) all three T_i can be nonzero. The stress still decays exponentially.

The vanishing of all three T_i for the side conditions (3.5a) which are equivalent to

$$\phi_1 = 0, \quad \phi_2 = 0, \quad \phi_3 = 0, \quad \text{at } x_2 = \pm 1, \tag{7.1}$$

is not sufficient for the stress to decay exponentially. The moment M_3 due to traction σ_{11} about the x_3-axis must also vanish. To see this, from $(2.3)_1$,

$$\begin{aligned} M_3 &= \int_{-1}^{+1} x_2 \sigma_{11}\, dx_2 = -\int_{-1}^{+1} x_2 \phi_{1,2}\, dx_2 \\ &= -x_2\phi_1 \Big|_{-1}^{+1} + \int_{-1}^{+1} \phi_1\, dx_2 . \end{aligned}$$

The first term vanishes due to (7.1). The second term is, using (3.4),

$$M_3 = -\sum_k \frac{C_k}{\lambda_k} \sum_{\alpha=1}^{3} \left\{ B_{1\alpha} \frac{1}{p_\alpha} e^{-\lambda_k p_\alpha x_2} q_{k\alpha} + \overline{B}_{1\alpha} \frac{1}{\overline{p}_\alpha} e^{-\lambda_k \overline{p}_\alpha x_2} h_{k\alpha} \right\}_{-1}^{+1} .$$

Since $B_{1\alpha} = -p_\alpha B_{2\alpha}$ [11], again using (3.4) and (7.1),

$$M_3 = -\sum_k \frac{C_k}{\lambda_k} \phi_2^{(k)} \Big|_{-1}^{+1} = 0 .$$

Thus the moment M_3 about the x_3-axis must vanish. This means that the traction σ_{11} at $x_1 = 0$ cannot be prescribed arbitrarily. It must be in self-equilibrium, in agreement with the Saint-Venant principle.

For general anisotropic materials there is another moment M_1 about the x_1-axis which may not vanish,

$$M_1 = \int_{-1}^{+1} x_2\sigma_{13}\, dx_2 = \int_{-1}^{+1} x_2\phi_{3,2}\, dx_2$$
$$= x_2\phi_3\Big|_{-1}^{+1} - \int_{-1}^{+1} \phi_3\, dx_2 .$$

The first term vanishes in view of (7.1) but the second term is in general nonzero. Nevertheless the solution still decays exponentially. Imposition of $M_1 = 0$ may lead to nonexistence of a solution. A similar situation was found for anisotropic elastic wedges subject to a concentrated couple [19]. It should be pointed out that nonzero M_1 does not imply nonequilibrium of the strip. If the strip of height $-1 \le x_3 \le +1$ is considered, the nonzero M_1 due to σ_{13} at $x_1 = 0$ is balanced by the moment due to σ_{32} at $x_3 = \pm 1$.

8. Monoclinic materials with the symmetry plane at $x_3 = 0$. In this section we determine the exponential decay factor λ for monoclinic materials with the symmetry plane at $x_3 = 0$ subject to the side conditions (3.5a) or (7.1). The objective therefore is to find λ of (3.13) with **K** there replaced by **B**. As was discussed in the last two paragraphs in Sec. 3, if λ is a root of (3.13) so is $-\lambda$ and $\pm\bar{\lambda}$. It suffices therefore to find λ for which

$$\operatorname{Re}\lambda > 0, \qquad \operatorname{Im}\lambda > 0. \tag{8.1}$$

In particular, we are interested in the smallest $\operatorname{Re}\lambda$ that provides the slowest exponential decay.

