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KINEMATICAL APPROACH TO THE SHAKEDOWN ANALYSIS OF SOME STRUCTURES

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Abstract. From Koiter's kinematical shakedown theorem, a new variational problem is deduced, which does not contain integrals over a time parameter and gives an upper bound on the safety factor. For a broad class of practical problems, including planar bar systems subjected to combined axial and bending loads, it leads to the exact value of the shakedown factor. The possible inadaptation modes (incremental, alternating, or mixed) on the shakedown boundary are determined.

1. Introduction. The limit (Gvozdev [1], Drucker, Prager, and Greenberg [2], and Hill [3]) and shakedown (Melan [4] and Koiter [5]) theorems are among the most remarkable achievements of plasticity theory applied to structural analysis. While the limit criterion has already found broad applications in engineering practice, one is skeptical to say so about shakedown analysis, although it secures a safer criterion for structures subjected to variable external loads. The reason lies in greater mathematical complexity of the shakedown criterion. In limit analysis one deals with the instantaneous moment of collapse; in shakedown analysis one has to do with restricted but unspecified processes over a time interval. Nevertheless, efforts have been made toward resolving the mathematical difficulties and bringing them into practical applications [6]–[12]. Most applications have been to bar systems and primarily to trusses with axially loaded bars and frames whose members are beams in bending. In the analysis of frames, a scheme with a finite number of plastic hinges is usually admitted. One ought to emphasize that the assumption of ideal plastic hinges, which is good in rigid plastic limit analysis, may lead to inaccuracies in shakedown problems, because local plastic deformations in a part of the cross section may cause alternating plasticity. The same should be said about the generalized variable approach. For more complicated structures, results were obtained in special cases.

The main difficulty one meets when one applies the static shakedown theorem is how to construct a set of self-equilibrated stress fields restricted by the yield criterion that contains the optimal point for which one is looking. In plane stress problems the use of the Airy stress function satisfies the equilibrium equations automatically; so the difficulty is partly reduced. This approach was applied successfully by Belytschko

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[13] and Weichert and Gross-Weege [14]. On the other hand, one has to deal with time integrals when one chooses the kinematic approach, although there are fewer constraints on the variables. The shakedown factor k_s is determined from Koiter's theorem as

$$k_s^{-1} = \sup_{\mathbf{e}^p \in (2)} \left[\int_0^T dt \int_V D(\mathbf{e}^p)\, dV\right]^{-1} \cdot \left[\int_0^T dt \int_V \boldsymbol{\sigma}^e \cdot \mathbf{e}^p \, dV\right] \tag{1}$$

where

$$\boldsymbol{\varepsilon}^p = \int_0^T \mathbf{e}^p \, dt \tag{2}$$

is compatible in V; $\boldsymbol{\varepsilon}^p(x)$ and $\mathbf{e}^p(x, t)$ $(x \in V)$ are the tensors of plastic strain increment (over a cycle) and plastic strain rate; $D(\mathbf{e}^p)$ is the dissipation function; the stress tensor $\boldsymbol{\sigma}^e(x, t)$ is the imagined elastic response of the body to external loads, the envelope of which is obtained by solving the corresponding elastic problem and is given beforehand in shakedown analysis. $\mathbf{e}^p \in (2)$ means that the supremum is taken over all fields $\mathbf{e}^p$ satisfying condition (2).

Gokhfeld [10, 15] succeeded in reducing the time integrals in (1) by constructing a set of generally incompatible strain rate fields $\mathbf{e}^p(x, t)$ satisfying condition (2):

$$\mathbf{e}^p(x, t) = \Lambda(x, t) \cdot \boldsymbol{\varepsilon}^p(x), \qquad \int_0^T \Lambda(x, t)\, dt = 1, \qquad \Lambda(x, t) \geq 0, \tag{3}$$

where $\Lambda(x, t)$ is a scalar function and $\boldsymbol{\varepsilon}^p(x)$ is compatible in V; he generally obtained an upper bound on k. Sawczuk [16], Gokhfeld and Cherniavski [10], König [11], and Nguyen and Morelle (see [12]), among others, subsequently used this way to evaluate the safety factor in practical cases.

Conditions (3) mean that the principal plastic deformation directions do not rotate during the cycle while the plastic deformations at every point change proportionally and monotonously. If plastic deformations at some point x equal zero at the end of a cycle, then they also remain zero during the cycle.

The kinematic factor obtained by using (3) is called the incremental collapse criterion [8, 11]. Still, as we have already noted, (3) presents only a special mode of incremental collapse. We will see subsequently that there are cases where incremental collapse can occur with a shakedown factor lower than that derived by (3).

In practical analysis, given a kind of structure and a loading process, it is difficult to say if the structure fails by the mechanism (3). This limits the practical value of the criterion discussed.

Here we should mention the special case of a membrane shell for which Stumpf and Le [17] obtained the shakedown factor in closed form from the kinematical theorem.

In Sec. 2 of this paper, a set of plastic strain rate fields, which is broader than (3), is constructed. From the kinematic shakedown criterion (1), a new variational problem is deduced which does not contain integrals over time; therefore, the mathematical difficulty is reduced to the same order as that of a kinematic limit problem. In general, it yields an upper bound on k_s that is better than the bound of Gokhfeld. Moreover, the constructed set of strain rate fields contains all admissible plastic deformation mechanisms for a class of structures, which are the subject of the following sections. For those structures it determines the exact value of the shakedown factor as well as the possible deformation modes of inadaptation on the shakedown boundary.

In Sec. 3, the safety factors for axisymmetric thick-walled cylinder and hollow sphere subjected to variable internal pressure are derived, which are in exact agreement with those obtained by the static approach.

In the last section, planar bar systems subjected to combined axial and bending loads are considered. The problem of determining the shakedown factor is reduced to a maximization problem, which can be solved by available numerical methods in the general case while leaving room for further simplifications in special subclasses of structures.

