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THE COMPLETENESS THEOREM FOR ROSSBY NORMAL MODES OF A STABLY STRATIFIED FLAT OCEAN WITH AN ARBITRARY FORM OF SIDE BOUNDARY

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Abstract. A mid-latitude flat ocean on a β-plane has characteristic oscillations called Rossby normal modes, where the motion is governed by the quasigeostrophic vorticity equation. Although the relevant eigenvalue problem differs from the usual one of Hilbert-Schmidt type, a variational proof is obtained that the Rossby normal modes constitute a complete orthonormal set for a basin with an arbitrary profile of stable density stratification and an arbitrary form of side boundary. In particular, for each fixed vertical mode, the set of the horizontal modes is complete and orthonormal in a two-dimensional Hilbert space. General solutions are expressed in terms of Rossby normal modes, not only to the initial-value problem, but also to the response problem of the closed basin.

1. Introduction. On account of the unusual reflection of Rossby waves on nonzonal boundaries (LeBlond and Mysak [10]), it has never been easy to understand and predict the evolution or the response of a closed ocean even in linear cases (e.g., Lighthill [11]; Anderson and Gill [1]). The primary purpose of this paper is to provide a general linear solution to the initial-value problem and the response problem of a closed ocean at mid-latitudes through a study of the set of characteristic oscillations of a closed basin on a β-plane.

Those oscillations, known as Rossby normal modes, can be calculated rather easily together with the associated characteristic frequencies (e.g., Longuet-Higgins [12]; Veronis [20]; Phillips [18]). They are defined, however, as the eigenfunctions for an unfamiliar kind of eigenvalue problem, to which the ordinary completeness theory of Hilbert-Schmidt type is not applicable.

No strict argument therefore has been made, so far as the author knows, for the completeness of the whole set of Rossby normal modes, except for general conjectures based on physical intuition (Greenspan [8, 9]; Rhines and Bretherton [19]; Miller [15]). Recently an elementary proof was given of the completeness theorem for a

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special case of a rectangular basin (Masuda [13]). In that case, the eigenvalue problem was reduced to a one-dimensional problem by virtue of the simple rectangular form of the basin. Later the one-dimensional problem was generalized to a Sturm-Liouville-Rossby eigenfunction problem (Masuda [14]); it should be remarked that Mishoe [16] already studied a similar problem, though in a different way (see also Brauer [2]; DiPrima and G. J. Habetler [5]). Moreover, close relations turned out to exist between the Sturm-Liouville and the Sturm-Liouville-Rossby systems (Masuda [14]).

In the previous investigations, however, discussion was restricted to a one-dimensional problem or to the special case that can be reduced to a one-dimensional problem. This paper deals with an ocean with an arbitrary form of side boundary. The next section formulates the problem, and the third gives some preparations. In the fourth section we provide a proof of the completeness theorem for Rossby normal modes. Then, by virtue of the theorem, general solutions to the initial-value and the response problems are presented explicitly in terms of Rossby normal modes.

2. Formulation. We consider a closed ocean with a uniform depth H, where the (stable) density stratification and the side boundary may be arbitrary if they are appropriately smooth. The slow motion of the ocean at mid-latitudes is governed by the linear quasigeostrophic equation

$$\frac{\partial}{\partial t}\left\{\nabla^2\Psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2(z)}\frac{\partial \Psi}{\partial z}\right)\right\} + \beta\frac{\partial \Psi}{\partial x} = 0 \tag{1}$$

in a bounded region $R \times (-H, 0)$, where Ψ is the quasigeostrophic stream function, (x, y, z) the (eastward, northward, vertical) coordinates, ∇ the horizontal gradient operator, f_0 the characteristic value of the Coriolis parameter f, β the meridional gradient of f, $N(z)$ the positive buoyancy frequency belonging to $C^1(-H, 0)$, and R the two-dimensional area of the basin. The stream function must vanish at the side boundaries $\partial R \times (-H, 0)$ and satisfy the surface and bottom conditions:

$$\begin{cases} \Psi = 0 & \text{on } \partial R \times (-H, 0), \\ \dfrac{f_0^2}{N^2(0)}\dfrac{\partial^2\Psi}{\partial t\,\partial z} = -\dfrac{\operatorname{curl}\sigma}{\rho_0} & \text{at } z = 0, \\ \dfrac{f_0^2}{N^2(-H)}\dfrac{\partial^2\Psi}{\partial t\,\partial z} = 0 & \text{at } z = -H, \end{cases} \tag{2}$$

where σ denotes the wind-stress on the sea surface and ρ_0 the characteristic density of sea water. The side boundary ∂R is appropriately smooth; for example, it is sufficient that ∂R be composed of finite smooth curves (Courant and Hilbert [3]; Wloka [21]). Also, the stream function must satisfy the initial condition

$$\Psi = \Phi(x, y, z) \quad \text{at } t = 0. \tag{3}$$

For the above equations, the reader can refer to Gill [7] or Pedlosky [17].

