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THE GENERICITY OF FLUTTER ILL-POSEDNESS IN 3-DIMENSIONAL ELASTIC-PLASTIC MODELS

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0. Introduction. In elastic-plastic models for granular material, it is common that the governing partial differential equations become ill-posed as plastic deformations are accumulated. In dynamical formulations, ill-posedness occurs if the governing equations lose their hyperbolicity. Equivalently, ill-posedness occurs if wave speeds become complex. Ill-posedness due to wave speed becoming zero and then pure imaginary has been studied extensively [M, S]. It was believed that this type of ill-posedness is related to the formation of shear bands. In our paper, we shall investigate the case that wave speeds become equal (the equations are not strictly hyperbolic) and then complex with nonzero real part. Following Rice [R], we call it flutter ill-posedness.

For two-dimensional models, An and Schaeffer [A, A-S] investigated the same problem. It was found that, even in the simplest of elastic-plastic models, the condition for the onset of flutter ill-posedness—wave speeds being equal—may be reached. By a topological argument, it was shown that a generic perturbation leads to equations with flutter ill-posedness in a neighborhood of a certain hardening modulus. In these papers, a readily applicable criterion for the occurrence of flutter ill-posedness is derived. It is demonstrated that flutter ill-posedness occurs in widely accepted models.

Recently, Loret [L] extended the results to three-dimensional models. It was shown that, whatever the hardening modulus, the dynamical equations of motion are never strictly hyperbolic; that is, in some direction, two wave speeds are always equal. Moreover, for some discrete values of the hardening modulus, the three wave speeds become equal. By algebraic calculation, he showed that, when the flow rule deviates from deviatoric associativity, the governing equations could exhibit flutter ill-posedness in a neighborhood of the discrete values of the hardening modulus.

In our paper, we continue to discuss flutter ill-posedness in three-dimensional models. For the case of three wave speeds being equal, we employ a topological argument,

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which is different from Loret's approach. By studying the change of topological degree of a certain mapping, we conclude that, generically, deviation from deviatoric associativity in the flow rule leads to equations with flutter ill-posedness in a neighborhood of the discrete values of the hardening modulus. A sufficient condition is given for the occurrence of flutter ill-posedness.

For the case of two wave speeds being equal, we investigate more general perturbations (in fact, deviation from deviatoric associativity alone cannot cause flutter ill-posedness in this case). We show that a small perturbation could lead to equations with flutter ill-posedness over a large range of the hardening modulus. It is demonstrated that nondeviatoric associativity in the flow rule and the rotational terms in the Jaumann rate will cause flutter ill-posedness. In particular, for the yield vertex model, flutter ill-posedness occurs near the direction of coincident transverse wave speeds when the Jaumann rate is used.

Note that, in the latter case, the occurrence of flutter ill-posedness almost does not depend on the value of the hardening modulus. Therefore, flutter ill-posedness is more likely to occur in three-dimensional models than in two-dimensional models.

This paper is divided into four sections. In Sec. 1, the governing equations will be given and the eigenvalue problem formulated. In Sec. 2, the acoustic tensor, derived from the eigenvalue problem, is reduced in a moving coordinate; a preliminary lemma from algebra is given and coincident wave speeds are analyzed. In Sec. 3, we study the case of three coincident wave speeds under deviation from deviatoric associativity in the flow rule. In Sec. 4, we study the case of two coincident transverse wave speeds under general perturbations.

1. Formulation of the eigenvalue problem.

1.1. *The governing equations.* The unknowns consist of the bulk density ρ, Cauchy stress tensor T, and velocity vector v, subject to conservation of mass and momentum

$$d_t\rho + \rho\partial_j v_j = 0, \tag{1.1}$$

$$\rho d_t v_i + \partial_j T_{ij} = 0, \tag{1.2}$$

where $d_t = \partial_t + v_j\partial_j$ is the material derivative and the summation convention is employed. In our formulation, compressive stresses are assumed to be positive.

To formulate the constitutive relation, we decompose the strain rate tensor

$$V_{ij} = -\frac{1}{2}(\partial_i v_j + \partial_j v_i)$$

into elastic and plastic parts

$$V = V^{(e)} + V^{(p)}. \tag{1.3}$$

For the elastic part, we assume the linear strain-stress relation,

$$V_{ij}^{(e)} = C_{ijkl}\nabla_t T_{kl}, \tag{1.4}$$

where C is a fourth-order tensor whose inverse E can be expressed through the shearing modulus G and Poisson's ratio ν by

$$E_{ijkl} = \frac{2\nu G}{1-2\nu}\delta_{ij}\delta_{kl} + G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{1.5}$$

To get an objective measure of the rate of change of stress, we use the Jaumann co-rotational rate

$$\nabla_t T_{ij} = d_t T_{ij} - T_{ik}\omega_{kj} - T_{jk}\omega_{ki},$$

where

$$\omega_{kl} = \frac{1}{2}(\partial_k v_l - \partial_l v_k)$$

is the spin tensor. Later on we take the rotational terms in the Jaumann derivative as perturbation terms, since the magnitude of stress T is quite small compared with the shearing modulus G, typically

$$|T|/G \sim 0.01.$$

For the plastic part, we assume that

$$V_{ij}^{(p)} = \frac{1}{h}\Psi_{ij}\Phi_{kl}\nabla_t T_{kl}. \tag{1.6}$$

The derivation of (1.6) is sketched as follows. Firstly, the flow rule gives

$$V^{(p)} = \lambda\Psi, \tag{1.7}$$

where λ is a scalar variable and Ψ is a symmetric tensor indicating the direction of plastic deformation. Secondly, differentiating the yield function $\phi(T, \gamma) = 0$ gives

$$\frac{\partial\phi}{\partial T_{kl}}\nabla_t T_{kl} + \frac{\partial\phi}{\partial\gamma}d_t\gamma = 0, \tag{1.8}$$

where γ is the total shearing strain defined by

$$d_t\gamma = \left|\left(V^{(p)}\right)^D\right|, \tag{1.9}$$

and for 3×3 matrices $\left\{A_{ij}\right\}$, the deviator and the norm of A are defined as

$$A^D = A - \frac{1}{3}\operatorname{tr}(A)I, \qquad |A|^2 = \frac{1}{2}\Sigma_{ij}A_{ij}^2.$$

It follows from (1.7)–(1.9) that

$$\lambda = \frac{\left|\left(V^{(p)}\right)^D\right|}{\left|\Psi^D\right|} = \frac{1}{h}\Phi_{kl}\nabla_t T_{kl}, \tag{1.10}$$

where $h = -|\Psi^D|\frac{\partial\phi}{\partial\gamma}$ is the plastic hardening modulus which changes from $+\infty$ to 0 as plastic deformations are accumulated, and $\Phi = \frac{\partial\phi}{\partial T}$ is the normal direction to the yield surface. Finally, substituting (1.10) in (1.7) yields (1.6). For convenience, we normalize Ψ and Φ in the sense that

$$|\Psi^D| = |\Phi^D| = 1.$$

Then, we can write Ψ and Φ as

$$\Psi_{ij} = \Psi_{ij}^D - \beta\delta_{ij}, \qquad \Phi_{ij} = \Phi_{ij}^D - \mu\delta_{ij}.$$

In practice, Ψ and Φ might depend on history as well as stress. The parameter μ specifies the angle of internal friction of the material, and β specifies the amount of dilation. Typically, we assume that

$$1 > \mu > \beta > 0.$$

The flow rule satisfies deviatoric associativity if and only if $\Psi^D = \Phi^D$. The reader can consult [A] for deriving (1.6) in general cases and a physical description of its terms in greater detail.

Combining (1.3), (1.4), and (1.6), we have

$$V_{ij} = \left(C_{ijkl} + \frac{1}{h}\Psi_{ij}\Phi_{kl}\right)\nabla_t T_{kl}.$$

Since the fourth-order tensor C is invertible, the above relation can be rewritten (cf. Lemma 1.3 in [A]) as

$$\nabla_t T_{ij} = \left(E_{ijkl} - \frac{1}{H}E_{ijmn}\Psi_{mn}\Phi_{rs}E_{rskl}\right)V_{kl}, \tag{1.11}$$

where E is given in (1.5) and H is defined as

$$H = h + \Phi_{ij}E_{ijkl}\Psi_{kl},$$

where $H > 0$, an assumption that holds for a large class of materials.

1.2. *The eigenvalue problem.* In this subsection, we linearize the equations (1.1), (1.2), and (1.11) first. Then by looking for exponential solutions, we obtain an eigenvalue problem. The square of the wave speed is the eigenvalue of the derived acoustic tensor.

