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STEADY FLOWS OF NONLINEAR BIPOLAR VISCOUS FLUIDS BETWEEN ROTATING CYLINDERS

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Abstract. The boundary-value problem governing the steady Couette flow of a nonlinear bipolar viscous fluid is formulated and solved for particular, degenerate, values of the constitutive parameters; for the general situation, in which the relevant constitutive parameters are positive, we establish existence and uniqueness of solutions to the boundary-value problems. Continuous dependence of solutions, in appropriate norms, is also established with respect to the parameters governing the nonlinearity and multipolarity of the model as these constitutive parameters converge to zero.

1. Introduction. A classical problem in the study of motions of an incompressible viscous fluid is that of steady (or equilibrium) flow between rotating concentric circular cylinders, i.e., proper Couette flow. Under the assumption that the fluid is governed by the classical Stokes' law, namely

$$t_{ij} = 2\mu_0 e_{ij} \tag{1.1}$$

where $\mu_0 > 0$ is the classical viscosity, the t_{ij} are the components of the Cauchy stress tensor, and the e_{ij} are the components of the rate of deformation tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1.2}$$

the u_i being the components of the velocity vector, the solution of the problem of proper Couette flow may be found in most classical texts on fluid dynamics (e.g., [1], [2], or [3]); if $v(r)$ denotes the tangential velocity of the fluid, which is supposed to lie between rigid cylinders of radii r_1 and r_2 $(> r_1)$ that rotate with constant angular velocities Ω_1 and Ω_2, r being the radial distance from the center line of the inner cylinder to a point in the fluid, then

$$v(r) = \frac{1}{r}\cdot\left(\frac{\Omega_1 - \Omega_2}{r_1^{-2} - r_2^{-2}}\right) + r\cdot\left(\frac{\Omega_1 r_1^2 - \Omega_2 r_2^2}{r_1^2 - r_2^2}\right) \tag{1.3}$$

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is the resulting steady flow determined by Stokes' law and the no-slip boundary conditions

$$\Omega(r_1) \equiv \frac{v(r_1)}{r_1} = \Omega_1, \qquad \Omega(r_2) \equiv \frac{v(r_2)}{r_2} = \Omega_2. \tag{1.4}$$

Also, the pressure distribution required to maintain the flow (1.3) is given by

$$\rho^{-1}p(r) = \tfrac{1}{2}A^2r^2 - \tfrac{1}{2}B^2r^{-2} + 2A \cdot B \ln r \tag{1.5}$$

where ρ is the (constant) fluid density and

$$A = \frac{r_2^2\Omega_2 - r_1^2\Omega_1}{r_2^2 - r_1^2}, \qquad B = \frac{\Omega_2 - \Omega_1}{r_2^{-2} - r_1^{-2}}. \tag{1.6}$$

Finally, the frictional couple exerted, per unit length, across a cylindrical surface in the fluid of radius r, $r_1 < r < r_2$, is independent of r and is given by

$$2\pi r^2 t_{r\theta} = -r\pi\mu_0 \left(\frac{\Omega_1 - \Omega_2}{r_1^{-2} - r_2^{-2}} \right) \tag{1.7}$$

where $t_{r\theta}$ is the tangential stress. In deriving the relations (1.3), (1.5), (1.6), and (1.7), one writes the equilibrium Navier-Stokes equations (based on (1.1)) in cylindrical coordinates (r, θ, z) and looks for solutions, subject to (1.4), of the form

$$v_r = \dot{r} = 0, \qquad v_\theta = r\dot{\theta}(r), \qquad v_z = \dot{z} = 0 \tag{1.8}$$

which can be supported by a pressure distribution $p = p(r)$; in such a situation (1.1) reduces to

$$t_{r\theta} = 2\mu_0 e_{r\theta} \tag{1.9}$$

where

$$e_{r\theta} = \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} \right). \tag{1.10}$$