For monoclinic materials with the symmetry plane at $x_3 = 0$,

$$\mathbf{B} = \begin{bmatrix} -k_1p_1 & -k_2p_2 & 0 \\ k_1 & k_2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{B}^{-1} = \frac{1}{p_1 - p_2}\begin{bmatrix} -k_1^{-1} & -p_2k_1^{-1} & 0 \\ k_2^{-1} & p_1k_2^{-1} & 0 \\ 0 & 0 & p_1 - p_2 \end{bmatrix},$$

where k_1 and k_2 are the normalization constants [20]. Hence

$$\mathbf{B}^{-1}\overline{\mathbf{B}} = \frac{1}{p_1 - p_2}\begin{bmatrix} k_1^{-1}\bar{k}_1(\bar{p}_1 - p_2) & k_1^{-1}\bar{k}_2(\bar{p}_2 - p_2) & 0 \\ k_2^{-1}\bar{k}_1(p_1 - \bar{p}_1) & k_2^{-1}\bar{k}_2(p_1 - \bar{p}_2) & 0 \\ 0 & 0 & p_1 - p_2 \end{bmatrix}.$$

It is shown in [21] that $i\mathbf{B}^{-1}\overline{\mathbf{B}}$ is an orthogonal, positive definite Hermitian. Equation (3.13) with **K** replaced by **B** leads to

$$e^{2\lambda p_3} - e^{2\lambda\bar{p}_3} = 0, \tag{8.2}$$

and

$$\begin{Vmatrix} (\bar{p}_1 - p_2)(e^{2\lambda p_1} - e^{2\lambda\bar{p}_1}) & (\bar{p}_2 - p_2)(e^{2\lambda p_1} - e^{2\lambda\bar{p}_2}) \\ (p_1 - \bar{p}_1)(e^{2\lambda p_2} - e^{2\lambda\bar{p}_1}) & (p_1 - \bar{p}_2)(e^{2\lambda p_2} - e^{2\lambda\bar{p}_2}) \end{Vmatrix} = 0. \tag{8.3}$$

The in-plane displacement (u_1, u_2) and the anti-plane displacement u_3 are uncoupled for monoclinic materials with the symmetry plane at $x_3 = 0$. Equation (8.2) applies to anti-plane displacements, while (8.3) applies to in-plane displacements. From (8.2)

$$2\lambda(\operatorname{Im} p_3) = n\pi, \qquad n = \text{integer}.$$

The decay factor λ is real for anti-plane displacements.

As to (8.3), let α_k, β_k be the real and imaginary parts of p_k,

$$p_k = \alpha_k + i\beta_k, \qquad k = 1, 2.$$

Writing

$$e^{2\lambda p_k} = e^{2\lambda\alpha_k}[\cos(2\lambda\beta_k) + i\sin(2\lambda\beta_k)],$$

it can be shown that expansion of the determinant in (8.3) leads to

$$\begin{aligned}&[(\alpha_1-\alpha_2)^2 + (\beta_1-\beta_2)^2]\sin^2\lambda(\beta_1+\beta_2)\\ &\quad = [(\alpha_1-\alpha_2)^2 + (\beta_1+\beta_2)^2]\sin^2\lambda(\beta_1-\beta_2)\\ &\qquad + [(\beta_1+\beta_2)^2 - (\beta_1-\beta_2)^2]\sinh^2\lambda(\alpha_1-\alpha_2),\end{aligned} \tag{8.4}$$

or

$$(\hat{\alpha}^2 + \hat{\beta}^2)\sin^2\hat{\lambda} = (1+\hat{\alpha}^2)\sin^2(\hat{\lambda}\hat{\beta}) + (1-\hat{\beta}^2)\sinh^2(\hat{\lambda}\hat{\alpha}), \tag{8.5}$$

where

$$\hat{\lambda} = \lambda(\beta_1+\beta_2), \quad \hat{\alpha} = \frac{\alpha_1-\alpha_2}{\beta_1+\beta_2}, \quad \hat{\beta} = \frac{\beta_1-\beta_2}{\beta_1+\beta_2}.$$