Suppose that $\rho^{(0)}$, $v^{(0)}$, and $T^{(0)}$ are the homogeneous solutions (see [S] for its existence). We assume that the material undergoes continual loading beyond this uniform deformation. In fact, only accumulated plastic deformations can cause the governing equations to become ill-posed. Now we substitute

$$\rho = \rho^{(0)} + \bar{\rho}, \qquad v = v^{(0)} + \bar{v}, \qquad T = T^{(0)} + \bar{T}$$

into the equations (1.1), (1.2), and (1.11) and retain only terms of first order in the incremental variables $\bar{\rho}$, $\bar{v}$, and $\bar{T}$. Thus we have

$$\begin{aligned} d_t\bar{\rho} + \rho^{(0)}\operatorname{div}\bar{v} + (\operatorname{div} v^{(0)})\bar{\rho} + \bar{v}\cdot\operatorname{grad}\rho^{(0)} &= 0, \\ \rho^{(0)}d_t\bar{v} + \operatorname{div}\bar{T} + \rho^{(0)}\bar{v}\cdot\operatorname{grad} v^{(0)} + \bar{\rho}d_t\bar{v}^{(0)} &= 0, \\ d_t\bar{T}_{ij} + B_{ijkl}\partial_l\bar{v}_k + \bar{B}_{ijkl}\bar{T}_{kl} + \bar{v}_k\partial_k T_{ij}^{(0)} &= 0, \end{aligned}$$

where $\bar{B}_{ijkl}$ is not given explicitly because of its irrelevance and

$$B_{ijkl} = E_{ijkl} + J_{ijkl} - \frac{1}{H}E_{ijmn}\Psi_{mn}\Phi_{rs}E_{rskl},$$

in which J_{ijkl} comes from the Jaumann derivative

$$J_{ijkl} = -\frac{1}{2}(\delta_{ik}T_{jl} - \delta_{il}T_{jk} + \delta_{jk}T_{il} - \delta_{jl}T_{ik}).$$

Now we extract the principal part of the linearized equations and substitute the following form of solution in it:

$$\bar{\rho} = e^{i(x\cdot\xi)+\lambda t}\hat{\rho}, \qquad \bar{v} = ie^{i(x\cdot\xi)+\lambda t}\hat{v}, \qquad \bar{T} = e^{i(x\cdot\xi)+\lambda t}\hat{T}, \tag{1.12}$$

where $\hat{\rho}$, $\hat{v}$, and $\hat{T}$ are constants, $\xi = (\xi_1, \xi_2, \xi_3)$ is a vector in Fourier space, $(x\cdot\xi)$ is the inner product of x and ξ, and $\lambda \in \mathbb{C}$ is to be determined. We obtain the following eigenvalue problem:

$$\begin{pmatrix} 0 & \rho^{(0)}\xi^{\mathrm{T}} & 0 \\ 0 & 0 & -\frac{1}{\rho^{(0)}}L(\xi) \\ 0 & M(\xi) & 0 \end{pmatrix} \hat{U} = (\lambda + iv_j^{(0)}\xi_j)\hat{U},$$

where $\hat{U} = \left(\hat{\rho}, \hat{v}, (\hat{T}_{11}, \hat{T}_{22}, \hat{T}_{33}, \hat{T}_{12}, \hat{T}_{13}, \hat{T}_{23})\right)^{\mathrm{T}}$ and

$$L(\xi) = \begin{pmatrix} \xi_1 & 0 & 0 & \xi_2 & \xi_3 & 0 \\ 0 & \xi_2 & 0 & \xi_1 & 0 & \xi_3 \\ 0 & 0 & \xi_3 & 0 & \xi_1 & \xi_2 \end{pmatrix},$$

$$M(\xi) = \begin{pmatrix} B_{111l}\xi_l & B_{112l}\xi_l & B_{113l}\xi_l \\ B_{221l}\xi_l & B_{222l}\xi_l & B_{223l}\xi_l \\ B_{331l}\xi_l & B_{332l}\xi_l & B_{333l}\xi_l \\ B_{121l}\xi_l & B_{122l}\xi_l & B_{123l}\xi_l \\ B_{131l}\xi_l & B_{132l}\xi_l & B_{133l}\xi_l \\ B_{231l}\xi_l & B_{232l}\xi_l & B_{233l}\xi_l \end{pmatrix}.$$

Excepting for one zero eigenvalue, the rest of the eigenvalues will be determined from

$$Q = \begin{pmatrix} 0 & -\frac{1}{\rho^{(0)}}L(\xi) \\ M(\xi) & 0 \end{pmatrix}.$$

It is easy to see that Q has three zero eigenvalues, whose eigenvectors are $(0, 0, 0, \bar{r}_i)^{\mathrm{T}}$ $(i = 1, 2, 3)$ where $\bar{r}_i^{\mathrm{T}} \in \mathbb{R}^6$ $(i = 1, 2, 3)$ form the kernel of $L(\xi)$. The remaining six eigenvalues of Q can be found by studying the following matrix

$$\frac{1}{\rho^{(0)}}L(\xi)M(\xi) = \frac{1}{\rho^{(0)}}\xi_i B_{ijkl}\xi_l. \tag{1.13}$$

Suppose that (1.13) has eigenvalues μ_i $(i = 1, 2, 3)$ with eigenvectors e_i $(i = 1, 2, 3)$. Then Q has eigenvalues $\pm\sqrt{\mu_i}$ $(i = 1, 2, 3)$ whose corresponding eigenvectors are

$$\left(e_i^{\mathrm{T}}, \pm\mu_i^{-\frac{1}{2}}(M(\xi)e_i)^{\mathrm{T}}\right)^{\mathrm{T}}, \qquad i = 1, 2, 3$$

(summation convention is not used here).

The eigenvalue problem can also be derived in the following way. Substituting (1.12) in the linearized equations, we obtain

$$\rho^{(0)}(\lambda + iv_r^{(0)}\xi_r)\hat{v}_j + \xi_i\hat{T}_{ij} = 0, \tag{1.14}$$

$$(\lambda + iv_r^{(0)}\xi_r)\hat{T}_{ij} = B_{ijkl}\xi_l\hat{v}_k. \tag{1.15}$$

Substituting (1.15) in (1.14) gives

$$\rho^{(0)}(\lambda + iv_r^{(0)}\xi_r)^2\hat{v}_j + \xi_i B_{ijkl}\xi_l\hat{v}_k = 0. \tag{1.16}$$

To obtain nontrivial solutions $\hat{v}$, it is necessary that $\mu_j^2 = (\lambda_j + iv_r^{(0)}\xi_r)^2$ $(j = 1, 2, 3)$ are eigenvalues of $-\frac{1}{\rho^{(0)}}\xi_i B_{ijkl}\xi_l$.

In the engineering literature $\left(\frac{1}{\rho^{(0)}}\xi_i B_{ijkl}\xi_l\right)$ in (1.13) is called the acoustic tensor. The wave speeds are actually equal to $\mu_i^{\frac{1}{2}}/|\xi|$. Since μ_i $(i = 1, 2, 3)$ are homogeneous of degree two in ξ, the wave speeds depend on the direction of ξ but not on the magnitude of ξ. Let

$$S^2 = \left\{\xi \in \mathbb{R}^3 : |\xi| = 1\right\}.$$

Then the acoustic tensor is a function of ξ and h, defined on $S^2 \times \mathbb{R}^+$.

2. Analysis of the acoustic tensor.

2.1. *Reduction of the acoustic tensor.* To study the eigenvalues of the acoustic tensor, it is convenient to formulate the tensor under a moving coordinate and then to subtract the obtained tensor by a multiple of the identity.

Choose two unit vectors η, ζ that are perpendicular to ξ and also are perpendicular to each other. Let

$$R = \begin{pmatrix} \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \\ \xi_3 & \eta_3 & \zeta_3 \end{pmatrix}. \tag{2.1}$$