In order to solve the problem defined by (1)–(3), we can expand the stream function in terms of the vertical mode $h_k(z)$ as

$$\Psi(x, y, z, t) = \sum_{k=0}^{\infty} h_k(z) g_k(x, y, t), \tag{4}$$

where g_k indicates the Fourier coefficient for h_k. Here the vertical mode is defined by the differential equation

$$\frac{d}{dz}\left\{\frac{f_0^2}{N^2(z)}\frac{d}{dz}h_k(z)\right\} + \nu_k^2 h_k(z) = 0, \tag{5}$$

where ν_k^2 denotes the eigenvalue that is nonnegative. The boundary conditions are

$$\frac{d}{dz}h_k(z) = 0 \quad \text{at } z = 0 \text{ and } -H, \tag{6}$$

and the normalization is made so that

$$\int_{-H}^{0} |h_k(z)|^2\, dz = 1. \tag{7}$$

Since (5) and (6) generate a Sturm-Liouville problem with the second kind of boundary condition, $\{h_k(z) \mid k = 0 \text{ or } k \in \mathbf{N}\}$ is a complete orthonormal set of the Hilbert space $H_0(-H, 0)$, where $\mathbf{N}$ is the set of positive integers. Hence the expression (4) is valid. The eigenfunction h_0 associated with the zero eigenvalue $(\nu_k = 0)$ is called the barotropic mode, the first eigenfunction h_1 the first baroclinic mode, and so on.

Multiplying (1) by $h_k(z)$ and integrating over z from $-H$ to 0, we find that

$$\frac{\partial}{\partial t}\{\nabla^2 g_k - \nu_k^2 g_k\} + \beta\frac{\partial g_k}{\partial x} = \frac{\operatorname{curl}\sigma}{\rho_0 h_k(0)}, \tag{8}$$

where the boundary conditions (2) have been used. To solve (8), we have another eigenvalue problem defined by

$$\begin{cases} \nabla^2\psi - \nu_k^2\psi + i\lambda\dfrac{\partial\psi}{\partial x} = 0 & \text{in } R, \\ \psi = 0 & \text{on } \partial R, \end{cases} \tag{9}$$

where i denotes $\sqrt{-1}$, λ the eigenvalue, and ψ the eigenfunction. Note that the eigenvalue λ is multiplied by *a derivative of the eigenfunction* rather than *the eigenfunction itself*. This difference makes it impossible to apply the usual completeness theory of Hilbert-Schmidt type. If the basin is rectangular with its two sides directed eastward, (9) can be reduced to a one-dimensional form, to which the theory of the *Sturm-Liouville-Rossby* equation is applied to assure the completeness of Rossby normal modes (Mishoe [16]; Masuda [13, 14]). For the other forms of basins such as a circular one, however, access to the completeness theorem by the methods used in the previous papers is hard. We therefore adopt a different approach; the proof obtained here, by the use of a variational principle, is of mathematical interest in itself (Courant and Hilbert [3, 4]).

3. Eigenfunctions and their orthogonality. Let $D_\infty(R)$ be a space of C^∞-class complex-valued functions that have compact supports in R. We define

$$(f, g) = \iint_R f(x, y) g^*(x, y)\, dx\, dy, \tag{10}$$

$$(f, g)_1 = (f, g) + \left(\frac{\partial f}{\partial x}, \frac{\partial g}{\partial x}\right) + \left(\frac{\partial f}{\partial y}, \frac{\partial g}{\partial y}\right), \tag{11}$$

$$\langle f|g\rangle = l^2(f, g) + \left(\frac{\partial f}{\partial x}, \frac{\partial g}{\partial x}\right) + \left(\frac{\partial f}{\partial y}, \frac{\partial g}{\partial y}\right), \tag{12}$$

where an asterisk denotes the complex conjugate. The constant l^2 is zero for the barotropic mode and positive for the baroclinic modes. The corresponding norms are written as

$$\|f\| = (f, f)^{1/2}, \tag{13}$$

$$\|f\|_1 = (f, f)_1^{1/2}, \tag{14}$$

$$\langle\langle f\rangle\rangle = \langle f|f\rangle^{1/2}. \tag{15}$$

Hilbert spaces $H_0(R)$ and $H_1(R)$ are defined as the completions of $D_\infty(R)$ with respect to the norms (13) and (14), respectively. As is well known, if f is a function in $H_1(R)$, we have inequalities

$$\|f\| \le F \min\left\{\left\|\frac{\partial f}{\partial x}\right\|, \left\|\frac{\partial f}{\partial y}\right\|\right\} \le F\|f\|_1, \tag{16}$$

$$\|f\| \le F\langle\langle f\rangle\rangle \tag{17}$$

for a positive constant F.

We return to the eigenvalue problem (9). For the convenience of the following description, we rewrite (9) as

$$\begin{cases} \mu(\nabla^2\psi - l^2\psi) + i\dfrac{\partial\psi}{\partial x} = 0 & \text{in } R, \\ \psi = 0 & \text{on } \partial R, \end{cases} \tag{18}$$

where we regard the inverse of the *original* eigenvalue λ as the *present* eigenvalue μ. Our concern is the completeness of the set of all the eigenfunctions of (18) in $H_0(R)$ and $H_1(R)$.

Let B be a bilinear functional on $H_1(R)$ defined by

$$B(f, g) = -i\iint_R f\frac{\partial g^*}{\partial x}\, dx\, dy \tag{19}$$

for f and g belonging to $H_1(R)$. It is easy to confirm the following relations:

$$\|B(f, g)\| \le \|f\|\langle\langle g\rangle\rangle \le F\langle\langle f\rangle\rangle\langle\langle g\rangle\rangle, \tag{20}$$

$$B(f, g) = B(g, f)^*, \tag{21}$$

$$B(f^*, g^*) = -B(g, f), \tag{22}$$

$$B(f+g, f+g) = B(f, f) + B(g, g) + 2\Re\{B(f, g)\}, \tag{23}$$

$$B(f+ig, f+ig) = B(f, f) + B(g, g) - 2\Im\{B(f, g)\}, \tag{24}$$

where $\Re$ and $\Im$ denote the real and the imaginary part, respectively.