2. Upper bound on the safety factor. We take a set of possible plastic strain rate fields $\mathbf{e}^{\mathrm{p}}$ as follows:

$$\mathbf{e}^{\mathrm{p}}(x, t) = \begin{cases} \Lambda(x, t) \cdot \boldsymbol{\varepsilon}^{\mathrm{p}}(x), & \int_0^T \Lambda(x, t) = 1, \quad x \in V_{\mathrm{p}}; \\ \Lambda(x, t) \cdot \hat{\boldsymbol{\varepsilon}}(x), & \int_0^T \Lambda(x, t) = 0, \quad x \in V_0, \end{cases} \tag{4}$$

where

$$\boldsymbol{\varepsilon}^{\mathrm{p}}(x) = \int_0^T \mathbf{e}^{\mathrm{p}}(x, t)\, dt \tag{5}$$

is a compatible field in V, $\Lambda(x, t)$ is a scalar function, $\hat{\boldsymbol{\varepsilon}}(x)$ is an arbitrary symmetric tensor function,

$$V_{\mathrm{p}} = \{x \in V \mid \boldsymbol{\varepsilon}^{\mathrm{p}}(x) \neq 0\}, \quad \text{and} \quad V_0 = \{x \in V \mid \boldsymbol{\varepsilon}^{\mathrm{p}}(x) = 0\}.$$

Assumption (4) presents all admissible plastic deformation fields where the plastic principal deformations at every point $x \in V$ do not rotate but change proportionally during the cycle.

Substituting (4) into (1), we get

$$k_{\mathrm{U}}^{-1} = \sup_{\substack{\boldsymbol{\varepsilon}^{\mathrm{p}} \in (5), \hat{\boldsymbol{\varepsilon}} \\ \Lambda \in (4)}} \frac{\int_{V_{\mathrm{p}}} dV \int_0^T \Lambda \cdot \boldsymbol{\sigma}^{\mathrm{e}} \cdot \boldsymbol{\varepsilon}^{\mathrm{p}}\, dt + \int_{V_0} dV \int_0^T \Lambda \cdot \boldsymbol{\sigma}^{\mathrm{e}} \cdot \hat{\boldsymbol{\varepsilon}}\, dt}{\int_{V_{\mathrm{p}}} dV \int_0^T |\Lambda| \cdot D(\boldsymbol{\varepsilon}^{\mathrm{p}})\, dt + \int_{V_0} dV \int_0^T |\Lambda| \cdot D(\hat{\boldsymbol{\varepsilon}})\, dt}. \tag{6}$$

It is obvious that $k_{\mathrm{s}}^{-1} \geq k_{\mathrm{U}}^{-1}$ $(k_{\mathrm{s}} \leq k_{\mathrm{U}})$. In deriving (6) we suggested that

$$D(\Lambda \cdot \boldsymbol{\varepsilon}^{\mathrm{p}}) = |\Lambda| \cdot D(\boldsymbol{\varepsilon}^{\mathrm{p}})$$

—this is the case of the Mises or Tresca yield condition.

Denote

$$U(\boldsymbol{\varepsilon}^{\mathrm{p}}(x), \hat{\boldsymbol{\varepsilon}}(x)) = \begin{cases} \max_t[\boldsymbol{\sigma}^{\mathrm{e}}(x, t) \cdot \boldsymbol{\varepsilon}^{\mathrm{p}}(x)] = \boldsymbol{\sigma}^{\mathrm{e}}(x, t_{\mathrm{U}x}) \cdot \boldsymbol{\varepsilon}^{\mathrm{p}}(x), & x \in V_{\mathrm{p}}, \\ \max_t[\boldsymbol{\sigma}^{\mathrm{e}}(x, t) \cdot \hat{\boldsymbol{\varepsilon}}(x)] = \boldsymbol{\sigma}^{\mathrm{e}}(x, t'_{\mathrm{U}x}) \cdot \hat{\boldsymbol{\varepsilon}}(x), & x \in V_0; \end{cases} \tag{7}$$

$$L(\boldsymbol{\varepsilon}^{\mathrm{p}}(x), \hat{\boldsymbol{\varepsilon}}(x)) = \begin{cases} \min_t[\boldsymbol{\sigma}^{\mathrm{e}}(x, t) \cdot \boldsymbol{\varepsilon}^{\mathrm{p}}(x)] = \boldsymbol{\sigma}^{\mathrm{e}}(x, t_{\mathrm{L}x}) \cdot \boldsymbol{\varepsilon}^{\mathrm{p}}(x), & x \in V_{\mathrm{p}}, \\ \min_t[\boldsymbol{\sigma}^{\mathrm{e}}(x, t) \cdot \hat{\boldsymbol{\varepsilon}}(x)] = \boldsymbol{\sigma}^{\mathrm{e}}(x, t'_{\mathrm{L}x}) \cdot \hat{\boldsymbol{\varepsilon}}(x), & x \in V_0. \end{cases} \tag{8}$$

STATEMENT 1.

$$k_U^{-1} = \sup_{\substack{\boldsymbol{\varepsilon}^p \in (5),\, \hat{\boldsymbol{\varepsilon}} \\ S(x) \geq 0}} \frac{\int_{V_p} [(S+1) \cdot U - S \cdot L]\, dV + \int_{V_0} S \cdot (U - L)\, dV}{\int_{V_p} (2S+1) \cdot D(\boldsymbol{\varepsilon}^p)\, dV + \int_{V_0} 2S \cdot D(\hat{\boldsymbol{\varepsilon}})\, dV}. \tag{9}$$

Proof. We have $\Lambda = \frac{\Lambda + |\Lambda|}{2} - \frac{|\Lambda| - \Lambda}{2}$.

