

NOTE ON THE NUMERICAL CONSTRUCTION OF GEODESICS AND RAY PATHS

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Summary. This note discusses the choice of parameter in the numerical construction of a geodesic, and in particular the role of arc length.

Problem description. One often encounters the problem of minimizing an integral of the form

$$I = \int_A^B \phi \, ds \tag{1}$$

over paths in n -dimensional space (x_1, \dots, x_n) linking the two end points A and B . Here ϕ is a function of the (x_i) and ds is the element of path length, given by

$$ds^2 = g_{ij} dx_i dx_j \tag{2}$$

where the coefficients g_{ij} are given functions of the (x_i) . (In (2), repeated indices are summed from 1 to n ; we use this summation convention throughout. As usual, we also require g_{ij} to be symmetric with respect to its indices.) For example, the case $\phi = 1$ might correspond to the problem of finding the shortest path (a geodesic) along a surface embedded in a higher-dimensional space. Alternatively, the choice $\phi = 1/c$, where c is a position-dependent velocity, corresponds to a least-time ray path problem. There are some aspects of the numerical solution of the Euler questions associated with (1) that do not seem to be discussed in the literature.

Let a possible path linking A and B be described parametrically by $x_i = x_i(t)$, $i = 1, 2, \dots, n$, where the parameter t ranges from t_0 to t_1 . Writing \dot{x}_i for dx_i/dt , (1) becomes

$$I = \int_{t_0}^{t_1} \phi(x_1(t), \dots, x_n(t)) \sqrt{g_{ij} \dot{x}_i \dot{x}_j} \, dt. \tag{3}$$

Denoting $g_{ij} \dot{x}_i \dot{x}_j$ by $F(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n)$, the Euler equations are

$$\frac{d}{dt} \left[\frac{\phi}{2\sqrt{F}} \{2g_{ij} \dot{x}_j\} \right] = \phi_{,i} \sqrt{F} + \frac{\phi}{2\sqrt{F}} g_{pq,i} \dot{x}_p \dot{x}_q, \quad i = 1, 2, \dots, n, \tag{4}$$

where a comma indicates a partial derivative with respect to the corresponding x_i coordinate. This is a set of n second-order differential equations; it is probably well

Received April 22, 1992.

1991 *Mathematics Subject Classification.* Primary 34, 49, 65, 78.

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known that a direct numerical solution may encounter difficulties. Write (4) in the form

$$A_{ij} \ddot{x}_j = G_i(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n) \quad (5)$$

where

$$A_{ij} = \frac{\phi}{\sqrt{F}} \left(g_{ij} - \frac{1}{F} g_{ir} g_{pj} \dot{x}_r \dot{x}_p \right), \quad i = 1, 2, \dots, n. \quad (6)$$

To construct numerically a geodesic, with some chosen initial direction at A , a natural approach would be to take steps Δt and use the values of $\{\ddot{x}_j\}$ from (5) to obtain new values of the $\{\dot{x}_j\}$ at each mesh point. However, this procedure—if successful—would assign a precise value of t to each point on the solution path, and this result contradicts the fact that, in principle, t can be assigned arbitrarily along any given path. In other words, the matrix (A_{ij}) must be singular, and in fact this is easily proved directly by observing from (6) that $A_{ij} \dot{x}_i = 0$, which corresponds to the vanishing of a linear combination of rows.

An easy way around this difficulty is to specialize the parameter t —so as to correspond to one of the x_i , or to arc length, say. The structure of (5) then alters, and the $\{\ddot{x}_j\}$ may be solved for. It is common practice to choose the parameter t so that, for the solution curve, it will equal the arc distance s from A . Then $F = 1$, and (4) simplifies to

$$\phi_{,k} \dot{x}_k g_{ij} \dot{x}_j + \phi g_{ij,k} \dot{x}_k \dot{x}_j + \phi g_{ij} \ddot{x}_j = \phi_{,i} + \frac{1}{2} \phi g_{pq,i} \dot{x}_p \dot{x}_q, \quad i = 1, 2, \dots, n, \quad (7)$$

where \dot{x}_i now represents dx_i/ds .

A typical numerical approach to the solution of the set (7) is to start at one end point and choose a tentative initial direction as specified by $\{\dot{x}_j\}$ values at that point (satisfying $F = 1$, of course). Equations (7) are then solved using some step size Δs , so as to give the path emanating from the initial point with that particular choice of direction; the idea is to iteratively modify the choice of initial direction until the path passes through the other end point.

At this point, an interesting question arises. In the set (7), s is simply a parameter and the requirement that it represent arc length is not explicitly incorporated. Intuitively, one may well feel that s will be found to correspond to physical distance along the numerically-generated solution curve; however, it may be of interest to provide a simple proof that this will indeed be the case. In mathematical terms, one could say that the set (7) represents a necessary condition that t correspond to arc length; we will show that it is also a sufficient condition. (A generalization to other choices for t will be made subsequently.)

Thus we consider the set (7), where s is simply some parameter, whose relation to arc length is to be determined. From the definition of F , we have

$$\frac{dF}{ds} = g_{ij,k} \dot{x}_i \dot{x}_j \dot{x}_k + 2g_{ij} \dot{x}_i \ddot{x}_j. \quad (8)$$

Since $F = 1$ at the initial point $s = 0$, we need only use (7) to show that (8) implies that $dF/ds = 0$. Our approach is to construct a differential equation for F . Assuming that ϕ vanishes nowhere, (7) may be divided by ϕ , multiplied by \dot{x}_i ,

and summed over the repeated index i to give

$$\begin{aligned} g_{ij,k} \dot{x}_i \dot{x}_j \dot{x}_k + 2g_{ij} \dot{x}_i \ddot{x}_j &= 2 \frac{\phi_{,i} \dot{x}_i}{\phi} (1 - g_{ij} \dot{x}_i \dot{x}_j) \\ &= 2 \frac{\phi_{,i} \dot{x}_i}{\phi} (1 - F). \end{aligned} \tag{9}$$

Thus (8) becomes

$$\frac{dF}{ds} = 2 \frac{\phi_{,i} \dot{x}_i}{\phi} (1 - F). \tag{10}$$

The unique solution of (10), corresponding to the initial condition $F = 1$, is clearly $F \equiv 1$. Thus, apart from round-off and truncation error, the parameter s encountered in the numerical solution of (7) must coincide with arc length.

More generally, any reasonable relation between t and s may be imposed. Write

$$\left(\frac{ds}{dt}\right)^2 = F = \psi(t) \tag{11}$$

where the positive function $\psi(t)$ may be specified arbitrarily (except that it should be finite and nonzero, to avoid possible pathologies). Then a substitution of $F = \psi(t)$ in (4), followed by a calculation similar to that given above, leads to

$$\frac{dF}{dt} = 2 \frac{\phi_{,r} \dot{x}_r}{\phi} (\psi - F) + \frac{\dot{\psi}}{\psi} F \tag{12}$$

where a superposed dot now once more denotes d/dt . This differential equation for F has the solution

$$F = \psi + C \frac{\psi}{\phi^2} \tag{13}$$

where C is an integration constant. Since the initial direction (\dot{x}_i) must be chosen so that $F = \psi$ at the point A , it follows that $C = 0$, so that the expected result $F \equiv \psi$ is again obtained.

The special choice $F = \psi = \phi^2$ is sometimes made in seismology; the replacement of ϕ/\sqrt{F} by unity then simplifies (4). Note that the general solution of the associated differential equation, as given by (13), is then also simplified.