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ON A PERIODIC DELAY POPULATION MODEL

By

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Abstract. The existence of a positive periodic solution for

$$N'(t) = N(t)\left[a(t) + b(t)N(t - m\omega) - c(t)N^2(t - m\omega)\right]$$

is established. Sufficient conditions are given for the periodic solution to be globally attractive.

1. Introduction. The scalar autonomous ordinary delay differential equation

$$\frac{dN(t)}{dt} = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right], \tag{1.1}$$

commonly known as the logistic equation with time delay τ, is most frequently employed in modelling the dynamics of population of a single species, where $N(t)$ is the population at time t, r is the growth rate of the species, and K is the carrying capacity of the habitat. The per-capita growth rate in (1.1) is a linear function of the population N (can be termed density of the population). The term $[K - N(t-\tau)]/K$ denotes the feedback mechanism which takes τ units of time to respond to changes in the population size. As Cunningham [3] suggested, the model (1.1) can be used to describe certain control systems. Similar equations can also be used in economic studies of business cycles. One can use such models in mathematical ecology. The effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.). It has been suggested by Nicholson [7] that any periodic change of climate tends to impose its period upon oscillations of internal origin or to cause such oscillations to have a harmonic relation to periodic climatic changes. In view of this it is realistic to assume that a, b, c are periodic

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functions of period ω and that the delay is an integral multiple of periodicity of the environment. One can refer to Pianka [8] for a discussion of the relevance of periodic environment to evolutionary theory. Zhang and Gopalsamy [9] have also studied the periodic delay logistic equation. Recently Gopalsamy and Ladas [4] proposed a model of a single-species population exhibiting the so-called Allee effect [1] in which the per-capita growth rate is a quadratic function of the density and subject to delays. In particular, they studied the oscillatory and asymptotic behavior of the positive solution of

$$N'(t) = N(t)\left[a + bN(t-\tau) - cN^2(t-\tau)\right], \tag{1.2}$$

where $a, c \in (0, \infty)$, $b \in R$, and $\tau \in [0, \infty)$. If $a > 0$, $b < 0$, and $c = 0$, then (1.2) reduces to an equation of the type (1.1). According to Gopalsamy and Ladas [4] one can interpret (1.2) as a single-species model with a quadratic per-capita growth rate and such a per-capita growth rate is the first-order nonlinear approximation of more general types of nonlinear growth rates with single humps.

The purpose of this article is to consider the following model:

$$N'(t) = N(t)\left[a(t) + b(t)N(t-m\omega) - c(t)N^2(t-m\omega)\right], \tag{1.3}$$

with an initial condition

$$N(t) = \phi(t) \quad \text{for } -m\omega \le t \le 0, \quad \phi \in C\left[[-m\omega, 0], R^+\right], \phi(0) > 0,$$

where a, b, c are periodic continuous functions with period $\omega > 0$, a and c are positive functions, and m is a positive integer.

In Sec. 2 we discuss the existence of a unique positive periodic solution $N^*(t)$ of (1.3), which is globally attractive for all other positive solutions of (1.3).

2. Existence of a positive solution. First, we consider (1.3) without delay. That is, we study

$$N'(t) = N(t)\left[a(t) + b(t)N(t) - c(t)N^2(t)\right]. \tag{2.1}$$

In the sequel we use the following notation:

$$\begin{aligned} a_0 &= \min_{0\le t\le\omega} a(t), & a^0 &= \max_{0\le t\le\omega} a(t) \\ b_0 &= \min_{0\le t\le\omega} b(t), & b^0 &= \max_{0\le t\le\omega} b(t) \\ c_0 &= \min_{0\le t\le\omega} c(t), & c^0 &= \max_{0\le t\le\omega} c(t), \end{aligned} \tag{2.2}$$

and we define

$$A = \frac{b^0 + \sqrt{b^{0^2} + 4a^0 c_0}}{2c_0}, \qquad B = \frac{b_0 + \sqrt{b_0^2 + 4a_0 c^0}}{2c^0}. \tag{2.3}$$

It is obvious that $A > B$. Our first result is the following.

THEOREM 2.1. There exists a unique ω-periodic positive solution of Eq. (2.1).