Multiplying the acoustic tensor left by R^{T} and right by R, we have

$$\frac{1}{\rho^{(0)}}\left\{E_{1ij1} + J_{1ij1}(\tilde{T}) - \frac{1}{H}E_{1imn}\tilde{\Psi}_{mn}\tilde{\Phi}_{rs}E_{rsj1}\right\}, \tag{2.2}$$

where $\tilde{T} = R^{\mathrm{T}}TR$, $\tilde{\Psi} = R^{\mathrm{T}}\Psi R$, and $\tilde{\Phi} = R^{\mathrm{T}}\Phi R$. During the calculation, we use the fact that E satisfies the isotropy condition

$$E_{mnrs} = E_{ijkl}R_{mi}R_{nj}R_{rk}R_{sl}.$$

Subtracting

$$\frac{1}{\rho^{(0)}}\left(G - \frac{1}{2}\tilde{T}_{11} - \frac{1}{6}T_{kk}\right)\delta_{ij}$$

from (2.2), we obtain $(\rho^{(0)})^{-1}$ times

$$A_{ij}(\xi, h) = \begin{pmatrix} \gamma + \frac{1}{2}\tilde{T}_{11}^D - \frac{1}{H}b_1c_1 & \tilde{T}_{12} - \frac{1}{H}b_1c_2 & \tilde{T}_{13} - \frac{1}{H}b_1c_3 \\ -\frac{1}{H}b_2c_1 & \frac{1}{2}\tilde{T}_{22}^D - \frac{1}{H}b_2c_2 & \frac{1}{2}\tilde{T}_{23} - \frac{1}{H}b_2c_3 \\ -\frac{1}{H}b_3c_1 & \frac{1}{2}\tilde{T}_{23} - \frac{1}{H}b_3c_2 & \frac{1}{2}\tilde{T}_{33}^D - \frac{1}{H}b_3c_3 \end{pmatrix}, \tag{2.3}$$

where $\gamma = \frac{G}{1-2\nu}$ and $\tilde{T}_{ij}^D$ is the (i, j)-component of the deviatoric part of $\tilde{T}$. Let $\kappa = \frac{1+\nu}{1-2\nu}$. We have

$$\begin{aligned} b_i &= E_{1imn}\tilde{\Psi}mn = -2G(\kappa\beta\delta_{1i} - \tilde{\Psi}_{1i}^D), \\ c_j &= \tilde{\Phi}_{rs}E_{rsj1} = -2G(\kappa\mu\delta_{1j} - \tilde{\Phi}_{1j}^D), \\ H &= h + 2G(3\kappa\mu\beta + \Psi_{ij}^D\Phi_{ij}^D). \end{aligned} \tag{2.4}$$

REMARK 2.1. Under this moving coordinate, when A is diagonal, we actually have longitudinal and transverse waves. In fact, it follows from (1.16) that

$$-\frac{1}{\rho^{(0)}}\left\{R^{\mathrm{T}}\left(\xi_i B_{ijkl}\xi_l\right)R\right\}R^{\mathrm{T}}\hat{v} = \mu^2\left(R^{\mathrm{T}}\hat{v}\right).$$

When A is diagonal, $R^{\mathrm{T}}\hat{v} = \lambda e_i$, where λ is a constant and e_i is the ith column of the identity matrix. It follows immediately that

$$\hat{v} = \lambda R e_i.$$

So the eigenvector $\hat{v}$ is either parallel to ξ or perpendicular to ξ. But this is not true in general if A is not diagonal.

2.2. *Preliminary lemma from algebra.* Since the eigenvalues of a matrix A are roots of the characteristic polynomial of A, we shall review some knowledge about the discriminant of a third-order polynomial and the relation between the discriminant and the invariants of A.

For a 3×3 matrix $A = \left\{a_{ij}\right\}$, there are three basic invariants:

$$\begin{aligned} J_1 &= a_{11} + a_{22} + a_{33} = \operatorname{tr}(A), \\ J_2 &= a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32}, \\ J_3 &= \det(A). \end{aligned}$$

The characteristic polynomial $\det(a_{ij} - c\delta_{ij})$ of A can be written as

$$\det(a_{ij} - c\delta_{ij}) = -c^3 + J_1 c^2 - J_2 c + J_3$$

whose roots correspond to the eigenvalues of A. The discriminant of the characteristic polynomial is equal to

$$\operatorname{disc}(A) = J_1^2 J_2^2 - 4J_2^3 - 4J_1^3 J_3 + 18 J_1 J_2 J_3 - 27 J_3^2. \tag{2.5}$$

LEMMA 2.2. Two of the eigenvalues of A are complex conjugate if and only if

$$\operatorname{disc}(A) < 0.$$

Proof. Let $f(c) = \det(a_{ij} - c\delta_{ij})$. Differentiating $f(c)$ with respect to c, we have

$$f'(c) = -3c^2 + 2J_1 c - J_2,$$

whose roots are

$$c_{1,2} = \frac{1}{3}\left(J_1 \pm \sqrt{J_1^2 - 3J_2}\right).$$

(1) In the case of $J_1^2 < 3J_2$, roots c_1 and c_2 are complex conjugate. The derivative $f'(c)$ does not change sign, and $f(c) = 0$ has a pair of complex conjugate roots. Since $f(\bar{c}) = \overline{f(c)}$ where $\bar{a}$ means the conjugate of a, we have

$$f(c_1)\cdot f(c_2) = |f(c_1)|^2 > 0.$$

(2) In the case of $J_1^2 = 3J_2$, we have $c_1 = c_2$ and

$$f(c) = -\left(c - \frac{1}{3}J_1\right)^3 - \frac{1}{27}J_1^3 + J_3.$$

When $J_1^3 = 27J_3$, all three roots are the same and $f(c_1) = 0$. When $J_1^3 \neq 27J_3$, there are a pair of complex conjugate roots and

$$f(c_1) \cdot f(c_2) = f^2(c_1) > 0.$$

(3) In the case of $J_1^2 > 3J_2$, the function $f(c)$ achieves a local maximum at c_1 and a local minimum at c_2. The function $f(c)$ has a pair of complex conjugate roots if and only if $f(c_1)$ and $f(c_2)$ have the same sign, that is,

$$f(c_1) \cdot f(c_2) > 0.$$

Combining these three cases, we have the fact that $f(c) = 0$ has a pair of complex conjugate roots if and only if $f(c_1) \cdot f(c_2) > 0$. It is easy to check that

$$\operatorname{disc}(A) = -\frac{1}{27}f(c_1) \cdot f(c_2),$$

and the conclusion follows.

2.3. *Coincident wave speeds.* In this subsection we discuss the simplest case in which the flow rule satisfies deviatoric associativity and the rotation terms in the Jaumann derivative are not included. In this case, flutter ill-posedness does not occur. But the onset of flutter ill-posedness is reached and wave speeds become equal in certain circumstances.

THEOREM 2.3. (1) The two eigenvalues corresponding to transverse waves are equal if and only if ξ is one of eigenvectors of Ψ.

(2) The three eigenvalues of the acoustic tensor are equal if and only if ξ is one of the eigenvectors of Ψ and the hardening modulus h is equal to one of the following values:

$$h_i = 2G\left[2(1-2\nu)(\kappa\mu - p_i)(\kappa\beta - p_i) - 3\kappa\mu\beta - 2\right], \qquad i = 1, 2, 3,$$

where p_i $(i = 1, 2, 3)$ are eigenvalues of Ψ^D.

In the proof, we shall see that when ξ is one of eigenvectors of Ψ, the matrix A is diagonal. From Remark 2.1 it follows that we have a natural decomposition into longitudinal and transverse waves. In this event, one eigenvalue corresponds to the longitudinal wave and the remaining two eigenvalues correspond to transverse waves.

Proof of Theorem 2.3. To find out when coincident eigenvalues could occur it is sufficient to see when the discriminant is equal to zero. Without the rotation terms in the Jaumann derivative (terms containing stress T are absent in (2.3)), the determinant of $A(\xi, h)$ is zero $(J_3 \equiv 0)$ and

$$J_1 = \gamma - \frac{1}{H}(b_1c_1 + b_2c_2 + b_3c_3),$$
$$J_2 = -\frac{\gamma}{H}(b_2c_2 + b_3c_3).$$

It gives

$$\begin{aligned}\operatorname{disc}(A) &= J_2^2(J_1^2 - 4J_2)\\ &= \frac{\gamma^2}{H^2}(b_2c_2 + b_3c_3)^2\left\{\left(\gamma - \frac{1}{H}(b_1c_1 + b_2c_2 + b_3c_3)\right)^2 + \frac{4\gamma}{H}(b_2c_2 + b_3c_3)\right\}.\end{aligned} \tag{2.6}$$

When the flow rule satisfies deviatoric associativity $(\Psi^D = \Phi^D)$, it follows from (2.4) that

$$b_2 = c_2, \qquad b_3 = c_3.$$

Therefore, we have $\operatorname{disc}(A) \geq 0$ and it will be zero if and only if $b_2 = b_3 = 0$ or, equivalently,

$$\tilde{\Psi}_{12} = \tilde{\Psi}_{13} = 0.$$

LEMMA 2.4. Suppose that A is a 3×3 symmetric matrix and ξ, η, and ζ are three mutually orthogonal unit vectors. Then $\xi^{\mathrm{T}}A\eta = 0$, $\xi^{\mathrm{T}}A\zeta = 0$ if and only if ξ is an eigenvector of A. Moreover, $\xi^{\mathrm{T}}A\xi$ is the eigenvalue of A corresponding to ξ.

Proof. Suppose that R is the matrix whose columns consist of ξ, η, and ζ. If $\eta^{\mathrm{T}}A\xi = 0$ and $\zeta^{\mathrm{T}}A\xi = 0$, then

$$A\xi = RR^{\mathrm{T}}A\xi = R(\xi^{\mathrm{T}}A\xi, \eta^{\mathrm{T}}A\xi, \zeta^{\mathrm{T}}A\xi) = (\xi^{\mathrm{T}}A\xi)\xi.$$

If $A\xi = \lambda\xi$, then

$$\xi^{\mathrm{T}}A\xi = \lambda, \qquad \eta^{\mathrm{T}}A\xi = \lambda\eta^{\mathrm{T}}\xi = 0, \qquad \zeta^{\mathrm{T}}A\xi = \lambda\zeta^{\mathrm{T}}\xi = 0.$$

The proof is complete.