In recent work [4] Nečas and Shilhavý have examined the foundations of a continuum mechanical theory for the response of a multipolar viscous fluid; the theory that is proposed in [4] is consistent with the second law of thermodynamics, in the form of the Clausius-Duhem inequality, and builds upon earlier work of Toupin [5], Green and Rivlin [6, 7], and Bleustein and Green [8]. Bellout, Bloom, and Nečas [9] explored several of the consequences of the theory formulated in [4] with emphasis on the isothermal, incompressible, bipolar case that is described below; special consideration was given in [9] to the nature of the velocity profiles predicted for several important cases of laminar flows at low viscosity.

The mathematical theory of multipolar fluids (not to be confused with earlier models of micropolar fluid response [10]) generalizes the usual Navier-Stokes model in three important respects: it allows for nonlinear constitutive relations between the viscous part of the stress tensor and velocity gradients (this aspect is also present in the work of Ladyzhenskaya [11, 12], Kaniel [13], and Du and Gunzburger [14]), it allows for a dependence of the viscous stress on velocity gradients of order two or higher (an aspect implicit in various regularizations of the Navier-Stokes equations that have been studied by Lions [15], Temam [16], and Ou and Sritharan [17, 18]) and it introduces constitutive relations for higher-order stress tensors (moments of stress) which must be present in the balance of energy equation as soon as higher-order velocity gradients are admitted into the theory.

For an isothermal, incompressible, bipolar viscous fluid, the constitutive relations studied in [9] assume the form

$$t_{ij} = -p\delta_{ij} + 2\mu_0(\varepsilon + |e|^2)^{-\alpha/2} e_{ij} - 2\mu_1 \Delta e_{ij}, \tag{1.11a}$$

$$t_{ijk} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_h} \tag{1.11b}$$

where $e = (e_{ij}e_{ij})^{1/2}$, δ_{ij} is the Kronecker delta, the t_{ijk} are the components of the (first) multipolar stress tensor and ε, μ_0, μ_1, and α are constitutive parameters. It is assumed in [9] that ε, μ_0, and μ_1 are positive and that $0 \le \alpha < 1$; for $\alpha = \mu_1 = 0$, (1.11a,b) reduce to Stokes' law (1.1); the constitutive relations (1.11a,b) with $\mu_1 = 0$ and $\alpha < 0$, i.e., $\alpha = 2-p$ with $p > 2$, have been considered in the text of Ladyzhenskaya [11], as well as in the recent papers by Du and Gunzburger [14] and Bellout, Bloom, and Nečas [19], as non-Newtonian generalizations of Stokes' law which involve the nonlinear viscosity ($p > 2$ in [11], [14], and [19])

$$\mu(|e|) = \mu_0(\varepsilon + |e|^2)^{\frac{p-2}{2}}. \tag{1.12}$$

For the non-Newtonian model of viscous flow generated by (1.12) it is possible to exhibit the existence of a unique regular weak solution to the associated boundary-value problem for space dimension $n = 2$ when $p \ge 2$ (for the case of a bounded domain as well as in the space-periodic problem) and for space dimension $n = 3$ when $p \ge \frac{5}{2}$ (for the case of a bounded domain [11, 12]) and $p \ge \frac{11}{5}$ for the space-periodic problem [19]). Our concern in this paper, however, will be with the model governed by (1.11a,b) when $0 \le \alpha < 1$ and $\mu_1 > 0$.