Denote

$$S(x) = \int_0^T \frac{|\Lambda| - \Lambda}{2}\, dt, \qquad S(x) \geq 0. \tag{10}$$

From (4) one deduces

$$\int_0^T \frac{|\Lambda| + \Lambda}{2}\, dt = \begin{cases} S(x) + 1, & x \in V_p, \\ S(x), & x \in V_0. \end{cases} \tag{11}$$

Then (10) can be rewritten as

$$k_U^{-1} = \sup_{\substack{\boldsymbol{\varepsilon}^p \in (5),\, \hat{\boldsymbol{\varepsilon}} \\ S \geq 0}} \sup_{\Lambda \in (10-11)} \frac{\int_{V_p} dV \int_0^T \Lambda \cdot \boldsymbol{\sigma}^e \cdot \boldsymbol{\varepsilon}^p\, dt + \int_{V_0} dV \int_0^T \Lambda \cdot \boldsymbol{\sigma}^e \cdot \hat{\boldsymbol{\varepsilon}}\, dt}{\int_{V_p} (2S+1) \cdot D(\boldsymbol{\varepsilon}^p)\, dV + \int_{V_0} 2S \cdot D(\hat{\boldsymbol{\varepsilon}})\, dV}. \tag{12}$$

Let us consider first the case $x \in V_p$:

$$\begin{aligned} \int_0^T \Lambda \cdot \boldsymbol{\sigma}^e \cdot \boldsymbol{\varepsilon}^p\, dt &= \int_0^T \left(\frac{\Lambda + |\Lambda|}{2} - \frac{|\Lambda| - \Lambda}{2} \right) \cdot \boldsymbol{\sigma}^e \cdot \boldsymbol{\varepsilon}^p\, dt \\ &\leq U \cdot \int_0^T \frac{\Lambda + |\Lambda|}{2}\, dt - L \cdot \int_0^T \frac{|\Lambda| - \Lambda}{2}\, dt \\ &= U \cdot (S+1) - L \cdot S. \end{aligned}$$

On the other hand, taking $\Lambda = (S+1) \cdot \delta(t - t_{Ux}) - S \cdot \delta(t - t_{Lx})$ ($\delta(t)$ is the Dirac function) satisfying (10), (11), we have

$$\int_0^T \Lambda \cdot \boldsymbol{\sigma}^e \cdot \boldsymbol{\varepsilon}^p\, dt = U \cdot (S+1) - L \cdot S.$$

Therefore, we conclude

$$\sup_{\Lambda \in (10-11)} \int_0^T \Lambda \cdot \boldsymbol{\sigma}^e \cdot \boldsymbol{\varepsilon}^p\, dt = U \cdot (S+1) - L \cdot S, \qquad x \in V_p. \tag{13}$$

Similarly one obtains

$$\sup_{\Lambda \in (10-11)} \int_0^T \Lambda \cdot \boldsymbol{\sigma}^e \cdot \hat{\boldsymbol{\varepsilon}}\, dt = U \cdot S - L \cdot S, \qquad x \in V_0. \tag{14}$$

Equation (9) follows from (12), (13), (14). The proof is completed. Denote

$$\hat{U}(\boldsymbol{\varepsilon}^p, \hat{\boldsymbol{\varepsilon}}) = \max \left\{ \max_{x \in V_p} \frac{U(\boldsymbol{\varepsilon}^p(x)) - L(\boldsymbol{\varepsilon}^p(x))}{2D(\boldsymbol{\varepsilon}^p(x))},\ \max_{x \in V_0} \frac{U(\hat{\boldsymbol{\varepsilon}}(x)) - L(\hat{\boldsymbol{\varepsilon}}(x))}{2D(\hat{\boldsymbol{\varepsilon}}(x))} \right\} \tag{15}$$

and x_U is the point where the maximum is reached.

STATEMENT 2.

$$k_{\mathrm{U}}^{-1} = \sup_{\boldsymbol{\varepsilon}^{\mathrm{p}}\in(5),\,\hat{\boldsymbol{\varepsilon}}} \max\left\{\left[\int_{V_{\mathrm{p}}} D(\boldsymbol{\varepsilon}^{\mathrm{p}})\,dV\right]^{-1}\cdot\int_{V_{\mathrm{p}}} U\,dV,\ \hat{U}\right\}. \tag{16}$$

Proof. Denote

$$\widehat{S}(x) = \begin{cases} 2S\cdot D(\boldsymbol{\varepsilon}), & x\in V_{\mathrm{p}}, \\ 2S\cdot D(\hat{\boldsymbol{\varepsilon}}), & x\in V_0; \end{cases} \qquad X = \int_V \widehat{S}\,dV \geq 0. \tag{17}$$

Then (9) can be rewritten as

$$k_{\mathrm{U}}^{-1} = \sup_{\substack{\boldsymbol{\varepsilon}^{\mathrm{p}}\in(5),\,\hat{\boldsymbol{\varepsilon}}\\ X\geq 0}}\ \sup_{\widehat{S}(x)\in(17)} \frac{\int_{V_{\mathrm{p}}}\widehat{S}\cdot\frac{U-L}{2D(\boldsymbol{\varepsilon}^{\mathrm{p}})}\,dV + \int_{V_{\mathrm{p}}} U\,dV + \int_{V_0}\widehat{S}\cdot\frac{U-L}{2D(\hat{\boldsymbol{\varepsilon}})}\,dV}{X + \int_{V_{\mathrm{p}}} D(\boldsymbol{\varepsilon}^{\mathrm{p}})\,dV}. \tag{18}$$

From (18) and (15) it is easy to see that

$$k_{\mathrm{U}}^{-1} \leq \sup_{\substack{\boldsymbol{\varepsilon}^{\mathrm{p}}\in(5),\,\hat{\boldsymbol{\varepsilon}}\\ X\geq 0}} \left[X + \int_{V_{\mathrm{p}}} D(\boldsymbol{\varepsilon}^{\mathrm{p}})\,dV\right]^{-1}\cdot\left[\widehat{U}\cdot X + \int_{V_{\mathrm{p}}} U\,dV\right].$$

On the other hand, putting $\widehat{S} = X\cdot\delta(x - x_0)$, $x_0 \in V_{\mathrm{p}}$ or V_0, satisfying (17) into (18) and taking (15) into account we deduce

$$k_{\mathrm{U}}^{-1} \geq \sup_{\substack{\boldsymbol{\varepsilon}^{\mathrm{p}}\in(5),\,\hat{\boldsymbol{\varepsilon}}\\ X\geq 0}} \left[X + \int_{V_{\mathrm{p}}} D(\boldsymbol{\varepsilon}^{\mathrm{p}})\,dV\right]^{-1}\cdot\left[\widehat{U}\cdot X + \int_{V_{\mathrm{p}}} U\,dV\right].$$

Thus,

$$k_{\mathrm{U}}^{-1} = \sup_{\substack{\boldsymbol{\varepsilon}^{\mathrm{p}}\in(5),\,\hat{\boldsymbol{\varepsilon}}\\ X\geq 0}} \left[X + \int_{V_{\mathrm{p}}} D(\boldsymbol{\varepsilon}^{\mathrm{p}})\,dV\right]^{-1}\cdot\left[\widehat{U}\cdot X + \int_{V_{\mathrm{p}}} U\,dV\right].$$

The expression after sup depends monotonically on $X \in [0, +\infty)$; therefore, the supremum is attained at $X = 0$ or at $X = +\infty$; thus we obtain (16).