According to Lemma 2.4, the facts of $\tilde{\Psi}_{12} = \xi^{\mathrm{T}}\Psi^D\eta = 0$, $\tilde{\Psi}_{13} = \xi^{\mathrm{T}}\Psi^D\zeta = 0$ imply that ξ is one of the eigenvectors of Ψ^D as well as of Ψ. In this case, the matrix A in (2.3) is diagonal and two transverse wave speeds are equal (two eigenvalues with eigenvectors η and ζ are equal). Note that the transverse wave speed is nonzero since we subtract a multiple of the identity. When, in addition to ξ being an eigenvector of Ψ,

$$\gamma - \frac{1}{H}b_1c_1 = 0, \tag{2.7}$$

the longitudinal wave speed is also equal to the transverse wave speed. In this case, the second factor in (2.6) is equal to zero. The equality (2.7) holds when, from (2.4),

$$H_i = \frac{4G^2}{\gamma}(\kappa\beta - \tilde{\Psi}_{11}^D)(\kappa\mu - \tilde{\Psi}_{11}^D) = \frac{4G^2}{\gamma}(\kappa\beta - p_i)(\kappa\mu - p_i), \qquad i = 1, 2, 3, \tag{2.8}$$

or in terms of h,

$$h_i = 2G\left[2(1 - 2\nu)(\kappa\mu - p_i)(\kappa\beta - p_i) - 3\kappa\mu\beta - 2\right], \qquad i = 1, 2, 3, \tag{2.9}$$

where $p_i = \xi^{\mathrm{T}}\Psi^D\xi$ ($i = 1, 2, 3$, ξ is one of eigenvectors of Ψ^D) are eigenvalues of Ψ^D satisfying

$$p_1^2 + p_2^2 + p_3^2 = 2, \qquad p_1 + p_2 + p_3 = 0.$$

The proof of Theorem 2.3 is complete.

REMARK 2.5. From the proof we also see that the equality $J_2 = 0$ corresponds to two coincident transverse wave speeds and that both $J_2 = 0$ and $J_1 = 0$ correspond to three coincident wave speeds.

3. Deviation from deviatoric associativity in the flow rule.

3.1. *Statement of results.* In this section, we only allow deviation from deviatoric associativity ($\Psi^D \neq \Phi^D$) and the rotational terms in the Jaumann derivative are excluded. In this case,

$$\mathrm{disc}(A) = J_2^2(J_1^2 - 4J_2).$$

The value of $\mathrm{disc}(A)$ could become negative only if $J_1^2 - 4J_2$ becomes negative. So flutter ill-posedness could occur near h_i ($i = 1, 2, 3$) given by (2.9) where three wave speeds are equal. This is the case analyzed by Loret [L]. In this section, we shall use a different approach to prove the following conclusion.

THEOREM 3.1. Generically, the nondeviatoric associativity in the flow rule will make the discriminant become negative in a neighborhood of the discrete values h_i ($i = 1, 2, 3$) given by (2.9). In other words, generically, the nondeviatoric associativity in the flow rule will lead to equations with flutter ill-posedness near h_i.

The proof will be given in the next subsection 3.2.

In general, under perturbation, $J_1^2 - 4J_2$ could become strictly positive. But our topological arguments show that nondeviatoric associativity in the flow rule cannot perturb $J_1^2 - 4J_2$ away from zero in the positive direction and, in fact, that such perturbations are likely to even make $J_1^2 - 4J_2$ become negative somewhere. In terms of terminology from the dynamical bifurcation theory, this grazing of the stability boundary is "structurally stable".

For those who may not completely trust generic arguments, we shall give a sufficient condition for the occurrence of flutter ill-posedness in subsection 3.3. We also compare our results with Loret's there.

3.2. *Topological proof of Theorem* 3.1. We derive from (2.3) and (2.4) that

$$\begin{aligned} J_1^2 - 4J_2 = \frac{16\gamma G^2}{H} * \Bigg\{ \Bigg[\frac{1}{4}\sqrt{\frac{H}{G(1-2\nu)}} - \sqrt{\frac{G(1-2\nu)}{H}} \\ \times \left((\kappa\mu - \tilde{\Psi}_{11}^D)(\kappa\beta - \tilde{\Phi}_{11}^D) + \tilde{\Psi}_{12}\tilde{\Phi}_{12} + \tilde{\Psi}_{13}\tilde{\Phi}_{13} \right) \Bigg]^2 \\ + \frac{1}{4}\left(\tilde{\Psi}_{12} + \tilde{\Phi}_{12}\right)^2 + \frac{1}{4}\left(\tilde{\Psi}_{13} + \tilde{\Phi}_{13}\right)^2 \\ - \frac{1}{4}\left(\tilde{\Psi}_{12} - \tilde{\Phi}_{12}\right)^2 - \frac{1}{4}\left(\tilde{\Psi}_{13} - \tilde{\Phi}_{13}\right)^2 \Bigg\}. \end{aligned}$$

Now we study the mapping $F_H : S^2 \to \mathbb{R}^5$ given by

$$y_1 = \frac{1}{4}\sqrt{\frac{H}{G(1-2\nu)}} - \sqrt{\frac{G(1-2\nu)}{H}}\left((\kappa\mu - \tilde{\Psi}_{11}^D)(\kappa\beta - \tilde{\Phi}_{11}^D) + \tilde{\Psi}_{12}\tilde{\Phi}_{12} + \tilde{\Psi}_{13}\tilde{\Phi}_{13}\right),$$

$$y_2 = \frac{1}{2}\left(\tilde{\Psi}_{12} + \tilde{\Phi}_{12}\right), \qquad y_3 = \frac{1}{2}\left(\tilde{\Psi}_{13} + \tilde{\Phi}_{13}\right),$$

$$y_4 = \frac{1}{2}\left(\tilde{\Psi}_{12} - \tilde{\Phi}_{12}\right), \qquad y_5 = \frac{1}{2}\left(\tilde{\Psi}_{13} - \tilde{\Phi}_{13}\right), \tag{3.1}$$

where $\tilde{\Psi}_{ij}^D$ and $\tilde{\Phi}_{ij}^D$ are functions on $S^2 = \left\{\xi \in \mathbb{R}^3 : |\xi| = 1\right\}$. Note that $J_1^2 - 4J_2 < 0$ if and only if $y^{\mathrm{T}}Ny < 0$, where

$$N = \begin{pmatrix} I_3 & 0 \\ 0 & -I_2 \end{pmatrix} \tag{3.2}$$

and I_2, I_3 are block identity matrices (subscript indicates its order). Let

$$\Gamma = \left\{y \in \mathbb{R}^5 : y^{\mathrm{T}}Ny < 0\right\}.$$

Flutter ill-posedness does not occur if and only if

$$F_H(S^2) \in \mathbb{R}^5 \setminus \Gamma \quad \text{for all } H > 0,$$

where $F_H(S^2)$ is the image of the mapping F_H.

It is easy to check that

$$\mathbb{R}^5 \setminus \bar{\Gamma} \cong \mathbb{R}^3 \setminus \{0\} \cong S^2,$$

where $\cong$ denotes the homotopy relation and $\bar{\Gamma}$ is the closure of Γ. Studying topological properties of the image $F_H(S^2)$ in $\mathbb{R}^5 \setminus \bar{\Gamma}$ is equivalent to studying topological properties of $\bar{F}_H(S^2)$ in S^2 where $\bar{F}_H : S^2 \to S^2$ is the induced mapping defined by

$$z_i = \frac{y_i}{\sqrt{y_1^2 + y_2^2 + y_3^2}}, \qquad i = 1, 2, 3.$$

For the mapping $\bar{F}_H : S^2 \to S^2$, we define a topological degree [Mi],

$$\deg(\bar{F}_H, z_0) = \sum_{y_0 \in \bar{F}_H^{-1}(z_0)} \operatorname{sign}(d\bar{F}_H)_{y_0}$$

where $\operatorname{sign}(d\bar{F}_H)_{y_0}$ is the sign of the Jacobi matrix $d\bar{F}_H$ of the mapping $\bar{F}_H$ at y_0. This degree is well defined since it does not depend on the choice of the regular value z_0 and is homotopically invariant. Specifically, if $(z_1^2 + z_2^2 + z_3^2)(\xi, H) > 0$ for $H \in [H_1, H_2]$, then $\bar{F}_{H_1}$ is homotopic to $\bar{F}_{H_2}$, i.e., $\bar{F}_{H_1} \sim \bar{F}_{H_2}$ and

$$\deg(\bar{F}_{H_1}, z_0) = \deg(\bar{F}_{H_2}, z_0).$$

Intuitively, the degree is like the times of $F_H(S^2)$ encircling Γ. If the degree changes as H passes H_0, then F_{H_1} $(H_1 < H_0)$ is not homotopic to F_{H_2} $(H_2 > H_0)$ and $F_{H_0}(S^2)$ must intersect $\bar{\Gamma}$. In the "best case", the image $F(S^2 \times \mathbb{R}^+)$

might intersect $\bar{\Gamma}$ only at its vertex $y = (0, 0, 0, 0, 0)$ at the origin, in which case the governing equations would exhibit coincident wave speeds at isolated points in parameter space. In fact, this is the case for the flow rule satisfying deviatoric associativity.