The following two simple lemmas for the bilinear functional B are frequently referred to later.

LEMMA 1. If f belongs to $H_1(R)$ and $\|f\| = 0$, then $\|f\|_1$ and $\langle\langle f\rangle\rangle$ are zero.

Proof. For $\tilde{f}$ in $D_\infty(R)$, we have

$$\begin{aligned}\left\|\frac{\partial f}{\partial x}\right\|^2 &\le \left|\left(\frac{\partial f}{\partial x}, \frac{\partial \tilde{f}}{\partial x}\right)\right| + \left|\left(\frac{\partial f}{\partial x}, \frac{\partial (f-\tilde{f})}{\partial x}\right)\right| \\ &\le \left|\left(f, \frac{\partial}{\partial x}\left(\frac{\partial \tilde{f}}{\partial x}\right)\right)\right| + \left\|\frac{\partial f}{\partial x}\right\| \cdot \left\|\frac{\partial (f-\tilde{f})}{\partial x}\right\| \\ &\le \|f\| \cdot \left\|\frac{\partial^2 \tilde{f}}{\partial x^2}\right\| + \langle\langle f\rangle\rangle \cdot \langle\langle f-\tilde{f}\rangle\rangle\end{aligned} \tag{25}$$

using the triangular and the Schwartz inequalities. Since $\|f\| = 0$ and f can be approximated arbitrarily well in $H_1(R)$ by a suitable $\tilde{f}$ in $D_\infty(R)$, $\|\partial f/\partial x\|$ must be zero. Likewise $\|\partial f/\partial y\|$ vanishes, whence $\|f\|_1$ and $\langle\langle f\rangle\rangle$ are zero.

LEMMA 2. If f belongs to $H_1(R)$ and $B(f, g)$ is zero for any g in $H_1(R)$, then $\langle\langle f\rangle\rangle = \|f\|_1 = 0$.

Proof. In the same way as in Lemma 1, we have

$$\left\|\frac{\partial f}{\partial x}\right\|^2 \le \left|B\left(f, \frac{\partial \tilde{f}}{\partial x}\right)\right| + \langle\langle f\rangle\rangle \cdot \langle\langle f-\tilde{f}\rangle\rangle \to 0 \tag{26}$$

by choosing adequate $\tilde{f}$ in $D_\infty(R)$. Inequality (16) yields $\|f\| = 0$. Since f belongs to $H_1(R)$, Lemma 1 yields $\langle\langle f\rangle\rangle = \|f\|_1 = 0$.

Now we can consider the supremum μ_1 of $B(f, f)$ for any f with unit norm, since $B(f, f)$ is real and bounded from (21) and (20), respectively:

$$\mu_1 = \sup B(f, f) \tag{27}$$

for f in $H_1(R)$ and $\langle\langle f\rangle\rangle = 1$.

First we show that μ_1 is positive. If μ_1 is negative, there is a function f in $H_1(R)$ such that $\langle\langle f\rangle\rangle = 1$, and $B(f, f) \le \mu_1$. Eq. (22) shows that $B(f^*, f^*) \ge |\mu_1| > 0$. Hence μ_1 must be nonnegative. If μ_1 equals zero, then $B(f, f) = 0$ for any f in $H_1(R)$. The identities (23) and (24) show that $B(f, g)$ is zero for any f and g in $H_1(R)$. Lemma 2 implies that any function in $H_1(R)$ is zero, which is incorrect. Thus, μ_1 must be positive.

There exists a maximizing sequence $\{f_n \mid n \in \mathbf{N}\}$ of functions of unit norm in $H_1(R)$ for which the sequence $\{B(f_n, f_n)\}$ converges to μ_1.

PROPOSITION 1. Let $\{f_n\}$ be the above maximizing sequence. If $\{g_n \mid n \in \mathbf{N}\}$ is in $H_1(R)$ and $\langle\langle g_n\rangle\rangle \le 1$, then

$$\lim_{n\to\infty} |\mu_1 \langle f_n | g_n\rangle - B(f_n, g_n)| = 0. \tag{28}$$

Proof. From the definition of μ_1, we have

$$\begin{aligned} 0 &\le \mu_1 \langle\langle f_n + \varepsilon g_n \rangle\rangle^2 - B(f_n + \varepsilon g_n, f_n + \varepsilon g_n) \\ &= [\mu_1 \langle\langle f_n \rangle\rangle^2 - B(f_n, f_n)] + \varepsilon^2 [\mu_1 \langle\langle g_n \rangle\rangle^2 - B(g_n, g_n)] \\ &\quad + 2\varepsilon \Re\{\mu_1 \langle f_n | g_n \rangle - B(f_n, g_n)\} \end{aligned} \tag{29}$$

for any real ε. From the definition of μ_1, (17), and (20) we have

$$0 \le \mu_1 \langle\langle g_n \rangle\rangle^2 - B(g_n, g_n) \le \mu_1 + F. \tag{30}$$

For sufficiently large n, we have

$$0 \le \mu_1 \langle\langle f_n \rangle\rangle^2 - B(f_n, f_n) \le \varepsilon^2 (\mu_1 + F), \tag{31}$$

because $\{f_n\}$ is the maximizing sequence. Choosing the sign of ε so that $\varepsilon \Re\{\mu_1 \langle f_n | g_n \rangle - B(f_n, g_n)\}$ is negative, we obtain from (29)–(31) that

$$|\Re\{\mu_1 \langle f_n | g_n \rangle - B(f_n, g_n)\}| \le |\varepsilon| (\mu_1 + F). \tag{32}$$

Since $|\varepsilon|$ can be arbitrarily small for large n, the left-hand side of (32) tends to zero. Repeating the same argument with g_n replaced by $-ig_n$, the imaginary part also vanishes as $n \to \infty$.