Equation (16) can be rewritten as

$$\begin{aligned} k_{\mathrm{U}}^{-1} &= \max_{V_{\mathrm{p}},\,V_0\subset V}\left\{\sup_{\boldsymbol{\varepsilon}^{\mathrm{p}}\in(5)}\left[\int_{V_{\mathrm{p}}} D(\boldsymbol{\varepsilon}^{\mathrm{p}})\,dV\right]^{-1}\cdot\int_{V_{\mathrm{p}}} U\,dV,\ \sup_{\boldsymbol{\varepsilon}^{\mathrm{p}}\in(5),\,\hat{\boldsymbol{\varepsilon}}}\widehat{U}\right\} \\ &= \max\left\{\sup_{\boldsymbol{\varepsilon}^{\mathrm{p}}\in(5)}\left[\int_V D(\boldsymbol{\varepsilon}^{\mathrm{p}})\,dV\right]^{-1}\cdot\int_V U\,dV,\ \sup_{\hat{\boldsymbol{\varepsilon}}}\frac{U(\hat{\boldsymbol{\varepsilon}}) - L(\hat{\boldsymbol{\varepsilon}})}{2D(\hat{\boldsymbol{\varepsilon}})}\right\}. \end{aligned}$$

From the last formulae one arrives at

CONCLUSION. $k_{\mathrm{U}}^{-1} = \max\{W, \widehat{W}\}$ where

$$W = \sup_{\boldsymbol{\varepsilon}^{\mathrm{p}}\in(5)}\left[\int_V D(\boldsymbol{\varepsilon}^{\mathrm{p}})\,dV\right]^{-1}\cdot\int_V U\,dV, \qquad U(\boldsymbol{\varepsilon}^{\mathrm{p}}) = \max_t[\boldsymbol{\sigma}^{\mathrm{e}}(x, t)\cdot\boldsymbol{\varepsilon}^{\mathrm{p}}(x)],$$

$$\widehat{W} = \sup_{\hat{\boldsymbol{\varepsilon}}} \frac{U(\hat{\boldsymbol{\varepsilon}}) - L(\hat{\boldsymbol{\varepsilon}})}{2D(\hat{\boldsymbol{\varepsilon}})} = \sup_{\hat{\boldsymbol{\varepsilon}}, t_1, t_2} \frac{[\boldsymbol{\sigma}^{\mathrm{e}}(x, t_1) - \boldsymbol{\sigma}^{\mathrm{e}}(x, t_2)] \cdot \hat{\boldsymbol{\varepsilon}}}{2D(\hat{\boldsymbol{\varepsilon}})}. \tag{19}$$

Note that the final expression of W^{-1} coincides with Gokhfeld's factor.

For structures under loading conditions, where the compatible plastic strain rate cycle has the special form (4), the upper bound k_{U} should coincide with the exact shakedown factor k_{s}:

$$k_{\mathrm{s}}^{-1} = k_{\mathrm{U}}^{-1} = \max\{W, \widehat{W}\},$$

and the equation $k_{\mathrm{s}} = 1$ determines the load domain; inside the structure will shakedown, while outside it will not. It is interesting to note that two local modes of plastic deformations in (4) are formally separated in the final expressions of W and $\widehat{W}$ in (19); so on the boundary of the shakedown domain, where $1 = W > \widehat{W}$, the structure fails because of incremental collapse, while at $1 = \widehat{W} > W$ the alternating plasticity mode might take place. At the same time we should not rule out the existence of mixed collapse modes on the shakedown boundary. In fact, it is possible that at $1 = \widehat{W} > W$ we have even perfect incremental collapse as well as alternating plasticity. Looking back at (15), (19) we can find out that this would happen when

$$\widehat{W} = \sup_{\hat{\boldsymbol{\varepsilon}}, t_1, t_2} \frac{[\boldsymbol{\sigma}^{\mathrm{e}}(\mathbf{x}, t_1) - \boldsymbol{\sigma}^{\mathrm{e}}(\mathbf{x}, t_2)]\hat{\boldsymbol{\varepsilon}}}{2D(\hat{\boldsymbol{\varepsilon}})} = \sup_{\boldsymbol{\varepsilon}^{\mathrm{p}} \in (5), \mathbf{x}, t_1, t_2} \frac{[\boldsymbol{\sigma}^{\mathrm{e}}(\mathbf{x}, t_1) - \boldsymbol{\sigma}^{\mathrm{e}}(\mathbf{x}, t_2)]\boldsymbol{\varepsilon}^{\mathrm{p}}}{2D(\boldsymbol{\varepsilon}^{\mathrm{p}})}. \tag{20}$$

This is the case we encounter in the next section.

3. Thick-walled cylinder and sphere. Let us consider a thick-walled hollow cylinder of inner and outer radii a and b, respectively. The cylinder is subjected to an internal pressure P, which may vary arbitrarily in time within the limit $0 \leq P \leq P_0$.