For a small deviation from deviatoric associativity, when H is away from the discrete values H_i given in (2.8), the degree does not change at all. So we only carry out calculation of the degree for the case of the flow rule satisfying deviatoric associativity.

PROPOSITION 3.2. Assume that Ψ^D has three distinct eigenvalues $p_1 > p_2 > p_3$. Then when H passes H_i given in (2.8), the degree of the mapping $\bar{F}_H$ will change.

Proof. The mapping $\bar{F}_H : S^2 \to S^2$ is given by

$$z_1 = \frac{y_1}{\sqrt{y_1^2 + \tilde{\Psi}_{12}^2 + \tilde{\Psi}_{13}^2}}, \qquad z_2 = \frac{\tilde{\Psi}_{12}}{\sqrt{y_1^2 + \tilde{\Psi}_{12}^2 + \tilde{\Psi}_{13}^2}}, \qquad z_3 = \frac{\tilde{\Psi}_{13}}{\sqrt{y_1^2 + \tilde{\Psi}_{12}^2 + \tilde{\Psi}_{13}^2}},$$

where

$$y_1 = \sqrt{\frac{G(1-2\nu)}{H}} \left\{ \frac{H}{4G(1-2\nu)} - \left[\tilde{\Psi}_{11}^D - \frac{\kappa}{2}(\mu+\beta)\right]^2 + \frac{\kappa^2}{4}(\mu-\beta)^2 - \tilde{\Psi}_{12}^2 - \tilde{\Psi}_{13}^2 \right\}.$$

Suppose that, in the reference coordinate, Ψ^D is diagonal,

$$\Psi^D = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix}.$$

Parametrize $\bar{F}_H(S^2)$ by $z = (\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$ where $0 < \theta' < \pi$ and $0 \le \phi' < 2\pi$, and parametrize S^2 by $\xi = (\frac{1}{\sqrt{2}}(\cos\theta - \sin\theta\cos\phi), \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta\cos\phi), \sin\theta\sin\phi)$ where $0 < \theta < \pi$ and $0 \le \phi < 2\pi$. In the latter, the singular point of parametrization is moved away from the principal directions of Ψ. Under this parametrization, $\eta = (\frac{-1}{\sqrt{2}}(\sin\theta + \cos\theta\cos\phi), \frac{-1}{\sqrt{2}}(\sin\theta - \cos\theta\cos\phi), \cos\theta\sin\phi)$ and $\zeta = (\frac{1}{\sqrt{2}}\sin\phi, \frac{-1}{\sqrt{2}}\sin\phi, \cos\phi)$. Consequently,

$$\begin{aligned}
\tilde{\Psi}_{11}^D &= \frac{p_1}{2}(\cos\theta - \sin\theta\cos\phi)^2 + \frac{p_2}{2}(\cos\theta + \sin\theta\cos\phi)^2 + p_3\sin^2\theta\sin^2\phi, \\
\tilde{\Psi}_{12} &= -\frac{1}{2}(p_1 - p_2)(\cos^2\theta - \sin^2\theta)\cos\phi - \frac{1}{2}(p_1 + p_2 - 2p_3)\sin\theta\cos\theta\sin^2\phi, \\
\tilde{\Psi}_{13} &= \frac{1}{2}\sin\phi\left[(p_1 - p_2)\cos\theta - (p_2 - p_3)\sin\theta\cos\phi\right].
\end{aligned} \tag{3.3}$$

Choose $z_0 = z(\theta' = \frac{\pi}{2}, \phi' = 0) = (1, 0, 0)$. The $\bar{F}_H^{-1}(z_0)$ consists of at most

$$(\theta_0, \phi_0) = \left(\frac{\pi}{4}, \pi\right), \left(\frac{3}{4}\pi, 0\right), \left(\frac{\pi}{4}, 0\right), \left(\frac{3}{4}\pi, \pi\right), \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3}{2}\pi\right). \tag{3.4}$$

Whether (θ_0, ϕ_0) belongs to $\bar{F}_H^{-1}(z_0)$ depends on whether

$$y_1 = y_1(\theta_0, \phi_0, H) > 0.$$

Without loss of generality, we assume that

$$\left(p_3 - \frac{\kappa}{2}(\mu+\beta)\right)^2 > \left(p_2 - \frac{\kappa}{2}(\mu+\beta)\right)^2 > \left(p_1 - \frac{\kappa}{2}(\mu+\beta)\right)^2.$$

For $H < H_3$, the set $\bar{F}_H^{-1}(z_0)$ is empty and

$$\deg(\bar{F}_H\,;\, z_0) = 0.$$

For $H_3 < H < H_2$, the set $\bar{F}_H^{-1}(z_0) = \{(\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, \frac{3}{2}\pi)\}$ and

$$\begin{aligned}\deg(\bar{F}_H, z_0) &= \operatorname{sign}\left(\frac{\partial(\theta', \phi')}{\partial(\theta, \phi)}\right)_{(\frac{\pi}{2}, \frac{\pi}{2})} + \operatorname{sign}\left(\frac{\partial(\theta', \phi')}{\partial(\theta, \phi)}\right)_{(\frac{\pi}{2}, \frac{3}{2}\pi)} \\ &= 2\operatorname{sign}\left[(p_1 - p_3)(p_2 - p_3)/(y_1|y_1|)\right] = 2.\end{aligned}$$

Similarly, for $H_2 < H < H_1$, we have $\deg(\bar{F}_H, z_0) = 0$, and for $H > H_1$, we have $\deg(\bar{F}_H, z_0) = 2$. Thus, when H passes H_i $(i = 1, 2, 3)$, the degree of the mapping will change. The proof is complete.

Generically, nondeviatoric associativity in the flow rule makes the image $F(S^2 \times \mathbb{R}^+)$ intersect the interior of Γ. The discriminant will be negative in a neighborhood of the discrete values of the hardening modulus. The degree of the mapping $\bar{F}_H$ will change as H crosses over the neighborhoods. Therefore, flutter ill-posedness is unavoidable in this case.

3.3. *A sufficient condition.* Return to the original mapping $F_H : S^2 \to \mathbb{R}^5$ given by (3.1). We shall give a sufficient condition in terms of the mapping F_H, which guarantees the occurrence of flutter ill-posedness.

PROPOSITION 3.3. Assume that s is the perturbation parameter and θ, ϕ are parametrization parameters of S^2. If $y^{\mathrm{T}}Ny|_{s=0} \geq 0$, $y^{\mathrm{T}}Ny|_{(\theta_0, \phi_0, h_0, 0)} = 0$ where N is given in (3.2) and if the rank of the Jacobian of the mapping F given by (3.1) (F also depends on s here) is four, i.e.,

$$\operatorname{rank}\left(\frac{\partial(y_1, y_2, y_3, y_4, y_5)}{\partial(\theta, \phi, h, s)}\right)_{(\theta_0, \phi_0, h_0, 0)} = 4,$$

then flutter ill-posedness occurs near $(\theta_0, \phi_0, h_0, 0)$.

Proof. Suppose

$$D = \left(\frac{\partial(y_1, y_2, y_3, y_4, y_5)}{\partial(\theta, \phi, h, s)}\right)_{(\theta_0, \phi_0, h_0, 0)} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

where D_{11} is a 3×3 matrix. Locally, we have the mapping $S^2 \times R^+ \times R \to \mathbb{R}^5$ defined by

$$y = D\begin{pmatrix} \bar{r} \\ s \end{pmatrix},$$

where $\bar{r} = (\theta - \theta_0, \phi - \phi_0, h - h_0)^{\mathrm{T}}$. The value of $J_1^2 - 4J_2$ is equal to $\frac{16\lambda G^2}{H} y^{\mathrm{T}}Ny$, where

$$y^{\mathrm{T}}Ny = (D_{11}\bar{r} + D_{12}s)^{\mathrm{T}}(D_{11}\bar{r} + D_{12}s) - (D_{21}\bar{r} + D_{22}s)^{\mathrm{T}}(D_{21}\bar{r} + D_{22}s). \tag{3.5}$$

From the facts that rank $\binom{D_{11}}{D_{21}} = 3$ and

$$(D_{11}\bar{r})^{\mathrm{T}}(D_{11}\bar{r}) - (D_{21}\bar{r})^{\mathrm{T}}(D_{21}\bar{r}) \geq 0,$$

it follows that D_{11} is nonsingular $(\det(D_{11}) \neq 0)$. When $s \neq 0$, we can choose $\bar{r} = D_{11}^{-1}D_{12}s$. Substituting this in (3.5), we have

$$y^{\mathrm{T}}Ny = -(D_{21}D_{11}^{-1}D_{12} + D_{22})^{\mathrm{T}}(D_{21}D_{11}^{-1}D_{12} + D_{22})s^2 < 0.$$

In fact, if $D_{21}D_{11}^{-1}D_{12}+D_{22} = 0$, then $\operatorname{rank}(D) = 3$. This contradicts the assumption. The proof is complete.