PROPOSITION 2. For the maximizing sequence $\{f_n\}$ above,

$$\lim_{m, n \to \infty} |\mu_1 \langle\langle f_m - f_n \rangle\rangle^2 - B(f_m - f_n, f_m - f_n)| = 0. \tag{33}$$

Proof. As in (29), the following inequality holds:

$$\begin{aligned} &|\mu_1 \langle\langle f_n - f_m \rangle\rangle^2 - B(f_n - f_m, f_n - f_m)| \\ &\quad \le |\mu_1 \langle\langle f_n \rangle\rangle^2 - B(f_n, f_n)| + |\mu_1 \langle\langle f_m \rangle\rangle^2 - B(f_m, f_m)| \\ &\qquad + 2|\Re\{\mu_1 \langle f_n | f_m \rangle - B(f_m, f_n)\}|. \end{aligned} \tag{34}$$

Since f_n is the maximizing sequence, the first and second terms of the right-hand side decrease to zero. Proposition 1 shows that the third term becomes arbitrarily small as m and n increase to infinity, because the norm of f_n is bounded by unity.

PROPOSITION 3. There is a subsequence $\{g_n\}$ of the maximizing sequence $\{f_n\}$ such that $\{g_n\}$ converges to a function g_0 in $H_1(R)$.

Proof. Since $\langle\langle f_n \rangle\rangle = 1$, Rellich's selection theorem (Courant and Hilbert [3, 4]) assures the existence of a subsequence g_n such that $\|g_m - g_n\|$ decreases to zero. For this subsequence, we obtain

$$\lim_{m, n \to \infty} |B(g_m - g_n, g_m - g_n)| = 0, \tag{35}$$

since

$$\begin{aligned} |B(g_m - g_n, g_m - g_n)| &\le |B(g_m - g_n, g_m)| + |B(g_m - g_n, g_n)| \\ &\le \|g_m - g_n\| \{\langle\langle g_m \rangle\rangle + \langle\langle g_n \rangle\rangle\} \\ &\le 2\|g_m - g_n\|. \end{aligned} \tag{36}$$

Since μ_1 is positive, Proposition 2 and (35) give

$$\begin{aligned}\langle\langle g_m - g_n\rangle\rangle^2 \le \frac{1}{\mu_1}|\mu_1\langle\langle g_m - g_n\rangle\rangle^2 - B(g_m - g_n, g_m - g_n)| \\ + \frac{1}{\mu_1}|B(g_m - g_n, g_m - g_n)| \to 0\end{aligned} \tag{37}$$

as m and n tend to infinity. From inequality (37) $\{g_n\}$ is a Cauchy sequence in the Hilbert space $H_1(R)$, which is complete. Hence, g_n converges to a certain g_0 in $H_1(R)$:

$$\lim_{n\to\infty}\langle\langle g_n - g_0\rangle\rangle = \lim_{n\to\infty}\|g_n - g_0\|_1 = 0. \tag{38}$$

PROPOSITION 4. The function $g_0(x, y)$ above is a weak solution of (18) and $\tilde{g}_0(x, y) = \exp\{-ix/2\mu_1\}g_0(x, y)$ is a weak solution of

$$(\nabla^2 - l^2)\tilde{g}_0 + \frac{1}{4\mu_1^2}\tilde{g}_0 = 0. \tag{39}$$

Proof. For any f in $H_1(R)$, we have

$$\mu_1\langle g_0|f\rangle - B(g_0, f) = 0 \tag{40}$$

since

$$\begin{aligned}|\mu_1\langle g_0|f\rangle - B(g_0, f)| &\le |\mu_1\langle g_0 - g_n|f\rangle - B(g_0 - g_n, f)| \\ &\quad + |\mu_1\langle g_n|f\rangle - B(g_n, f)| \\ &\le (\mu_1 + F)\langle\langle g_0 - g_n\rangle\rangle\langle\langle f\rangle\rangle + |\mu_1\langle g_n|f\rangle - B(g_n, f)| \to 0\end{aligned} \tag{41}$$

from (20), the Schwarz inequality, and Proposition 1. If f belongs to $D_\infty(R)$, integrating (40) by parts, we obtain

$$\iint_R g_0\left\{\mu_1(\nabla^2 - l^2) - i\frac{\partial}{\partial x}\right\} f^*\, dx\, dy = 0 \tag{42}$$

which shows g_0 to be a weak solution of (18). Replacing $f(x, y)$ by $\tilde{f}(x, y) = \exp\{-ix/2\mu_1\}f(x, y)$ and $g_0(x, y)$ by $\tilde{g}_0(x, y)$ in (42), we find

$$\iint_R \tilde{g}_0\left\{(\nabla^2 - l^2) + \frac{1}{4\mu_1^2}\right\}\tilde{f}^*\, dx\, dy = 0. \tag{43}$$

That is, $\tilde{g}_0$ is a weak solution of (39), because $\tilde{f}$ can be any function in $D_\infty(R)$.