The most striking aspect of the multipolar stress tensor t_{ijk} in (1.11b) is the manner in which it influences the formulation of the initial-boundary value problem for a bipolar fluid; indeed it has been demonstrated in [19] that the initial-boundary value problem based on the constitutive theory (1.11a,b) has the form (for a domain $\Omega \subseteq \mathbf{R}^3$):

$$\rho\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + 2\frac{\partial}{\partial x_j}(\mu(|e|)e_{ij}) - 2\mu_1\frac{\partial}{\partial x_j}(\Delta e_{ij}) + f_i \text{ in } \Omega \times [0, T) \tag{1.13}$$

(where μ is given by (1.12), with $1 < p \le 2$),

$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \times [0, T), \tag{1.14}$$

and

$$v_i = 0, \quad t_{ijk}\nu_j\nu_k = 0, \quad i = 1, 2, 3 \text{ on } \partial\Omega \times [0, T) \tag{1.15}$$

(with prescribed initial conditions relative to $\mathbf{u}$ and $\frac{\partial \mathbf{u}}{\partial t}$) where $\mathbf{v}$ is the exterior unit normal to $\partial\Omega$. In (1.13) ρ is the (constant) density and $\mathbf{f}$ is the external body force vector; the second set of boundary conditions is a direct consequence of the principal of virtual work (e.g., Toupin [5]) and expresses the condition that the first moments of the traction vanish on the boundary (similar boundary conditions are present in the work of Bleustein and Green [8]).

For the initial-boundary value problem (1.13–1.15), with μ as given by (1.12), and $1 < p \leq 2$, existence of unique solutions has been proved in [20] and the existence of a global compact attractor for the semigroup associated with (1.13–1.15) has been demonstrated in [21]; the analogous results for the case $p \geq 2$ (with $\mu_1 > 0$) may be found in [19] and [22]. Our concern in this paper, however, is with a much more mundane problem for the bipolar viscous fluid, namely, the existence of equilibrium solutions of the form (1.8) for the case where Ω is the domain bounded by two concentric circular cylinders of radii r_1 and r_2 that rotate with constant angular velocities Ω_1 and Ω_2. Our considerations, therefore, are focused, for the special problem referenced above, on the implications of the constitutive theory (1.11a,b), with $0 \leq \alpha < 1$, and $\mu_1 > 0$, which means that $1 < p \leq 2$ in the nonlinear viscosity (1.12); this appears to be a more interesting and physically relevant case than the one in which $p \geq 2$. Among the special problems considered in [9], with respect to the constitutive theory (1.11a,b), with $0 \leq \alpha < 1$, and $\mu_1 > 0$, was that of plane Poiseuille flow between parallel plates located (in the (x_1, x_2, x_3) Cartesian coordinate system) at, say, $x_2 = \pm a$; it is easily shown that the boundary-value problem associated with an equilibrium flow of this type has, for the bipolar viscous fluid, the form

$$\mu_0 \left[\left(\varepsilon + \tfrac{1}{2} u'^2(x_2) \right)^{-\alpha/2} u'(x_2) \right]' - \mu_1 u''''(x_2) = p_1 , \tag{1.16}$$

$$u(\pm a) = 0, \qquad u''(\pm a) = 0 \tag{1.17}$$

where the velocity field of the flow is $\mathbf{u} = (u(x_2), 0, 0)$ and p_1 is the (constant) pressure gradient. Solutions of (1.16), (1.17) are more properly denoted by $u(x_2; \varepsilon, \mu_1)$, for given α, $0 \leq \alpha < 1$, and such solutions are proven, rigorously, in [9], to exist (classically) and to be unique. It is also demonstrated in [9] that $u(x_2; \varepsilon, \mu_1)$ may be approximated, in the norm of $C^{1+\delta}(-a, a)$, $0 < \delta < \frac{1}{2}$, by the solutions $u_0(x_2) = u(x_2; 0, 0)$ of the boundary-value problem

$$\mu_0 \left[\left(\tfrac{1}{2} u_0'^2(x_2) \right)^{-\alpha/2} u_0'(x_2) \right]' = p_1 , \tag{1.18}$$

$$u_0(\pm a) = 0. \tag{1.19}$$

It is easily shown [23] that $u_0(x_2)$ has the representation

$$u_0(x_2) = d_\alpha \left[1 - \left[\frac{|x_2|}{a} \right]^{(2-\alpha)/(1-\alpha)} \right], \qquad -a \leq x_2 \leq a \tag{1.20}$$