Now we apply Proposition 3.3 to verify Loret's results [L]. He discussed two cases, the noncoaxial and coaxial case. We claim that the conditions of Proposition 3.3 are satisfied in the noncoaxial case but not in the coaxial case. Suppose that

$$\Phi^D = \Psi^D - 2s\Delta,$$

where Δ is a symmetric matrix satisfying $\operatorname{tr}(\Delta) = 0$. Then the mapping $F : S^2 \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^5$ can be rewritten as

$$y_1 = \sqrt{\frac{G(1-2\nu)}{H}}\left\{\frac{H}{4G(1-2\nu)} - (\kappa\mu - \tilde{\Psi}_{11}^D)(\kappa\beta - \tilde{\Psi}_{11}^D) - (\tilde{\Psi}_{12})^2 - (\tilde{\Psi}_{13})^2 \right.$$
$$\left. - 2s\left[\tilde{\Delta}_{11}(\kappa\mu - \tilde{\Psi}_{11}^D) - \tilde{\Psi}_{12}\tilde{\Delta}_{12} - \tilde{\Psi}_{13}\tilde{\Delta}_{13}\right]\right\},$$

$$y_2 = \tilde{\Psi}_{12} - s\tilde{\Delta}_{12}, \qquad y_3 = \tilde{\Psi}_{13} - s\tilde{\Delta}_{13},$$
$$y_4 = s\tilde{\Delta}_{12}, \qquad y_5 = s\tilde{\Delta}_{13},$$

where $\tilde{\Psi}_{11}^D$, $\tilde{\Psi}_{12}$, $\tilde{\Psi}_{13}$ are given in (3.3) and $\tilde{\Delta}_{11} = \xi^{\mathrm{T}}\Delta\xi$, $\tilde{\Delta}_{12} = \xi^{\mathrm{T}}\Delta\eta$, $\tilde{\Delta}_{13} = \xi^{\mathrm{T}}\Delta\zeta$. We knew from the proof of Theorem 2.3 that $y^{\mathrm{T}}Ny|_{s=0} \geq 0$. Also we knew that

$$y^{\mathrm{T}}Ny|_{(\theta_0, \phi_0, h_0, 0)} = 0,$$

where (θ_0, ϕ_0) is any one of the pairs in (3.4) and h_0 is any one of the values in (2.9). Let $(\theta_0, \phi_0) = (\frac{\pi}{4}, \pi)$; then it is easy to check $D_{21} = 0$ and

$$|D_{11}| = \left|\left(\frac{\partial(y_1, y_2, y_3)}{\partial(\theta, \phi, h)}\right)_{(\theta_0, \phi_0, h_0, 0)}\right| = \sqrt{\frac{G(1-2\nu)}{H}}\frac{\sqrt{2}}{4}(p_1 - p_2)(p_1 - p_3) \neq 0$$

($|D_{11}| \neq 0$ is also true for other choices of (θ_0, ϕ_0)), where $H_0 = h_0 + 2G(3\kappa\mu\beta + 2)$. Hence the rank of Jacobi of the mapping F is four if and only if

$$\left[\left(\frac{\partial y_4}{\partial s}\right)^2 + \left(\frac{\partial y_5}{\partial s}\right)^2\right]_{(\theta_0, \phi_0, h_0, 0)} = \left[(\tilde{\Delta}_{12})^2 + (\tilde{\Delta}_{13})^2\right]_{(\theta_0, \phi_0, h_0, 0)} \neq 0.$$

According to Lemma 2.4, both $\tilde{\Delta}_{12}$ and $\tilde{\Delta}_{13}$ are zero if and only if $\xi(\theta_0, \phi_0)$ is an eigenvector of Δ. In the noncoaxial case, $\xi(\theta_0, \phi_0)$ is not an eigenvector of Δ. In the coaxial case, $\xi(\theta_0, \phi_0)$ is an eigenvector of Δ. So $\operatorname{rank}(D) = 4$ in the noncoaxial case but not in the coaxial case. The claim is proved.

In fact, the coaxial case is not generic. The dimension of perturbation is restricted to be less than five. As a result, the transition from the state of non-strict hyperbolicity to loss of hyperbolicity is not smooth. Only large deviation could cause flutter ill-posedness in the coaxial case.

4. The case of two coincident transversal wave speeds.

4.1. *Statement of main results.* In the three-dimensional case, we knew from Theorem 2.3 that, whatever the hardening modulus is, two transversal wave speeds are equal when ξ is near principal directions of Ψ. But deviation from deviatoric associativity in the flow rule alone cannot cause flutter ill-posedness in this case, since $J_3 = \det(A) \equiv 0$. More general perturbations need to be considered. Another reason for considering more general perturbations is that the plastic part (1.6) of constitutive law should not be restricted to be of rank one under perturbations. In fact, the plastic part $V^{(p)}$ of constitutive law in the yield vertex model [C-H] is not of rank one. In this section, we shall show that flutter ill-posedness could occur under general perturbations in the case of two coincident transverse wave speeds.

Recall that, without perturbations, $J_2 = 0$ when two transverse wave speeds are equal. We can assume that

$$J_2 = J_3 = 0, \qquad J_1 = J_1^{(0)}$$

at $\theta = \theta_0$, $\phi = \phi_0$, $s = 0$ where s is a perturbation parameter. If

$$\left.\frac{\partial(J_1, J_2, J_3)}{\partial(\theta, \phi, s)}\right|_{(\theta_0, \phi_0, 0)} \neq 0, \tag{4.1}$$

then, according to the Inverse Function Theorem,

$$f : (\theta, \phi, s) \to (J_1, J_2, J_3)$$

maps a neighborhood N of $(\theta_0, \phi_0, 0)$ onto a neighborhood of $(J_1^{(0)}, 0, 0)$. In particular, on a subset N' of N,

$$J_2 = 0 \quad \text{and} \quad 4J_1^3 J_3 > 0.$$

It implies that

$$\operatorname{disc}(A) < 0 \quad \text{on } N'.$$

The condition (4.1) might not guarantee that flutter ill-posedness occurs for all $h \geq 0$ because the parameter s may be restricted to be positive. If J_3 can only be positive on

$$N' = \{(\theta, \phi, s) \in (\theta_0 - \delta, \theta_0 + \delta) \times (\phi_0 - \delta, \phi_0 + \delta) \times (0, \delta) : \ J_2(\theta, \phi, s) = 0\},$$

then flutter ill-posedness occurs on $N' \times M$ where

$$M = \left\{h \geq 0 : \ J_1(\theta, \phi, s, h) \geq 0, \quad (\theta, \phi, s) \in N'\right\}.$$

Similarly, if J_3 can only be negative on N', then flutter ill-posedness occurs on $N' \times M'$ where

$$M' = \left\{h \geq 0 : \ J_1(\theta, \phi, s, h) \leq 0, \quad (\theta, \phi, s) \in N'\right\}.$$

Note that, when $s = 0$ and two transversal wave speeds are equal,

$$J_1 = \frac{G}{(1-2\nu)H}\left[h + 2G(3\kappa\mu\beta + 2) - 4G(1-2\nu)(\kappa\beta - p_0)(\kappa\mu - p_0)\right],$$

where p_0 is one of the eigenvalues of Ψ^D. There always exists a value $h = h_0$ such that $J_1(\theta_0, \phi_0, 0, h,) > 0$ for $h > h_0$ and $J_1(\theta_0, \phi_0, 0, h,) < 0$ for $h < h_0$. So flutter ill-posedness could occur in a large range of the hardening modulus.

If the condition (4.1) does not hold, then more delicate theory about singularities should be used. We do not go further in this paper.

In the next two subsections 4.2 and 4.3, we shall demonstrate that flutter ill-posedness occurs near the direction of two coincident transverse wave speeds under specific perturbations.

4.2. *Models including the Jaumann rate and a nondeviatorically associative flow rule.* We assume that J_1 is away from zero. So the longitudinal wave speed cannot be equal to two transversal wave speeds. As was said in the end of subsection 1.1, we take $|T|/G$ as a perturbation parameter. Then we calculate J_3 to see whether J_3 could be nonzero on the set of (θ, ϕ) for which $J_2(\theta, \phi) = 0$.