The new decay factor $\hat{\lambda}$ depends on two material parameters $\hat{\alpha}$ and $\hat{\beta}$. While $\hat{\alpha}$ can assume any value, $\hat{\beta}$ is limited to $-1 < \hat{\beta} < +1$ because β_1, β_2, the imaginary parts of p_1, p_2, are positive and nonzero. Since $\hat{\lambda}$ is an even function of $\hat{\alpha}$ and $\hat{\beta}$ according to (8.5), it suffices to study $\hat{\lambda}$ for $\hat{\alpha} > 0$, $0 \le \hat{\beta} < 1$. Numerical calculations of $\hat{\lambda}$ for the smallest $\operatorname{Re}\hat{\lambda}$ are shown in Fig. 1 for $\operatorname{Re}\hat{\lambda}$ and in Fig. 2 (see p. 294) for the corresponding $\operatorname{Im}\hat{\lambda}$. It is seen that $\operatorname{Re}\hat{\lambda} = 2\pi$ is the largest $\operatorname{Re}\hat{\lambda}$ that occurs at $\hat{\alpha} = 0$, $\hat{\beta} = 1/2$, or $\alpha_1 = \alpha_2$, $\beta_1 = 3\beta_2$. The special cases $\hat{\alpha} = 0$ and $\hat{\beta} = 0$ are discussed below.

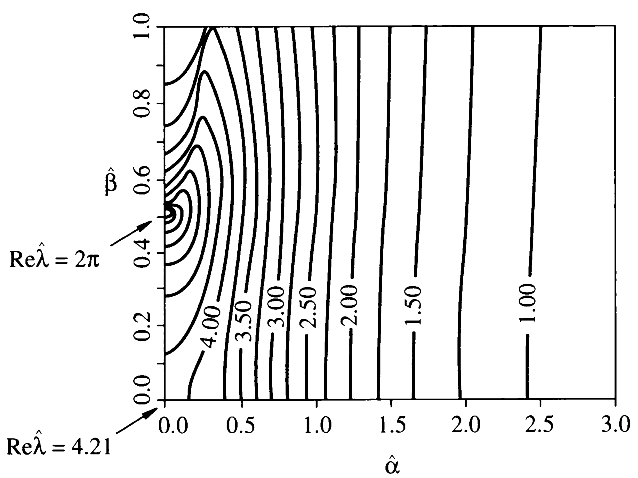

FIG. 1. Contour lines for constant $\operatorname{Re}\hat{\lambda}$.

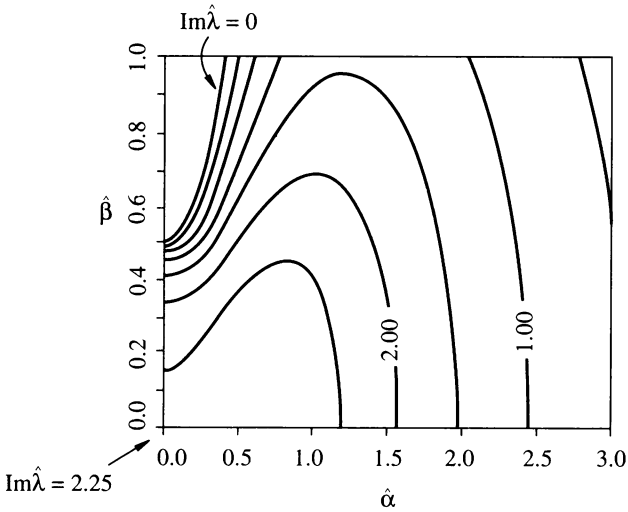

FIG. 2. Contour lines for constant $\operatorname{Im}\hat{\lambda}$.

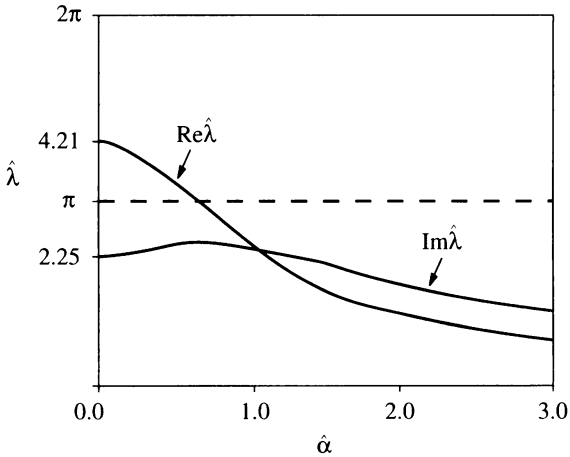

FIG. 3. Dependence of $\hat{\lambda}$ on $\hat{\alpha}$ when $\hat{\beta} = 0$.