For this rotationally symmetric case the Tresca yield condition takes the form

$$|\sigma_\varphi - \sigma_r| \leq \sigma_Y.$$

The elastic radial and circumferential stresses are

$$\sigma_r^{\mathrm{e}} = P \cdot a^2 \cdot (1 - b^2/r^2)/(b^2 - a^2), \qquad \sigma_\varphi^{\mathrm{e}} = P \cdot a^2 \cdot (1 + b^2/r^2)/(b^2 - a^2).$$

The material is supposed to be plastically incompressible; so the admissible plastic strain rate cycle has the form (4) (thus $k_{\mathrm{s}} = k_{\mathrm{U}}$), where

$$\varepsilon_\varphi^{\mathrm{p}} = -\varepsilon_r^{\mathrm{p}} = c/r^2, \qquad c = \mathrm{const}, \qquad \hat{\varepsilon}_\varphi = -\hat{\varepsilon}_r.$$

One has $D(\hat{\boldsymbol{\varepsilon}}) = \sigma_Y \cdot |\hat{\varepsilon}_r|$, $\sigma_Y =$ stress limit in tension,

$$D(\boldsymbol{\varepsilon}^{\mathrm{p}}) = \sigma_Y \cdot |\varepsilon_r^{\mathrm{p}}| = \sigma_Y \cdot c/r^2, \qquad \int_a^b D(\boldsymbol{\varepsilon}^{\mathrm{p}}) \cdot 2\pi \cdot r\,dr = 2\pi \cdot c \cdot \sigma_Y \cdot \ln(b/a).$$

From (19) one obtains

$$U = 2c \cdot P_0 \cdot a^2 \cdot b^2/(r^4(b^2 - a^2)), \qquad \int_a^b U \cdot 2\pi \cdot r\,dr = 2\pi \cdot c \cdot P_0,$$

$$W = P_0/(\sigma_Y \cdot \ln(b/a)), \qquad \widehat{W} = P_0 \cdot b^2/((b^2 - a^2) \cdot \sigma_Y),$$

$$k_{\mathrm{s}} = \sigma_Y/P_0 \cdot \min\{\ln(b/a), 1 - a^2/b^2\}.$$

This result coincides with that obtained earlier by the static shakedown approach (see, e.g., [11]). Note that in this case Gokhfeld's bound coincides with the limit factor $k_c = \sigma_Y/P_0 \cdot \ln(b/a)$. The cylinder will fail because of either alternating plasticity or incremental collapse if $k_s = 1$ and $\ln(b/a) > 1 - a^2/b^2$, as follows from (20) for this case.

A hollow sphere of inner and outer radii a and b, respectively, is subjected to an internal pressure P, which may vary arbitrarily in time within the limits $0 \le P \le P_0$. One has analogously

$$\sigma_r^e = -P \cdot (b^3/r^3 - 1)/(b^3/a^3 - 1), \qquad \sigma_\varphi^e = \sigma_\theta^e = P \cdot (b^3/2r^3 + 1)/(b^3/a^3 - 1).$$

The admissible plastic strain rate cycle has the form (4), where

$$\varepsilon_\varphi^p = \varepsilon_\theta^p = -\varepsilon_r^p/2 = c/r^3, \qquad c = \text{const} \ge 0, \qquad \hat{\varepsilon}_\varphi = \hat{\varepsilon}_\theta = -\hat{\varepsilon}_r/2,$$
$$D(\hat{\boldsymbol{\varepsilon}}) = \sigma_Y \cdot |\hat{\varepsilon}_r|, \qquad D(\boldsymbol{\varepsilon}^p) = 2\sigma_Y \cdot c/r^3,$$
$$\int_a^b D(\boldsymbol{\varepsilon}^p) \cdot 4\pi \cdot r^2\, dr = 8\pi \cdot c \cdot \sigma_Y \cdot \ln(b/a).$$

From (19) we derive

$$U = 3c \cdot P \cdot b^3 \cdot a^3/(r^6(b^3 - a^3)), \qquad \int_a^b U \cdot 4\pi \cdot r^2\, dr = 4\pi \cdot c \cdot P_0,$$
$$W = P_0/(2\sigma_Y \cdot \ln(b/a)), \qquad \widehat{W} = 3P_0 b^3/(4\sigma_Y(b^3 - a^3)),$$
$$k_s = (\sigma_Y/P_0) \min\{2\ln(b/a), (4/3)(1 - a^3/b^3)\}.$$

Again, we have instantaneous collapse at $1 = W > \widehat{W}$, while at $1 = \widehat{W} > W$ either alternating plasticity or incremental collapse mode occurs.

4. Planar bar-systems. Let us consider a planar bar-system (see Fig. 1 on p. 714) described in two-dimensional Euclidean space by

$$\mathbf{r}(\xi, z) = \boldsymbol{\rho}(\xi) + z \cdot \mathbf{n}(\xi), \qquad h^-(\xi) \le z \le h^+(\xi);$$

here $\boldsymbol{\rho}(\xi)$ describes the centroidal axis of a bar and $\mathbf{n}(\xi)$ is the unit vector normal to the centroidal axis. The hypothesis that plane cross sections remain plane and normal to the centroidal axis is assumed as usual; therefore, the compatible plastic strain increment has the form

$$\varepsilon_\xi^p(\xi, z) = \lambda(\xi) - z \cdot \kappa(\xi),$$
$$D(\varepsilon^p) = \sigma_Y \cdot |\varepsilon_\xi^p|, \qquad D(\varepsilon^p) = \sigma_Y \cdot |\hat{\varepsilon}_\xi|,$$

where $\lambda(\xi)$ stands for the strain of the centroidal axis and $\kappa(\xi)$ is the change of its curvature.

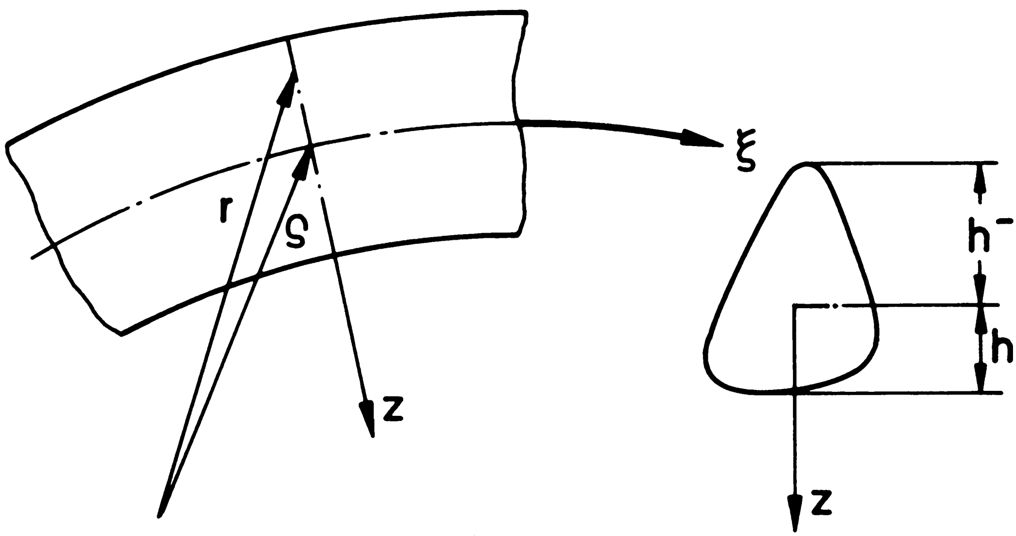

FIG. 1. Planar bar system

It is obvious that the plastic deformations of bars have the form (4); therefore, $k_s = k_U$.