As is well known, weak solutions of elliptic equations can be regarded as genuine solutions, which really satisfy the boundary conditions required, in the sense that the norm of the difference between the weak solution and the genuine smooth solution is zero in $H_0(R)$ (Weyl's Lemma, see, e.g., Wloka [21]). Proposition 3 indicates that the weak solutions g_0 and $\tilde{g}_0$ obtained above belong to $H_1(R)$. Lemma 1 shows that the weak solutions equal the smooth genuine solutions as functions in $H_1(R)$. Therefore $\tilde{g}_0$ and g_0 are considered regular functions in $H_1(R)$. Thus we obtain the following theorem.

THEOREM 1. The maximum value μ_1 is attained by an eigenfunction $u_1(x, y)$ of the partial differential equation (18).

Obviously the minimum value of $B(f, f)$ subject to the condition $\langle\langle f\rangle\rangle = 1$ is $\mu_{-1} = -\mu_1$, and $u_{-1} = u_1^*$ minimizes B. Also we have the orthgonality relations

$$\langle u_1|u_{-1}\rangle = B(u_1, u_{-1}) = 0. \tag{44}$$

From (21) and (40) it follows that if a function f in $H_1(R)$ is orthogonal to u_1 (i.e., $\langle f|u_1\rangle = 0$), then $B(f, u_1) = 0$ and vice versa; the same is true for u_{-1}.

Next, we consider the similar maximizing problem with additional conditions

$$\langle f|u_1\rangle = \langle f|u_{-1}\rangle = 0. \tag{45}$$

Let μ_2 be the supremum defined above:

$$\mu_2 = \sup B(f, f) \tag{46}$$

for f satisfying (45) and $\langle\langle f\rangle\rangle = 1$. Then we can see that μ_2 is positive as follows. The same argument as for μ_1 shows that μ_2 cannot be negative. Suppose that $\mu_2 = 0$. This implies that $B(f, f) = 0$ for any function f satisfying (45). Then, for any f and g subject to the above conditions, $B(f, g) = 0$ by virtue of (23) and (24). Now, let f be a function satisfying (45) and let g be a function in $H_1(R)$. If we define

$$\tilde{g} = g - \langle g|u_1\rangle u_1 - \langle g|u_{-1}\rangle u_{-1}, \tag{47}$$

then we have

$$\langle \tilde{g}|u_1\rangle = \langle \tilde{g}|u_{-1}\rangle = 0. \tag{48}$$

Since $\tilde{g}$ satisfies (45) and $\mu_2 = 0$ from the assumption, it follows that

$$B(f, \tilde{g}) = 0. \tag{49}$$

Hence we find

$$\begin{aligned} B(f, g) &= B(f, \tilde{g}) + \langle g|u_1\rangle^* B(f, u_1) + \langle g|u_{-1}\rangle^* B(f, u_{-1}) \\ &= 0. \end{aligned} \tag{50}$$

Lemma 2 implies that f equals zero. In other words, the functions orthogonal to u_1 and u_{-1} must be zero; this implies $H_1(R)$ is spanned by two vectors u_1 and u_{-1}, which leads to a contradiction. Thus, μ_2 must be positive.

There exists a maximizing sequence $\{f_n\}$ of unit norm subject to (45), for which $\{B(f_n, f_n)\}$ converges to μ_2. Then Proposition 5 holds in the same way as Proposition 1.

PROPOSITION 5. Let $\{f_n\}$ be the above (second) maximizing sequence and $\{g_n\}$ be functions in $H_1(R)$ satisfying $\langle\langle g_n\rangle\rangle \le 1$. Then,

$$\lim_{n\to\infty} |\mu_2\langle f_n|g_n\rangle - B(f_n, g_n)| = 0. \tag{51}$$

Proof. First, note that, if g_n is bounded by unity and is orthogonal to u_1 and u_{-1}, (51) holds in just the same way as in the proof of Proposition 2 for μ_1. Let

$\tilde{g}_n$ be defined by (47), which is orthogonal to u_1 and u_{-1}. By the same calculation as for (50), we have

$$\mu_2\langle f_n|g_n\rangle - B(f_n, g_n) = \mu_2\langle f_n|\tilde{g}_n\rangle - B(f_n, \tilde{g}_n). \tag{52}$$

Since $1 \geq \langle\langle g_n\rangle\rangle \geq \langle\langle \tilde{g}_n\rangle\rangle$ (Bessel's inequality), (52) guarantees the proposition.

Following the same procedure as for μ_1, we find that the second maximum μ_2 is the second eigenvalue of (18) and that μ_2 is attained by the associated eigenfunction u_2. It is easy to confirm

$$\langle u_1|u_2\rangle = \langle u_1|u_{-2}\rangle = 0, \tag{53}$$

$$B(u_1, u_2) = B(u_1, u_{-2}) = 0. \tag{54}$$

Thus we obtain the following result.