In terms of (2.3) and (2.4), we have

$$J_1 = \frac{G}{(1-2\nu)H}\left\{H - 4G(1-2\nu)\left[(\kappa\beta - \tilde{\Psi}^D_{11})(\kappa\mu - \tilde{\Phi}^D_{11}) + \tilde{\Psi}_{12}\tilde{\Phi}_{12} + \tilde{\Psi}_{13}\tilde{\Phi}_{13}\right]\right\},$$

$$J_2 = -\frac{4G^3}{H(1-2\nu)}\left\{\tilde{\Psi}_{12}\tilde{\Phi}_{12} + \tilde{\Psi}_{13}\tilde{\Phi}_{13} + O(s)\right\},$$

$$J_3 = -\frac{2G^4}{H(1-2\nu)}\left\{\tilde{\Psi}_{13}\tilde{\Phi}_{13}\frac{\tilde{T}^D_{22}}{G} + \tilde{\Psi}_{12}\tilde{\Phi}_{12}\frac{\tilde{T}^D_{33}}{G} - \tilde{\Psi}_{12}\tilde{\Phi}_{13}\frac{\tilde{T}_{23}}{G} - \tilde{\Psi}_{13}\tilde{\Phi}_{12}\frac{\tilde{T}_{23}}{G} + O\left(s^2\right)\right\},$$

where $s = \frac{|T|}{G}$ and $O(s)$ represents terms with the order s as $s \to 0$. Note that $J_3 = \frac{G^4}{H}O\left(\frac{|T|}{G}\right)$. If $\tilde{\Psi}_{12}\tilde{\Phi}_{12} + \tilde{\Psi}_{13}\tilde{\Phi}_{13} = 0$, then

$$\operatorname{disc}(A) = -4J_1^3 J_3 + \frac{G^{10}}{H^4}O\left(\left(\frac{|T|}{G}\right)^2\right).$$

To find flutter ill-posedness, we only need to prove that

$$\bar{J}_3 = \tilde{\Psi}_{13}\tilde{\Phi}_{13}\tilde{T}^D_{22} + \tilde{\Psi}_{12}\tilde{\Phi}_{12}\tilde{T}_{33} - \tilde{\Psi}_{12}\tilde{\Phi}_{13}\tilde{T}_{23} - \tilde{\Psi}_{13}\tilde{\Phi}_{12}\tilde{T}_{23} \tag{4.2}$$

is not zero on the set of (θ, ϕ) for which $\left(\tilde{\Psi}_{12}\tilde{\Phi}_{12} + \tilde{\Psi}_{13}\tilde{\Phi}_{13}\right)(\theta, \phi) = 0$.

Suppose that Ψ^D is diagonal in our reference coordinates with eigenvalues p_1, p_2, p_3. We parametrize S^2 by $\xi = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$. Under this parametrization, $\eta = (-\sin\theta, \cos\theta\cos\phi, \cos\theta\sin\phi)$ and $\zeta = (0, -\sin\phi, \cos\phi)$. Consequently,

$$\tilde{\Psi}_{12} = -\left[(p_1 - p_2) + (p_2 - p_3)\sin^2\phi\right]\sin\theta\cos\theta,$$

$$\tilde{\Psi}_{13} = -(p_2 - p_3)\sin\theta\sin\phi\cos\phi.$$

Let $\Phi = \Lambda^{\mathrm{T}}\Psi\Lambda$, where

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}.$$

In this case,

$$\tilde{\Phi}_{12} = -\left[(p_1 - p_2) + (p_2 - p_3)\sin^2(\phi + \alpha)\right]\sin\theta\cos\theta\,,$$
$$\tilde{\Phi}_{13} = -(p_2 - p_3)\sin\theta\sin(\phi + \alpha)\cos(\phi + \alpha).$$

If $\phi_0 = -\frac{\alpha}{2}$, then

$$\tilde{\Psi}_{12}\tilde{\Phi}_{12} + \tilde{\Psi}_{13}\tilde{\Phi}_{13} = \sin^2\theta\left[\left((p_1 - p_2) + (p_2 - p_3)\sin^2\frac{\alpha}{2}\right)^2\cos^2\theta - (p_2 - p_3)^2\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\right].$$

This will be zero when

$$\cos\theta_0 = \frac{(p_2 - p_3)\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{p_1 - p_2 + (p_2 - p_3)\sin^2\frac{\alpha}{2}}.$$

Now we calculate the value of J_3 at $\theta = \theta_0$, $\phi = \phi_0$. We assume

$$T^D = \bar{\Lambda}^{\mathrm{T}}\begin{pmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{pmatrix}\bar{\Lambda}, \qquad \bar{\Lambda} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2}\\ 0 & -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2}\end{pmatrix}.$$

Thus, we have

$$\bar{J}_3 = -\sin^2\theta_0(p_2 - p_3)^2\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\left((\sigma_1 - \sigma_2)\sin^2\theta_0 + \sigma_2 - \sigma_3\right) < 0.$$

Therefore, when (θ, ϕ) is near (θ_0, ϕ_0),

$$(\mathrm{disc}(A))(\theta, \phi, h) < 0$$

for $h > 4G(1 - 2\nu)\left(\kappa\beta - \tilde{\Psi}^D_{11}(\theta_0, \phi_0)\right)\left(\kappa\mu - \tilde{\Phi}^D_{11}(\theta_0, \phi_0)\right) - 2G(3\kappa\mu\beta + 2)$.

Here, we just choose a special perturbation. Nevertheless, we believe that the rotational terms in the Jaumann rate and nondeviatoric associativity in the flow rule will lead to equations with flutter ill-posedness near the direction of two coincident transverse speeds in general.

4.3. *The yield vertex model.* In this model, the plastic part of constitutive law is not of rank one. As shown in [A], the flow rule essentially satisfies nondeviatoric associativity. In this subsection, we shall show that, when rotational terms in the Jaumann derivative are included, flutter ill-posedness occurs near the direction of two coincident transverse wave speeds.

The plastic part of the constitutive law [S-S] is written as

$$V^{(p)}_{ij} = \Psi_{ij} - \beta|\Psi|\delta_{ij}, \tag{4.3}$$

where

$$\Psi(T, A) = \frac{\sigma}{G_p}\left\{PA + \alpha\frac{|(I - P)A|^2}{|A|}\frac{T^D}{|T^D|}\right\} + \frac{\sigma}{G_r}(I - P)A,$$

in which $A = \nabla_t \left(\frac{T^D}{\sigma}\right)$, $\sigma = \frac{1}{3}\operatorname{tr}(T)$, G_p and G_r are plastic moduli (subscripts p and r are mnemonic for "proportional" and "rotating" respectively), and P is the projection operator along the direction of T^D,

$$PA = \left(A, \frac{T^D}{|T^D|}\right)\frac{T^D}{|T^D|}.$$

Assume that, compared with the proportional loading, the rotational loading is small, i.e., $|PA| \approx |A|$ and $s = \frac{|(I-P)A|}{|PA|}$ is small. Then we obtain

$$|\Psi| \approx \frac{\sigma}{G_p}\left[1 + \left(\alpha + \frac{1}{2}\left(\frac{G_p}{G_r}\right)^2\right) s|(I-P)A|\right].$$

More explicitly, (4.3) is written as

$$V_{ij}^{(p)} = M_{ijkl}\nabla_t T_{kl},$$

where

$$\begin{aligned} M_{ijkl} = {} & \frac{1}{2G_p}(\tau_{ij} - \beta\delta_{ij})(\tau_{kl} - \mu\delta_{kl}) - \frac{1}{2G_r}\tau_{ij}\tau_{kl} \\ & + \frac{1}{G_r}\left(\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}\right) + \frac{1}{4G_p}\left(2\alpha\tau_{ij} - \bar{\beta}\delta_{ij}\right) s b_{kl} \end{aligned}$$

in which $\bar{\beta} = \left(2\alpha + \left(\frac{G_p}{G_r}\right)^2\right)\beta$, $\tau_{ij} = \frac{T_{ij}^D}{|T^D|}$, $\mu = \frac{2}{3}\frac{|T^D|}{\sigma}$, and

$$b_{kl} = \frac{\nabla_t T_{kl}^D - \left(\frac{T^D}{|T^D|}, \nabla_t T^D\right)\frac{T_{kl}^D}{|T^D|}}{\sqrt{|\nabla_t T^D|^2 - \left(\frac{T^D}{|T^D|}, \nabla_t T^D\right)^2}}.$$

It is easy to check that $\tau_{ij} b_{ij} = 0$.