The elastic stress response to the external loads has the form

$$\sigma_\xi^e(\xi, z, t) = M^e(\xi, t)/J_0(\xi) \cdot z + N^e(\xi, t)/F_0(\xi) = m^e(\xi, t) \cdot z + n^e(\xi, t) \qquad (21)$$

where $M^e(\xi, t)$ and $N^e(\xi, t)$ are moment and axial force and $J_0(\xi)$ and $F_0(\xi)$ are the moment of inertia and the area of the bar's cross section. From (19) and (21) one deduces

$$\begin{aligned}\widehat{W} &= \sup_{\hat{\varepsilon}_\xi, t_1, t_2} (2\sigma_Y \cdot |\hat{\varepsilon}_\xi|)^{-1} \cdot \{[m^e(\xi, t_1) - m^e(\xi, t_2)] \cdot z \\ &\quad + n^e(\xi, t_1) - n^e(\xi, t_2)\} \cdot \hat{\varepsilon}_\xi \\ &= \max_{z=h^+, h^-} \max_{\xi, t_1, t_2} 1/2\sigma_Y \cdot |[m^e(\xi, t_1) - m^e(\xi, t_2)] \cdot z + n^e(\xi, t_1) - n^e(\xi, t_2)| \\ &\int_V D(\varepsilon^p)\, dV = \int D_\xi\, d\xi,\end{aligned}$$

where

$$D_\xi = \sigma_Y \cdot \int_{F_0(\xi)} |\varepsilon_\xi^p|\, dF$$

$$= \begin{cases} \sigma_Y \cdot [|\lambda \cdot F(\lambda/\kappa, h^+) - \kappa \cdot I(\lambda/\kappa, h^+)| \\ \qquad + |\lambda \cdot F(h^-, \lambda/\kappa) - \kappa \cdot I(h^-, \lambda/\kappa)|] \quad \text{if } h^- < \lambda/\kappa < h^+, \\ \sigma_Y \cdot |\lambda| \cdot F_0 \quad \text{if otherwise.} \end{cases} \qquad (22)$$

$F(z_1, z_2)$ with $z_1 \le z_2$ denotes that part of the cross-sectional area that lies between the horizontal axes $z = z_1$ and $z = z_2$; $I(z_1, z_2)$, $J(z_1, z_2)$ are the corresponding static moment and moment of inertia.

$$W = \sup_{\lambda, \kappa} \left(\int D_\xi\, d\xi \right)^{-1} \cdot \int d\xi \int_{F_0(\xi)} U_\xi\, dF,$$

where

$$U_\xi = \max_t [m^e(\xi, t) \cdot z + n^e(\xi, t)] \cdot [\lambda(\xi) - z \cdot \kappa(\xi)]. \tag{23}$$

Thus, the problem of determining the shakedown factor (1) is reduced to the maximization problem: $k_s^{-1} = \max\{W, \widehat{W}\}$ (W, $\widehat{W}$ are defined in (22), (23)) with the variables $\lambda(\xi)$, $\kappa(\xi)$ satisfying the kinematic constraints of the particular problem. This can be solved by available numerical methods of mathematical programming.

Denote

$$m_L^e(\xi) = \min_t m^e(\xi, t), \qquad m_U^e(\xi) = \max_t m^e(\xi, t),$$
$$n_L^e(\xi) = \min_t n^e(\xi, t), \qquad n_U^e(\xi) = \max_t n^e(\xi, t).$$

For separated axial and bending deformations relations (22) and (23) are simplified drastically. For bending frames we have

$$\widehat{W} = \max_\xi\{(1/2)\sigma_Y \cdot [m_U^e(\xi) - m_L^e(\xi)] \cdot \max(h^+, |h^-|)\};$$
$$U_\xi = \begin{cases} \kappa \cdot m_U^e \cdot J_0 & \text{if } \kappa \geq 0, \\ \kappa \cdot m_L^e J_0 & \text{if } \kappa < 0; \end{cases} \qquad D_\xi = 2\kappa \cdot \sigma_Y \cdot I(0, h^+). \tag{24}$$

In the case of trusses we have

$$\widehat{W} = \max_\xi (1/2)\sigma_Y \cdot [n_U^e(\xi) - n_L^e(\xi)];$$
$$U_\xi = \begin{cases} \lambda \cdot n_U^e \cdot F_0 & \text{if } \lambda \geq 0, \\ \lambda \cdot n_L^e \cdot F_0 & \text{if } \lambda < 0; \end{cases} \qquad D_\xi = \sigma_Y \cdot |\lambda| \cdot F_0. \tag{25}$$

EXAMPLE 1. A cantilever beam of variable cross section (see Fig. 2) is subjected to variable distributed loads $q(\xi, t)$ and moment $M(\ell)$:

$$0 \leq q(\xi, t) \leq q_s(\xi), \qquad -M_s \leq M(\ell, t) \leq M.$$

The elastic response of the beam to the external loads is

$$M^e(\xi) = M(\ell) - \int_\xi^\ell q(\eta) \cdot (\eta - \xi)\, d\eta, \qquad \sigma_\xi^e = M^e(\xi)/J(\xi) \cdot z = m^e(\xi) \cdot z.$$