THEOREM 2. There are sequences of $\{\mu_n\}$ and $\{u_n\}$, where μ_n and u_n are the eigenvalue and the associated eigenfunction, respectively, of the partial differential equation (18). The following orthogonality conditions hold:

$$\langle u_m|u_n\rangle = \delta_{m,n}, \tag{55}$$

$$B(u_m, u_n) = \mu_m\delta_{m,n}, \tag{56}$$

where $\delta_{m,n}$ is the Kronecker delta function.

PROPOSITION 6. The dimension of the eigenspace associated with an eigenvalue is finite and μ_n decreases to zero as n increases to infinity.

Proof. Suppose a positive eigenvalue has an infinite-dimensional eigenspace or μ_n does not tend to zero. Then, there is a positive lower bound α for the sequence $\{\mu_n \mid n \in \mathbf{N}\}$. Since $\langle\langle u_n\rangle\rangle = 1$, Rellich's theorem guarantees the existence of a subsequence $\{v_n\}$ such that

$$\lim_{m,n\to\infty} \|v_m - v_n\| = 0. \tag{57}$$

Hence we have

$$\begin{aligned} |B(v_m - v_n, v_m - v_n)| &\leq \|v_m - v_n\|\{\langle\langle v_m\rangle\rangle + \langle\langle v_n\rangle\rangle\} \\ &= 2\|v_m - v_n\| \to 0. \end{aligned} \tag{58}$$

On the other hand, we have

$$\begin{aligned} B(v_m - v_n, v_m - v_n) &= B(v_m, v_m) + B(v_n, v_n) \\ &= \mu_m + \mu_n \geq 2\alpha \end{aligned} \tag{59}$$

from (55) and (56). This contradiction between (58) and (59) proves the proposition.

4. Completeness theorems and general solutions. By virtue of the preceding preparations, we can now prove the main theorems.

THEOREM 3 (The completeness theorem for a specified vertical mode). The set of the eigenfunctions of (18) is a complete orthonormal set in $H_1(R)$; any function f in $H_1(R)$ can be expanded as

$$f(x, y) = \lim_{n\to\infty} f_n(x, y), \tag{60}$$

where

$$f_n(x, y) = \sum_{\substack{j=-n \\ j\neq 0}}^{n} \langle f|u_j\rangle u_j(x, y) \tag{61}$$

in the sense that

$$\lim_{n\to\infty} \|f - f_n\| = \lim_{n\to\infty} \langle\langle f - f_n\rangle\rangle = 0. \tag{62}$$

That is, $\{f_n\}$ converges to f strongly (norm convergence).

Proof. From the orthogonality conditions, we have

$$\langle\langle f - f_n\rangle\rangle^2 = \langle\langle f\rangle\rangle^2 - \sum_{\substack{j=-n \\ j\neq 0}}^{n} |\langle f|u_j\rangle|^2 \geq 0, \tag{63}$$

$$B(f - f_n, f - f_n) = B(f, f) - \sum_{\substack{j=-n \\ j\neq 0}}^{n} |\langle f|u_j\rangle|^2 \mu_j. \tag{64}$$

Eq. (63) shows convergence of the following two series:

$$\sum_{\substack{j=-n \\ j\neq 0}}^{n} |\langle f|f_j\rangle|^2, \tag{65}$$

$$\sum_{\substack{j=-n \\ j\neq 0}}^{n} |\langle f|f_j\rangle|^2 |\mu_j| \leq |\mu_1| \sum_{\substack{j=-n \\ j\neq 0}}^{n} |\langle f|f_j\rangle|^2. \tag{66}$$

From the variational construction of the eigenfunctions u_j,

$$0 \leq |B(f - f_n, f - f_n)| \leq \mu_{n+1}\langle\langle f - f_n\rangle\rangle \leq \mu_{n+1}\langle\langle f\rangle\rangle. \tag{67}$$

Using (67), (23), and (24), we have, for any functions f and h in $H_1(R)$,

$$\begin{aligned} |B(f - f_n, h - h_n)| &\leq |\Re(B(f - f_n, h - h_n))| + |\Im(B(f - f_n, h - h_n))| \\ &\leq |B(f - f_n, f - f_n)| + |B(f - f_n, f - f_n)| \\ &\quad + \frac{1}{2}|B(f - f_n + i(h - h_n), f - f_n + i(h - h_n))| \\ &\quad + \frac{1}{2}|B(f - f_n - i(h - h_n), f - f_n - i(h - h_n))| \\ &\leq 2\mu_{n+1}[\langle\langle f\rangle\rangle^2 + \langle\langle h\rangle\rangle^2]. \end{aligned} \tag{68}$$

Since $(f - f_n)$ is orthogonal to h_n, $B(f - f_n, h - h_n) = B(f - f_n, h)$. Therefore we find that for any h in $H_1(R)$

$$\lim_{n\to\infty} B(f - f_n, h) = 0, \tag{69}$$

because μ_n decreases to zero as n increases to infinity by virtue of Proposition 6.

On the other hand, the absolute convergence of the series (65) implies

$$\lim_{m,n\to\infty} \langle\langle (f - f_n) - (f - f_m)\rangle\rangle = \lim_{m,n\to\infty} \langle\langle f_n - f_m\rangle\rangle = 0. \tag{70}$$

Since $(f-f_n)$ belongs to the complete space $H_1(R)$, $(f-f_n)$ converges to a function g in $H_1(R)$. Hence (69) is rewritten as

$$B(g, h) = 0, \tag{71}$$

for any h belonging to H_1. Lemma 2 is applied to show that $\|g\| = 0$. Since g is a function in $H_1(R)$, Lemma 1 shows that $\langle\langle g\rangle\rangle = \|g\|_1 = 0$. That is,

$$g = \lim_{n\to\infty}(f - f_n) = 0 \tag{72}$$

or

$$f = \lim_{n\to\infty} f_n. \tag{73}$$

The following is an almost trivial consequence of Theorem 3, (55)–(56), and Proposition 6. It corresponds to the converse of Theorem 1.