CASE 1. When $\beta_1 = \beta_2$, i.e., $\hat{\beta} = 0$, (8.5) reduces to

$$\hat{\alpha} \sin \hat{\lambda} = \pm \sinh(\hat{\lambda}\hat{\alpha}). \tag{8.6}$$

Numerical calculations show that the minus sign in (8.6) should be used for the smallest $\operatorname{Re}\hat{\lambda}$. The results for $\operatorname{Re}\hat{\lambda}$ and $\operatorname{Im}\hat{\lambda}$ are given in Fig. 3. It can be shown that

$$\frac{d\hat{\lambda}}{d\hat{\alpha}} = 0 \quad \text{at} \quad \hat{\lambda} = 0.$$

As is clear from Fig. 3, $\operatorname{Re}\hat{\lambda}$ is monotonic while $\operatorname{Im}\hat{\lambda}$ has a maximum at $\hat{\alpha} \cong 0.65$.

CASE 2. When $\alpha_1 = \alpha_2$, i.e., $\hat{\alpha} = 0$, we have

$$\hat{\beta} \sin \hat{\lambda} = \pm \sin(\hat{\lambda}\hat{\beta}). \tag{8.7}$$

Again the minus sign should be used for the smallest $\operatorname{Re}\hat{\lambda}$. The results as shown in

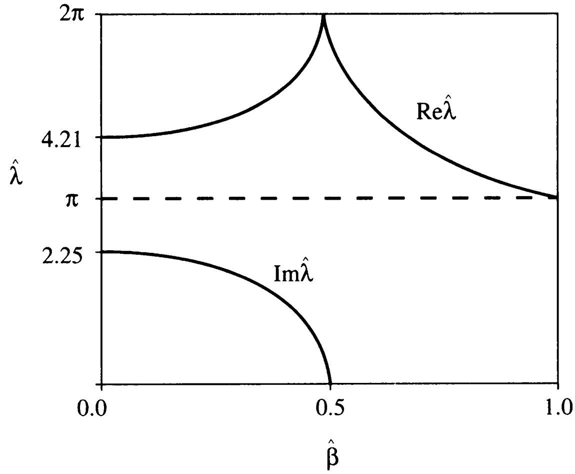

FIG. 4. Dependence of $\hat{\lambda}$ on $\hat{\beta}$ when $\hat{\alpha} = 0$.

Fig. 4 indicate that $\hat{\lambda}$ is real for $\hat{\beta} \geq 1/2$. It can be shown that

$$\hat{\lambda} = \begin{cases} 2\pi & \text{at } \hat{\beta} = 1/2, \\ \pi & \text{as } \hat{\beta} \to 1, \end{cases}$$

and

$$\frac{d\hat{\lambda}}{d\hat{\beta}} = \begin{cases} 0 & \text{at } \hat{\beta} = 0, \\ \infty & \text{at } \hat{\beta} = 1/2, \\ -\pi/2 & \text{as } \hat{\beta} \to 1. \end{cases}$$

CASE 3. $\hat{\alpha} = \hat{\beta} = 0$. By taking a limit, (8.6) or (8.7) reduces to (1.4), and $\hat{\lambda}$ for the smallest $\mathrm{Re}\,\hat{\lambda}$ is

$$\hat{\lambda} = 2\lambda \cong 4.21 + 2.25i. \tag{8.8}$$

The results obtained above apply to special materials such as transversely isotropic materials studied in [5, 8] and isotropic materials. It can be shown that (8.4), (8.6) presented above are equivalent to (18), (26), and (28) of [8], respectively, while (8.7) above is equivalent to (22) and its odd version in [8] as well as (48) and (50) of [5]. It should be pointed out that $\hat{\alpha} = \hat{\beta} = 0$ means that $p_1 = p_2$. For isotropic materials $p_1 = p_2 = i$. Thus (1.4) and λ for the smallest $\mathrm{Re}\,\lambda$ given in (8.8) apply to materials other than isotropic materials as long as $p_1 = p_2$.