Proof. Let $N(t, 0, N_0)$ denote the unique solution of (2.1) through $(0, N_0)$. We shall show that

$$N_0 \in [B, A] \quad \text{implies that} \quad N(t, 0, N_0) \in [B, A]. \tag{2.4}$$

In fact, $N(0) > 0$ implies that $N(t, 0, N_0) > 0$ for all $t > 0$. From (2.2)

$$N'(t) \leq N(t)\left[a^0 + b^0 N(t) - c_0 N^2(t)\right].$$

Set

$$\begin{aligned} F(u) &= a^0 + b^0 u - c_0 u^2 \\ &= c_0(A - u)\left(u - \frac{b^0 - \sqrt{b^{0^2} + 4a^0 c_0}}{2c_0}\right). \end{aligned}$$

Then $N_0 \in [B, A]$, $F(N_0) > 0$. By continuity there exists a neighbourhood C_δ of zero such that

$$F(N(t)) > 0 \quad \text{for } t \in C_\delta.$$

Let

$$t^* = \inf\{t > 0 : N(t) > A\}.$$

Then $N(t^*) = A$, $N'(t^*) > 0$. Thus,

$$\begin{aligned} 0 < N'(t^*) &= N(t^*)\left[a(t^*) + b(t^*)N(t^*) - c(t^*)N^2(t^*)\right] \\ &= N(t^*)c(t^*)\left[\frac{b(t^*) + \sqrt{b^2(t^*) + 4a(t^*)c(t^*)}}{2c(t^*)} - A\right] \\ &\quad \times \left[A - \frac{b(t^*) - \sqrt{b^2(t^*) + 4a(t^*)c(t^*)}}{2c(t^*)}\right] \\ &\leq 0, \end{aligned}$$

which, in view of the definition of A, is a contradiction. Thus, $N(t, 0, N_0) \leq A$ for all $t > 0$. In a similar manner, we can prove that $N(t, 0, N_0) \geq B$. In particular,

$$N_\omega = N(\omega, 0, N_0) \in [B, A].$$

Define a mapping $f: [B, A] \to [B, A]$ as follows:

$$f(N_0) = N_\omega.$$

Since the solution $N(t, 0, N_0)$ depends continuously on the initial value N_0, the mapping f is continuous and maps the interval $[B, A]$ into itself. Therefore, f has a fixed point N_0^*. Thus, the unique positive solution $N^*(t) = N(t, 0, N^*)$ of (2.2) through the initial point $(0, N_0^*)$ is periodic with period ω. This completes the proof of the theorem.

Next we establish the global attractivity of N^*.

THEOREM 2.2. If

$$b^0 < c_0 N_0^*, \text{ where } N_0^* = \min_{0 \le t \le \omega} N^*(t), \tag{2.4b}$$

then any other positive solution of $N(t)$ of (2.1) satisfies

$$\lim_{t \to \infty} [N(t) - N^*(t)] = 0. \tag{2.5}$$

Proof. Set

$$N(t) = N^*(t) e^{x(t)}. \tag{2.6}$$

We shall show that $x(t) \to 0$ as $t \to \infty$. In fact, from (2.1) and (2.6) we have

$$\frac{dx(t)}{dt} = b(t) N^*(t)[e^{x(t)} - 1] - c(t) N^{*^2}(t)[e^{2x(t)} - 1]. \tag{2.7}$$

Now we define a Lyapunov function

$$V(x(t)) = [e^{x(t)} - 1]^2.$$

Calculating the derivative of V along a solution of (2.1), we have

$$\begin{aligned} \frac{dV(t)}{dt} &= 2[e^{x(t)} - 1] e^{x(t)} x'(t) \\ &= 2e^{x(t)} N^*(t)[e^{x(t)} - 1]^2 [b(t) - c(t) N^*(t)(e^{x(t)} + 1)]. \end{aligned}$$

We note that

$$\begin{aligned} b(t) - c(t) N^*(t)(e^{x(t)} + 1) &\le b(t) - c(t) N^*(t) \\ &\le b^0 - c_0 N_0^*. \end{aligned}$$

Using the assumption (2.4b), we have $dV(t)/dt < 0$. Thus,

$$\frac{dV(t)}{dt} \le 2(b^0 - c_0 N_0^*) e^{x(t)} [e^{x(t)} - 1]^2$$

or

$$\frac{dV(t)}{dt} + 2(c_0 N_0^* - b^0) e^{x(t)} [e^{x(t)} - 1]^2 \le 0,$$

or

$$V(t) + 2(c_0 N_0^* - b^0) \int_0^t e^{x(s)} [e^{x(s)} - 1]^2 \, ds \le V(0) < \infty,$$

which implies that $e^{x(t)}[e^{x(t)} - 1]^2 \in L_1(0, \infty)$. It is easy to see that $x(t)$ and $x'(t)$ are both bounded. Consequently $e^{x(t)}[e^{x(t)} - 1]^2$ is uniformly continuous on $(0, \infty)$. By Barbalat's Lemma [2], $e^{x(t)} \to 1$ as $t \to \infty$. Thus, we have $\lim_{t \to \infty} x(t) = 0$, which completes the proof.