PROPOSITION 7. Let h be an eigenfunction of the differential equation (18) associated with an eigenvalue μ. Then, μ belongs to $\{\mu_n \mid n \in \mathbf{Z}'\}$ with $\mathbf{Z}'$ being the set of the integers except for 0, and h is expressed as a finite sum of u_n associated with the same eigenvalue μ.

By combining Theorem 3 for the *horizontal* modes with the usual completeness theorem for the vertical modes (Sec. 2), we obtain the completeness theorem for Rossby normal modes as follows. Let $h_k(z)$ be the kth vertical mode eigenfunction defined in Sec. 2 and let $u_{k,n}(x, y)$ be the nth eigenfunction defined in Sec. 3 associated with the vertical kth mode. We define the k, nth Rossby normal mode $v_{k,n}(x, y, z)$ by

$$v_{k,n}(x, y, z) = u_{k,n}(x, y)h_k(z). \tag{74}$$

Suppose $D_\infty(R\times\mathbf{R})$ and $D_\infty(R\times(-H, 0))$ are sets of C^∞-class complex-valued functions that have compact supports in $R\times\mathbf{R}$ and $R\times(-H, 0)$, respectively, where $\mathbf{R}$ is the set of real numbers. On these spaces we define three inner products:

$$(f, g)_D = \int_{-H}^{0}\iint_R [fg^*]\,dx\,dy\,dz, \tag{75}$$

$$\langle f|g\rangle_N = \int_{-H}^{0}\iint_R \left[\nabla f\cdot\nabla g^* + \frac{f_0^2}{N^2(z)}\frac{\partial f}{\partial z}\frac{\partial g^*}{\partial z}\right] dx\,dy\,dz, \tag{76}$$

$$[f, g]_D = (f, g)_D + \langle f|g\rangle_N; \tag{77}$$

the corresponding norms being written as $\|\cdot\|_D$, $\langle\langle\cdot\rangle\rangle_N$, and $[[\cdot]]_D$, respectively. The suffix N of (76) indicates the inner product relevant for *normal* modes; the square of the norm $\langle\langle f\rangle\rangle_N^2$ is the total energy (kinetic plus available potential) of the motion "f". The Hilbert space D we use here is defined as the completion of $D_\infty(R\times\mathbf{R})$ with respect to the norm $[[\cdot]]_D$. The two norms $\langle\langle\cdot\rangle\rangle_N$ and $[[\cdot]]_D$ are equivalent as a distance, because inequalities like (16) and (17) are valid in D as well.

THEOREM 4 (The completeness theorem and expansion theorem for Rossby normal modes). The Rossby normal modes (74) make a complete orthonormal set in D

with respect to $\langle\cdot|\cdot\rangle_N$. Any function f in D can be expanded as

$$f=\sum_{k=0}^{\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\langle f|v_{k,n}\rangle_N\, v_{k,n}(x,y,z), \tag{78}$$

where the right-hand side converges strongly.

Proof. Suppose that f belongs to D and that $\langle f|v_{k,n}\rangle_N=0$ for any k and n. As is well known, it suffices to show $\langle\langle f\rangle\rangle_N=[[f]]_D=0$ for such a function f. The orthogonality of f with $v_{k,n}$ is rewritten as

$$\int_{-H}^{0}\iint_R f(x,y,z)\frac{\partial u_{k,n}^*(x,y)}{\partial x}h_k(z)\,dx\,dy\,dz=0 \tag{79}$$

since $v_{k,n}(x,y,z)$ is a Rossby normal mode (eigenfunction). The integrand of (79) is absolutely integrable, so that Fubini's theorem can be applied to show that

$$g_k(x,y)=\int_{-H}^{0}f(x,y,z)h_k(z)\,dz \tag{80}$$

is significant for a.e. (x,y) in R, and (79) becomes

$$\iint_R g_k(x,y)\frac{\partial u_{k,n}^*}{\partial x}\,dz=0. \tag{81}$$

Again from Fubini's theorem and the Schwarz inequality, we obtain

$$\begin{aligned}&\iint_R |g_k(x,y)|^2\,dx\,dy\\ &\quad\le\iint_R\left\{\int_{-H}^{0}|f(x,y,z)|^2\,dz\times\int_{-H}^{0}|h_k(z)|^2\,dz\right\}dx\,dy\end{aligned} \tag{82}$$

which shows $g_k(x,y)$ is an element of $H_0(R)=L_2(R)$. In the same way, we can show that

$$\frac{\partial g_k(x,y)}{\partial x}=\int_{-H}^{0}\frac{\partial f(x,y,z)}{\partial x}h_k(z)\,dz, \tag{83}$$

$$\frac{\partial g_k(x,y)}{\partial y}=\int_{-H}^{0}\frac{\partial f(x,y,z)}{\partial y}h_k(z)\,dz \tag{84}$$