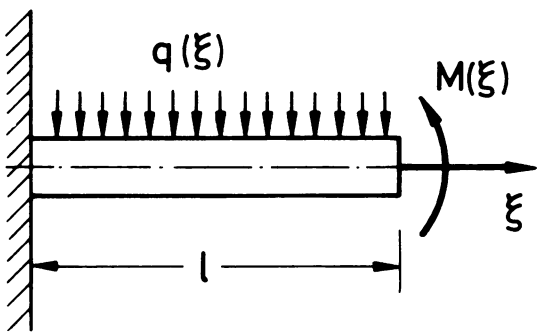

FIG. 2. Clamped beam subjected to constant load and singular moment

Denote

$$m_{\mathrm{s}}(\xi) = M_{\mathrm{s}}/J_0(\xi), \qquad p_{\mathrm{s}}(\xi) = \int_{\xi}^{\ell} q_{\mathrm{s}} \cdot (\eta - \xi)\, d\eta / J_0(\xi).$$

It is easy to see that

$$\begin{aligned} &\text{if } z \geq 0: \quad m_{\mathrm{U}} = m_{\mathrm{s}}, \; m_{\mathrm{L}} = -m_{\mathrm{s}} - p_{\mathrm{s}}; \\ &\text{if } z < 0: \quad m_{\mathrm{U}} = -m_{\mathrm{s}} - p_{\mathrm{s}}, \; m_{\mathrm{L}} = m_{\mathrm{s}}. \end{aligned}$$

Now from (24) one obtains

$$\begin{aligned} \widehat{W} &= \max_{\xi}\{(2m_{\mathrm{s}} + p_{\mathrm{s}}) \cdot \max(h^{+}, |h^{-}|)/(2\sigma_Y)\} \\ &= \max_{\xi}\left\{\left[2M_{\mathrm{s}} + \int_{\xi}^{\ell} q_{\mathrm{s}}(\eta) \cdot (\eta - \xi)\, d\eta\right] \cdot \max(h^{+}(\xi), |h^{-}(\xi)|)/(2\sigma_Y \cdot J_0(\xi))\right\}; \end{aligned}$$

$$D_{\xi} = \sigma_Y \cdot |\kappa| \cdot 2I(0, h^{+});$$

$$U_{\xi} = \begin{cases} \kappa \cdot m_{\mathrm{s}} \cdot J_0 & \text{if } \kappa \geq 0, \\ -\kappa \cdot (m_{\mathrm{s}} + p_{\mathrm{s}}) \cdot J_0 & \text{if } \kappa < 0. \end{cases}$$

The bar is statically determinate; therefore, W has the simple form

$$\begin{aligned} W &= \sup_{\kappa(\xi)}\left[\left(\int D_{\xi}\, d\xi\right)^{-1} \cdot \int U_{\xi}\, d\xi\right] = \max_{\xi} \sup_{\kappa}(U_{\xi}/D_{\xi}) \\ &= \max_{\xi}\{J_0 \cdot \max(m_{\mathrm{U}}^{\mathrm{e}}, -m_{\mathrm{L}}^{\mathrm{e}})/[\sigma_Y \cdot 2I(0, h^{+})]\} \\ &= \max_{\xi} J_0(\xi) \cdot [m_{\mathrm{s}}(\xi) + p_{\mathrm{s}}(\xi)]/[2\sigma_Y \cdot I(0, h^{+})] \\ &= \max_{\xi}\left[M_{\mathrm{s}} + \int_{\xi}^{\ell} q_{\mathrm{s}}(\eta) \cdot (\eta - \xi)\, d\eta\right] / [2\sigma_Y \cdot I(0, h^{+}(\xi))]. \end{aligned}$$

The shakedown factor is $k_{\mathrm{s}} = \min(W^{-1}, \widehat{W}^{-1})$.

For the bar of constant rectangular cross section $J_0 = 2h^3 \cdot b/3$, $I(0, h^{+}) = h^2 \cdot b/2$, and $q_{\mathrm{s}} = \mathrm{const}$, we have

$$\widehat{W} = (2M_{\mathrm{s}} + q_{\mathrm{s}} \cdot l^2/2) \cdot 3/(4h^2 \cdot b \cdot \sigma_Y), \qquad W = (M_{\mathrm{s}} + q_{\mathrm{s}} \cdot l^2/2)/(h^2 \cdot b \cdot \sigma_Y),$$

$$k_{\mathrm{s}} = (\sigma_Y \cdot h^2 \cdot b)/\max\{3M_{\mathrm{s}}/2 + 3l^2 \cdot q_{\mathrm{s}}/8, \; M_{\mathrm{s}} + l^2 \cdot q_{\mathrm{s}}/2\},$$

$$k_c = (\sigma_Y \cdot h^2 \cdot b)/(M_{\mathrm{s}} + l^2 \cdot q_{\mathrm{s}}/2).$$

EXAMPLE 2. Consider a system consisting of n parallel bars of cross sections F_i, lengths l_i, elastic moduli E_i, yield stresses σ_{Yi}, and coefficients of thermal expansion α_i $(i = 1, \ldots, n)$ (see Fig. 3). The system is subjected to a variable load $P_{\mathrm{L}} \leq P(t) \leq P_{\mathrm{U}}$ in a variable homogeneous temperature field $\theta_{\mathrm{L}} \leq \theta(t) \leq \theta_{\mathrm{U}}$. Assume that the material constants do not change in the temperature interval $[\theta_{\mathrm{L}}, \theta_{\mathrm{U}}]$ and that at $P = \theta = 0$ the system is free from internal stresses. The kinematic constraints for the strains λ_i of the bars are

$$\lambda_i \cdot l_i = u \qquad (i = 1, \ldots, n),$$

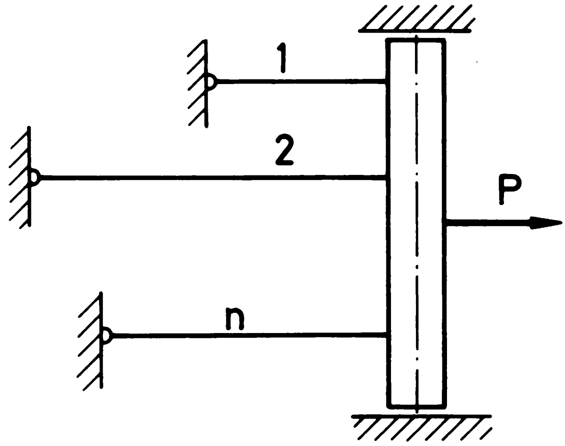

FIG. 3. Parallel bar system

where u is the horizontal displacement of the right ends of the bars. The imagined elastic stress response to the external load and temperature field in the i-bar is expressed as

$$n_i^{\mathrm{e}} = (\lambda_i - \alpha_i \cdot \theta) \cdot E_i = u^{\mathrm{e}} \cdot E_i/l_i - \alpha_i \cdot E_i \cdot \theta.$$