9. Discussion and concluding remarks. The Stroh formalism for two-dimensional anisotropic elasticity is employed to study the decay of stress in a semi-infinite anisotropic elastic strip. It is shown that the loading at the end $x_1 = 0$ need not be self-equilibrated for certain side conditions at $x_2 = \pm 1$. The decay factor for monoclinic materials with the plane of symmetry at $x_3 = 0$ is presented for the case when the sides $x_2 = \pm 1$ are traction free.

Discussion of the completeness of the eigenfunctions, the convergence of the series solution, and the approximation of the truncated system for the coefficients of the infinite series are beyond the scope of the present paper. However, the associated problem for an isotropic elastic strip has been studied by Gregory [22, 23]. The solutions of infinite systems of linear equations by truncation are investigated by Kantorovich and Krylov [24].

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Appendix. To prove that $\mathbf{K}$ of (3.12) is nonsingular, consider the matrix $\mathbf{X}$,

$$\mathbf{X} = 2i(\mathbf{I}_u\mathbf{A} + \mathbf{I}_\phi\mathbf{B})(\mathbf{I}_u\mathbf{A} - \mathbf{I}_\phi\mathbf{B})^{\mathrm{T}}. \tag{A.1}$$

Carrying out the matrix products and making use of (3.10) and the identities [14, 15, 17],

$$2i\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{H}, \quad -2i\mathbf{B}\mathbf{B}^{\mathrm{T}} = \mathbf{L}, \quad 2i\mathbf{A}\mathbf{B}^{\mathrm{T}} = \mathbf{S} + i\mathbf{I},$$

where $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$ are real matrices, we have

$$\mathbf{X} = \mathbf{I}_u\mathbf{H}\mathbf{I}_u + \mathbf{I}_\phi\mathbf{L}\mathbf{I}_\phi + \mathbf{I}_\phi\mathbf{S}^{\mathrm{T}}\mathbf{I}_u - \mathbf{I}_u\mathbf{S}\mathbf{I}_\phi \tag{A.2}$$

which is real. The fact that $\mathbf{L}$ and $\mathbf{H}$ are symmetric and positive definite implies that

$$\mathbf{a}^{\mathrm{T}}\mathbf{X}\mathbf{a} = \boldsymbol{\xi}^{\mathrm{T}}\mathbf{H}\boldsymbol{\xi} + \boldsymbol{\eta}^{\mathrm{T}}\mathbf{L}\boldsymbol{\eta} > 0, \qquad \boldsymbol{\xi} = \mathbf{I}_u\mathbf{a}, \quad \boldsymbol{\eta} = \mathbf{I}_\phi\mathbf{a}, \tag{A.3}$$

for any nonzero vector $\mathbf{a}$. Hence $\mathbf{X}$ is positive definite, though not necessarily symmetric. If $\mathbf{X}$ is singular and $\mathbf{a}$ is the right null vector, $\mathbf{X}\mathbf{a} = 0$ means that $\mathbf{a}^{\mathrm{T}}\mathbf{X}\mathbf{a} = 0$ which contradicts (A.3). Hence $\mathbf{X}$ is nonsingular. By (A.1), $\mathbf{I}_u\mathbf{A} + \mathbf{I}_\phi\mathbf{B} = \mathbf{K}$ is nonsingular.

Following the same procedure in deriving (A.2), it is readily shown that

$$2i\mathbf{K}\mathbf{K}^{\mathrm{T}} = \mathbf{I}_u\mathbf{H}\mathbf{I}_u - \mathbf{I}_\phi\mathbf{L}\mathbf{I}_\phi + \mathbf{I}_\phi\mathbf{S}^{\mathrm{T}}\mathbf{I}_u + \mathbf{I}_u\mathbf{S}\mathbf{I}_\phi.$$

Hence $\mathbf{K}\mathbf{K}^{\mathrm{T}}$ is symmetric and purely imaginary.

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