REMARK 2.1. If $b(t) \le 0$ then (2.4b) is satisfied automatically.

Now we consider Eq. (1.3) with initial condition

$$N(t) = \phi(t), \qquad -m\omega \le t \le 0, \qquad \phi(0) > 0. \tag{2.8}$$

It is easy to see that the unique positive solution $N^*(t)$ of (2.1) is also a positive periodic solution of (1.3). We want to find sufficient conditions for the attractive behaviour of $N^*(t)$. To that end we first estimate the upper and lower bounds of a solution of (1.3). In the following we assume that $b(t) \le 0$. We note that if $b(t) < 0$ and $c(t) \equiv 0$, then (1.3) reduces to a logistic equation.

THEOREM 2.3. If $N(t)$ is a solution of the initial value problem (1.3) and (2.8), then there exists a number $T = T(\phi)$ such that

$$\underline{N} \le N(t) \le \overline{N} \quad \text{for } t \ge T, \tag{2.9}$$

where

$$\overline{N} = Ae^{m\omega\langle a\rangle}, \tag{2.10}$$

$$\underline{N} = Be^{m\omega(\langle a\rangle + \langle b\rangle\overline{N} - \langle c\rangle\overline{N}^2)}, \qquad \langle g\rangle = \frac{1}{m\omega}\int_0^{m\omega} g(t)\,dt. \tag{2.11}$$

Proof. From (1.3)

$$\begin{aligned} N'(t) &\le N(t)\left[a^0 + b^0 N(t-m\omega) - c_0 N^2(t-m\omega)\right] \\ &= c_0 N(t)\left[A - N(t-m\omega)\right]\left[N(t-m\omega) - \frac{b^0 - \sqrt{b^{0^2} + 4a^0 c_0}}{2c_0}\right]. \end{aligned} \tag{2.12}$$

Now either $N(t)$ is oscillatory about A or it is nonoscillatory. In case $N(t)$ is oscillatory about A, we let $\{t_n\}$ be the sequence such that $\lim_{t\to\infty} t_n = \infty$ and $A - N(t_n) = 0$. Let $N(t_n^*)$ be the maximum of $N(t)$ on (t_n, t_{n+1}). Then

$$\begin{aligned} 0 \le N'(t_n^*) \le c_0 N(t_n^*)&\left[A - N(t_n^* - m\omega)\right] \\ &\times \left[N(t_n^* - m\omega) - \frac{b_0 - \sqrt{b_0^2 + 4a^0 c_0}}{2c_0}\right]. \end{aligned}$$

Now $N(t_n^* - m\omega) \le A$; so let $\overline{\xi}$ be the first zero of $A - N(t)$ in $[t_n^* - m\omega, t_n^*]$ and $\xi = \max\{\overline{\xi}, t_n\}$. By integrating (1.3) from ξ to t_n^*, we have

$$\log\frac{N(t_n^*)}{N(\xi)} \le \int_\xi^{t_n^*} a(t)\,dt,$$

or

$$N(t_n^*) \le A\exp\left[\int_{t_n^* - m\omega}^{t_n^*} a(t)\,dt\right] = Ae^{m\omega\langle a\rangle},$$

that is,

$$N(t) \le Ae^{m\omega\langle a\rangle} \quad \text{for } t \ge T.$$

Next we suppose that $N(t)$ is nonoscillatory about A. Since $N(t) > A$ implies that $N'(t) < 0$, for every ε there exists a $T_1 = T(\varepsilon)$ such that $N(t) < A + \varepsilon$ for $t > T_1$. Consequently,

$$N(t) \le Ae^{m\omega\langle a\rangle} = \overline{N} \quad \text{for } t \ge T_2 = \max\{T, T_1\}.$$

On the other hand, $N'(t) \ge N(t)[a(t) + b(t)\overline{N} - c(t)\overline{N}^2]$.