Reversing the relation

$$V_{ij} = \left(C_{ijkl} + M_{ijkl}\right)\nabla_t T_{kl},$$

we obtain

$$\nabla_t T_{ij} = B_{ijkl} V_{kl},$$

where

$$\begin{aligned} B_{ijkl} = {} & \tilde{\lambda}\delta_{ij}\delta_{kl} + \tilde{G}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \tilde{\nu}\tau_{ij}\tau_{kl} \\ & - \frac{4G^2}{H}(\kappa\beta\delta_{ij} - \tau_{ij})(\kappa\mu\delta_{kl} - \tau_{kl}) + \tilde{s}(u\delta_{ij} - w\tau_{ij})b_{kl} \end{aligned}$$

in which $H = 2G_p + 4G + 6G\kappa\mu\beta$ and

$$\tilde{\lambda} = \frac{2G}{1-2\nu} \cdot \frac{3\nu G_r + 2G(1+\nu)}{3(G_r + 2G)}, \qquad \tilde{G} = \frac{GG_r}{G_r + 2G},$$

$$\tilde{s} = \frac{4sG^3 G_r}{G_p(G_r + 2G)H}, \qquad \tilde{\nu} = \frac{2G^2}{G_r + 2G},$$

$$u = \left[\frac{\alpha G_p}{G} + \left(1 + \frac{G_p}{2G}\right)\left(\frac{G_p}{G_r}\right)^2\right]\kappa\beta, \qquad w = \frac{2\alpha G_p}{G} - 3\kappa\left(\frac{G_p}{G_r}\right)^2 \mu\beta.$$

As in subsection 2.1, we reduce the acoustic tensor

$$-\frac{1}{\rho^{(0)}}\xi_i(B_{ijkl} + J_{ijkl})\xi_l,$$

where J_{ijkl} is given in subsection 1.2. Finally, we wind up studying the following matrix:

$$\begin{aligned} A_{ij}(\xi, G_p, G_r) = {} & (\tilde{\lambda} + \tilde{G})\delta_{1i}\delta_{j1} + \tilde{\nu}\tilde{\tau}_{1i}\tilde{\tau}_{j1} - \frac{4G^2}{H}(\kappa\beta\delta_{1i} - \tilde{\tau}_{1i})(\kappa\mu\delta_{1j} - \tilde{\tau}_{1j}) \\ & + \tilde{s}(u\delta_{1i} - w\tilde{\tau}_{1i})\tilde{b}_{j1} - \frac{|T^D|}{2}(\delta_{1j}\tilde{\tau}_{i1} - \delta_{1i}\tilde{\tau}_{1j} - \tilde{\tau}_{ij}), \end{aligned} \tag{4.4}$$

where $\tilde{\tau} = R^{\mathrm{T}}\tau R$, $\tilde{b} = R^{\mathrm{T}}bR$, and R is given in (2.1).

Note that the last term in (4.4) comes from the Jaumann derivative. Without this term, $J_3 = \det(A) \equiv 0$. In this case, flutter ill-posedness could occur only when three wave speeds are equal.

From (4.4), we can obtain

$$J_2 = m(\tilde{\tau}_{12}^2 + \tilde{\tau}_{13}^2) - n\tilde{s}(\tilde{b}_{12}\tilde{\tau}_{12} + \tilde{b}_{13}\tilde{\tau}_{13}) + O\left(|T^D|\right),$$

$$\begin{aligned} J_3 = {} & \frac{1}{2}|T^D|\big[m(\tilde{\tau}_{22}\tilde{\tau}_{13}^2 + \tilde{\tau}_{33}\tilde{\tau}_{12}^2 - 2\tilde{\tau}_{23}\tilde{\tau}_{12}\tilde{\tau}_{13}) \\ & - n\tilde{s}\left(\tilde{b}_{12}(\tilde{\tau}_{12}\tilde{\tau}_{33} - \tilde{\tau}_{13}\tilde{\tau}_{23}) + \tilde{b}_{13}(\tilde{\tau}_{22}\tilde{\tau}_{13} - \tilde{\tau}_{12}\tilde{\tau}_{23})\right)\big] + O\left(|T^D|^2\right), \end{aligned}$$

where

$$\begin{aligned} m = {} & \left[\tilde{\lambda} + \tilde{G} + \tilde{\nu}\tilde{\tau}_{11}^2 - \frac{4G^2}{H}(\kappa\beta - \tilde{\tau}11)(\kappa\mu - \tilde{\tau}11) + \tilde{s}(u - w\tilde{\tau}_{11})\tilde{b}_{11}\right]\left(\tilde{\nu} - \frac{4G^2}{H}\right) \\ & - \left[\tilde{\nu}\tilde{\tau}_{11} + \frac{4G^2}{H}(\kappa\beta - \tilde{\tau}_{11})\right]\left[\tilde{\nu}\tilde{\tau}_{11} + \frac{4G^2}{H}(\kappa\mu - \tilde{\tau}_{11}) - \tilde{s}w\tilde{b}_{11}\right], \end{aligned}$$

$$\begin{aligned} n = {} & \left[\tilde{\lambda} + \tilde{G} + \tilde{\nu}\tilde{\tau}_{11}^2 - \frac{4G^2}{H}(\kappa\beta - \tilde{\tau}_{11})(\kappa\mu - \tilde{\tau}_{11})\right]w \\ & + \left[\tilde{\nu}\tilde{\tau}_{11} + \frac{4G^2}{H}(\kappa\mu - \tilde{\tau}_{11})\right](u - w\tilde{\tau}_{11}). \end{aligned}$$

In general, when $s \neq 0$, the vector ξ can be chosen such that (1) ξ is not an eigenvector of T and (2) the leading order in J_2 is zero. To complete the proof, it

remains to show that, for the same ξ, the determinant $J_3 \neq 0$. In fact, the leading order terms of J_2 and J_3 are simultaneously equal to zero if and only if

$$\Lambda = \begin{vmatrix} \tilde{\tau}_{12}^2 + \tilde{\tau}_{13}^2 & \tilde{b}_{12}\tilde{\tau}_{12} + \tilde{b}_{12}\tilde{\tau}_{13} \\ \tilde{\tau}_{22}\tilde{\tau}_{13}^2 + \tilde{\tau}_{33}\tilde{\tau}_{12}^2 - 2\tilde{\tau}_{23}\tilde{\tau}_{12}\tilde{\tau}_{13} & \tilde{b}_{12}(\tilde{\tau}_{12}\tilde{\tau}_{33} - \tilde{\tau}_{13}\tilde{\tau}_{23}) + \tilde{b}_{13}(\tilde{\tau}_{22}\tilde{\tau}_{13} - \tilde{\tau}_{12}\tilde{\tau}_{23}) \end{vmatrix} = 0.$$

Suppose that

$$\tau = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

in the reference coordinate. Parametrize S^2 by $\xi = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$. Then we have

$$\Lambda = (\tilde{b}_{12}\tilde{\tau}_{13} - \tilde{b}_{13}\tilde{\tau}_{12})(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)(\sigma_3 - \sigma_1)\sin^2\theta\sin\phi\cos\phi\cos\theta.$$

When $(\tilde{b}_{12}\tilde{\tau}_{13} - \tilde{b}_{13}\tilde{\tau}_{12})(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)(\sigma_3 - \sigma_1) \neq 0$, the value of Λ is not zero for ξ being away from the eigendirection of T. So, in general, $J_3 \neq 0$.

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References

[A] Lianjun An, *Loss of hyperbolicity in elastic-plastic material at finite strains*, SIAM. J. Appl. Math. **53**, 621–654 (1993)

[A-S] Lianjun An and D. Schaeffer, *The flutter instability in granular flow*, J. Mech. Phys. Solids **40**, 683–698 (1992)

[C-H] J. Christoffersen and J. Hutchinson, *A class of phenomenological corner theories of plasticity*, J. Mech. Phys. Solids **27**, 465–487 (1979)

[L] B. Loret, *Does deviation from deviatoric associativity lead to the onset of flutter instability?*, preprint, 1991

[M] J. Mandel, *Condition de stabilité et postulat de Drucker*, Rheology and soil mechanics (J. Kravtchenko and P. Sirieys, eds.), IUTAM Symposium at Grenoble, Springer, 1966, pp. 58–68

[Mi] J. Milnor, *Topology from the differentiable viewpoint*, The University Press of Virginia, 1965

[R] J. Rice, *The localization of plastic deformation*, Proc. 14th IUTAM Congress (W. Koiter, ed.), Delft, 1976, pp. 207–220

[S] D. Schaeffer, *Instability and ill-posedness in the deformation of granular materials*, Internat. J. Numer. Anal. Mech. Geomech. **14**, 253–278 (1990)

[S-S] D. Schaeffer and M. Shearer, *Loss of hyperbolicity in yield vertex plasticity models under nonproportional loading*, Nonlinear evolution equations that change type (B. Keyfitz and M. Shearer, eds.), IMA Vol. Math. Appl., vol. 27, Springer-Verlag, New York, 1990, pp. 192–217