are functions in $H_0(R)$, where $\partial f/\partial x$ and $\partial f/\partial y$ denote weak derivatives of f. It remains to confirm that (83) and (84) express the weak first derivatives of $g_k(x,y)$. Let $s(x,y)$ be a function in $D_\infty(R)$. It is approximated by $s(x,y)\cdot t(z)$ in $D_\infty(R\times(-H,0))$ with $t(z)$ an appropriate smooth function in $R\times(-H,0)$ such that $\|s\cdot t-s\|_D\to 0$. Since st belongs to $D_\infty(R\times(-H,0))$ and f has weak first derivatives in $R\times(-H,0)$, we have

$$\begin{aligned}&\int_{-H}^{0}\iint_R\frac{\partial f(x,y,z)}{\partial x}h_k(z)s(x,y)t(z)\,dx\,dy\,dz\\ &\quad=-\int_{-H}^{0}\iint_R f(x,y,z)h_k(z)\frac{\partial s(x,y)}{\partial x}t(z)\,dx\,dy\,dz.\end{aligned} \tag{85}$$

As the limit of (85), (83) represents the weak derivative of g_k with respect to x. The same is true for (84).

Since f belongs to D, there is a sequence $\{f_m \mid m \in \mathbf{N}\}$ of elements in $D_\infty(R \times (-H, 0))$ that converges to f in D. Let us define

$$g_{k,m}(x, y) = \int_{-H}^{0} f_m(x, y, z) h_k(z)\, dz, \tag{86}$$

which is an element of $D_\infty(R)$. Obviously we have

$$\lim_{m\to\infty} \|g_{k,m}(x, y) - g_k(x, y)\|_1 = 0, \tag{87}$$

which implies that $g_k(x, y)$ belongs to $H_1(R)$.

From the use of Lemma 2 and Theorem 3 for each vertical mode, it follows from (81) that $\|g_k\|_1 = \|g_k\| = 0$. Hence $g_k(x, y)$ is 0 for a.e. (x, y) in R. Since (80) holds for any k, the completeness theorem for the vertical modes assures that, for a.e. (x, y) in R, $f(x, y, z) = 0$ at a.e. z in $(-H, 0)$; $f(x, y, z) = 0$ for a.e. (x, y, z) in $R \times (-H, 0)$ and consequently $\|f\|_D = 0$.

With respect to $[[\cdot]]_D$, $\{f_m \in D_\infty(R \times \mathbf{R}) \mid m \in \mathbf{N}\}$ converges to f. For each m, there are sequences $\{f_{l,m,x} \mid l \in \mathbf{N}\}$, $\{f_{l,m,y} \mid l \in \mathbf{N}\}$ and $\{f_{l,m,z} \mid l \in \mathbf{N}\}$ in $D_\infty(R \times (-H, 0))$, such that

$$\lim_{l\to\infty} \left\| \frac{\partial f_m}{\partial x} - f_{l,m,x} \right\|_D = 0, \tag{88}$$

$$\lim_{l\to\infty} \left\| \frac{\partial f_m}{\partial y} - f_{l,m,y} \right\|_D = 0, \tag{89}$$

$$\lim_{l\to\infty} \left\| \frac{\partial f_m}{\partial z} - f_{l,m,z} \right\|_D = 0. \tag{90}$$

Then, an argument similar to the one in Lemma 1 shows that $\langle\langle f\rangle\rangle_N = [[f]]_D = 0$.

Now we write down the general solution to the problem given by (1)–(3). The corresponding characteristic frequency of the basin becomes

$$\omega_{k,n} = \beta \mu_{k,n}. \tag{91}$$

According to the usual calculation (Masuda [13]), we obtain the following general solution to the initial-value problem and the response problem (1)–(3):

$$\Psi(x, y, z, t) = \sum_{k=0}^{\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} v_{k,n}(x, y, z) \times \left[\langle \Phi | u_{k,n} \rangle_N e^{-i\omega_{k,n} t} + \int_0^t \left(-\frac{\operatorname{curl}\sigma}{\rho_0 h_k(0)}, u_{k,n} \right) e^{-i\omega_{k,n}(t-\tau)}\, d\tau \right]. \tag{92}$$

In particular, the response of the basin to an instantaneous torque located at (ξ, η) at time $t = \tau$ is found by setting $\Phi = 0$ and $\operatorname{curl}\sigma/\rho_0 = \delta(x-\xi)\delta(y-\eta)\delta(t-\tau)$, where δ denotes the delta function. That is, Green's function formally becomes

$$G(x, y, z, t; \xi, \eta, \tau) = -\sum_{k=0}^{\infty} \frac{h_k(z)}{h_k(0)} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} u_{k,n}(x, y) u_{k,n}(\xi, \eta) e^{-\omega_{k,n}(t-\tau)}. \tag{93}$$

5. Discussion. In this paper, a proof is obtained of the completeness of Rossby normal modes in a basin with an arbitrary form of side boundary. This investigation, however, is only a beginning of the study of completeness of characteristic oscillations appearing in geophysical fluid dynamics (see LeBlond and Mysak [10], for a variety of wave motions found in the atmosphere and the ocean). In fact, the present results apply merely to quasigeostrophic dynamics in basins with a uniform depth. When the bottom is uneven and the ocean is stratified, the problem becomes essentially three-dimensional. If ageostrophy is taken into consideration, the problem becomes that of vector-valued eigenfunctions (Masuda [14]). When the ocean has a boundary of zero depth or an infinite domain, a careful analysis of singularities is required. These are subjects to be studied in the future.

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