Let $N(t)$ be an eventually oscillatory solution of (1.3) about B and let $\{s_n\}$ be a sequence such that $\lim_{n\to\infty} s_n = \infty$ and $N(s_n) = B$. Suppose that $N(s_n^*)$ is a local minimum of $N(t)$ on (s_n, s_{n+1}). Then

$$0 = N'(s_n^*) \geq N(s_n^*)\left[A_0 + b_0 N(s_n^* - m\omega) - c^0 N^2(s_n^* - m\omega)\right]$$
$$= c^0 N(s_n^*)\left[B - N(s_n^* - m\omega)\right]\left[N(s_n^* - m\omega) - \frac{b_0 - \sqrt{b_0^2 + 4a_0c^0}}{2c^0}\right].$$

So $B - N(s_n^* - m\omega) \leq 0$, that is, there exists a point $\overline{\eta} \in [s_n^* - m\omega, s_n^*]$ such that $N(\overline{\eta}) = B$. Let $\eta = \max\{s_n, \overline{\eta}\}$; then

$$\log\frac{N(s_n^*)}{B} \geq \int_\eta^{s_n^*}\left[a(t) + b(t)\overline{N} - c(t)\overline{N}^2\right]dt,$$

$$a(t) + b(t)\overline{N} - c(t)\overline{N}^2 \leq a^0 + b^0\overline{N} - c_0\overline{N}^2$$
$$= c_0[A - \overline{N}]\left[\overline{B} - \frac{b^0 - \sqrt{b^{0^2} + 4a^0c_0}}{2c_0}\right]$$
$$\leq 0.$$

So

$$N(s_n^*) \geq B\exp\left(\int_{s_n^*-m\omega}^{s_n^*}\left[a(t) + b(t)\overline{N} - c(t)\overline{N}^2\right]dt\right),$$

that is,

$$N(s_n^*) \geq Be^{\{m\omega\langle a\rangle + \overline{N}m\omega\langle b\rangle - \overline{N}^2 m\omega\langle c\rangle\}} = \underline{N}.$$

This completes the proof.

Finally, we derive a sufficient condition for the global attractivity of $N^*(t)$ with respect to all other positive solutions of (1.3) and (2.8).

We set

$$N(t) = N^*(t)e^{x(t)}. \tag{2.13}$$

Then

$$\frac{dx(t)}{dt} = b(t)N(t - m\omega) - c(t)N^2(t - m\omega)$$
$$- b(t)N^*(t - m\omega) + c(t)N^{*^2}(t - m\omega).$$

Put

$$G(u) = b(t)N^*(t)e^u - c(t)N^{*^2}(t)e^{2u}. \tag{2.14}$$

Then

$$G'(u) = b(t)N^*(t)e^u - 2c(t)N^*(t)e^{2u}$$

and

$$\frac{dx(t)}{dt} = G(x(t - m\omega)) - G(0). \tag{2.15}$$

By the mean value theorem, we can write (2.15) in the form

$$\frac{dx(t)}{dt} = -F(t)x(t - m\omega), \tag{2.16}$$

where

$$\begin{aligned} F(t) &= -G'(\xi(t)) \\ &= -b(t)N^*(t)e^{\xi(t)} + 2c(t)N^*(t)^2 e^{2\xi(t)} \\ &= -b(t)\eta(t) + 2c(t)\eta^2(t), \end{aligned}$$

and $\eta(t)$ lies between $N^*(t)$ and $N(t - m\omega)$. By Theorem 2.3, we have

$$-b(t)\underline{N} + 2c(t)\underline{N}^2 \le F(t) \le -b(t)\overline{N} + 2c(t)\overline{N}^2, \tag{2.17}$$

from which one can prove the following.

THEOREM 2.4. If

$$-m\omega\langle b\rangle\overline{N} + 2m\omega\langle c\rangle\overline{N}^2 < \frac{\pi}{2}, \tag{2.18}$$

then every solution of (1.3) and (2.8) satisfies

$$\lim_{t\to\infty}[N(t) - N^*(t)] = 0.$$

Proof. From the above discussion, if we let

$$N(t) = N^*(t)e^{x(t)}, \tag{2.19}$$

then we have

$$\frac{dx(t)}{dt} + F(t)x(t - m\omega) = 0, \tag{2.20}$$

where $F(t) > 0$. By a known result [6], if

$$\lim_{t\to\infty}\int_{t-m\omega}^{t} F(s)\,ds < \frac{\pi}{2}, \tag{2.21}$$

then every solution $x(t)$ of (2.20) tends to zero as $t \to \infty$. For (2.21) to hold it is sufficient that

$$\int_{t-m\omega}^{t}(-b(t)\overline{N} + 2c(t)\overline{N}^2)\,dt < \frac{\pi}{2};$$

that is,

$$-m\omega\langle b\rangle\overline{N} + 2m\omega\langle c\rangle\overline{N}^2 < \frac{\pi}{2}.$$

Therefore, $x(t)$ tends to zero as $t \to \infty$ giving us the desired conclusion.

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