From the equilibrium equation

$$\begin{aligned} P &= \sum_j F_j \cdot n_j^{\mathrm{e}} \\ &= \sum_j u^{\mathrm{e}} \cdot E_j \cdot F_j/l_j - \sum_j E_j \cdot F_j \cdot \alpha_j \cdot \theta, \end{aligned}$$

one finds

$$u^{\mathrm{e}} = \left(P + \sum_j E_j \cdot F_j \cdot \alpha_j \cdot \theta\right) \Big/ \left(\sum_j E_j \cdot F_j/l_j\right).$$

Thus

$$\begin{aligned} n_{\mathrm{U}i}^{\mathrm{e}} &= \max_{P,\theta} n_i^{\mathrm{e}} \\ &= \frac{P_{\mathrm{U}} \cdot E_i/l_i}{\sum_j E_j F_j/l_j} + \frac{E_i}{l_i} \cdot \max\left\{\theta_{\mathrm{U}} \cdot \left[\frac{\sum_j E_j F_j \alpha_j}{\sum_j E_j F_j/l_j} - l_i \alpha_i\right], \theta_{\mathrm{L}} \cdot \left[\frac{\sum_j E_j F_j \alpha_j}{\sum_j E_j F_j/l_j} - l_i \alpha_i\right]\right\}, \\ n_{\mathrm{L}i}^{\mathrm{e}} &= \min_{P,\theta} n_i^{\mathrm{e}} \\ &= \frac{P_{\mathrm{L}} \cdot E_i/l_i}{\sum_j E_j F_j/l_j} + \frac{E_i}{l_i} \cdot \min\left\{\theta_{\mathrm{U}} \cdot \left[\frac{\sum_j E_j F_j \alpha_j}{\sum_j E_j F_j/l_j} - l_i \alpha_i\right], \theta_{\mathrm{L}} \cdot \left[\frac{\sum_j E_j F_j \alpha_j}{\sum_j E_j F_j/l_j} - l_i \alpha_i\right]\right\}. \end{aligned}$$

From the above formulae and (25) we have

$$\widehat{W} = \max_i \frac{1}{2\sigma_{Yi}} \cdot \left\{ \frac{(P_U - P_L) \cdot E_i/l_i}{\sum_j E_j F_j / l_j} + (\theta_U - \theta_L) \cdot \frac{E_i}{l_i} \cdot \left| \frac{\sum_j E_j F_j \alpha_j}{\sum_j E_j F_j / l_j} - l_i \alpha_i \right| \right\},$$

$$W = \frac{1}{\sum_i \sigma_{Yi} F_i} \cdot \max \left\{ P_U + \sum_i \frac{E_i F_i}{l_i} \cdot \max \left[\theta_U \cdot \left[\frac{\sum_j E_j F_j \alpha_j}{\sum_j E_j F_j / l_j} - l_i \alpha_i \right], \theta_L \cdot \left[\frac{\sum_j E_j F_j \alpha_j}{\sum_j E_j F_j / l_j} - l_i \alpha_i \right] \right], - P_L - \sum_i \frac{E_i F_i}{l_i} \cdot \min \left[\theta_U \cdot \left[\frac{\sum_j E_j F_j \alpha_j}{\sum_j E_j F_j / l_j} - l_i \alpha_i \right], \theta_L \cdot \left[\frac{\sum_j E_j F_j \alpha_j}{\sum_j E_j F_j / l_j} - l_i \alpha_i \right] \right] \right\}. \tag{26}$$

The shakedown factor is

$$k_s = \min(\widehat{W}^{-1}, W^{-1}). \tag{27}$$

For this problem the collapse factor is obtained as $k_c = \sum_i \sigma_{Yi} \cdot F_i / \max(P_U, -P_L)$.

We see that, in contrast to the shakedown factor, the collapse factor does not depend on self-equilibrated thermal stresses.

For two bars of the same material and of the same constant cross section F_0 with lengths l and $2l$ and with $P_L = -2P_s/3$, $P_U = P_s$, $\theta \equiv 0$, (27) leads to the same result as the one obtained by the static approach (see [11]):

$$k_s = \min(W^{-1}, \widehat{W}^{-1}) = \min(2F_0 \cdot \sigma_Y / P_s, \ 9F_0 \cdot \sigma_Y / 5P_s) = 9F_0 \cdot \sigma_Y / 5P_s.$$

Conclusions. As in limit analysis, also in shakedown analysis, the kinematic and static approaches might be equally usable to solve practical problems. The advantage of one approach over the other depends on the particular problem. For those problems where the structure is subjected to various kinematic constraints, the kinematic approach seems to be preferable. This is the case for the last example considered in the previous section. For the constrained bar system we could obtain the shakedown factor in an explicit analytical form ((26), (27)) using the kinematic approach. With the static approach we would need to solve a mathematical programming problem for every particular case. Another advantage of the kinematic approach is the fact that it gives a picture of the deformations of the structures at collapse and possible inadaptation modes (incremental, alternating, or mixed) as is shown by the examples considered in this paper. On the other side, the static approach has the advantage of giving a possible distribution of the residual stresses when the structure shakes down.

The main objection to the application of Koiter's theorem is its complexity (one has to deal with time integrals over a compatible plastic strain rate cycle). In our study, this difficulty could be overcome, although only for the class of problems under consideration. We hope that this work can contribute toward a broader application of the kinematic shakedown approach.

While efforts were made to generalize Melan's linear shakedown theorem to geometrically nonlinear problems (see [18] and the literature cited therein), similar efforts concerning the Koiter theorem are not known to the best of our knowledge.

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