with

$$d_\alpha = \left(\frac{1-\alpha}{2-\alpha} \right) \left(\frac{|p_1| a^{2-\alpha}}{\mu_0 2^{\alpha/2}} \right)^{\frac{1}{1-\alpha}} \tag{1.21}$$

and that the mean velocity associated with (1.20) is $\bar{u}_0 = (\frac{2-\alpha}{3-2\alpha}) d_\alpha$. By writing the associated time-dependent problem for plane Poiseuille flow of a bipolar viscous fluid in nondimensional form it may be shown [23] that there are two generalized Reynolds numbers associated with our problem, namely,

$$R_0^{(\alpha)} = \frac{U^{\alpha+1}}{\nu_0 a^{\alpha-1}} \quad \text{and} \quad R_1 = \frac{a^3 U}{\nu_1} \tag{1.22}$$

where U is a representative (mean) velocity, $\nu_0 = \mu_0/\rho$, $\nu_1 = \mu_1/\rho$; for $\alpha = 0$, $R_0^{(\alpha)}$ reduces to the usual Reynolds number associated with the Stokes' constitutive law (1.1). What is most interesting about the solution (1.20), (1.21) are the following observations; namely, if $\lim_{a\to 1^-} R_0^{(\alpha)} > \sqrt{2}$ then

$$\lim_{\alpha\to 1^-} u_0'(a) = -\infty, \qquad \lim_{\alpha\to 1^-} u_0'(-a) = +\infty \tag{1.23}$$

while

$$\left.\frac{d^n}{dx_2^n} u_0(x_2)\right|_{x_2=0} = 0 \quad \text{for } \alpha > \alpha_n \tag{1.24}$$

with $0 < \alpha_n < 1$, for each integer n, and $\alpha_n \to 1^-$ as $n \to \infty$; these results may also be found in [23]. They indicate that for plane Poiseuille flow, the bipolar viscous model based on (1.11a,b), with $0 \le \alpha < 1$, predicts profiles, as $\alpha \to 1^-$, which are in accord with those produced by the Prandtl boundary-layer theory associated with the Stokes' model, in the laminar régime, at sufficiently high Reynolds numbers; more precisely, the profiles given by (1.20), (1.21) flatten, as $\alpha \to 1^-$, both at the planes at $x_2 = \pm a$, as well as with respect to $x_2 = 0$. Similar observations are made in [9] for the case of a proper (equilibrium) Poiseuille flow of a bipolar viscous fluid in a circular pipe. For the case of plane (equilibrium) Poiseuille flow the observations referenced above for the profiles $u_0(x_2)$ remain valid for the profiles $u(x_2; \varepsilon, \mu_1)$, when ε and μ_1 are sufficiently small, because of the continuous dependence result of [9] which we have alluded to above. In fact, this continuous dependence on ε and μ_1 has been made explicit in [23] where it is proven that $\exists\, C_+$, C_1, C_2, all positive and independent of both ε and μ_1, such that

$$\begin{aligned} -\left(1 + \frac{1}{\sqrt{1-\alpha}}\right) a\sqrt{\varepsilon} - \frac{\sqrt{aC_1}}{1-\alpha}(\sqrt{\varepsilon} + C_2)^\alpha \tilde{\mu}^{1/2} \\ \le u(x_2; \varepsilon, \mu_1) - u_0(x_2) \\ \le \left(1 + \frac{1}{\sqrt{1-\alpha}}\right) a\sqrt{\varepsilon} + \frac{aC}{1-\alpha} + (\sqrt{\varepsilon} + C_2)^\alpha \tilde{\mu} \end{aligned} \tag{1.25}$$

where $\tilde{\mu} = \mu_1/\mu_0$. For the problem of plane Poiseuille flow, within the context of the bipolar model (1.11a,b), with $0 \le \alpha < 1$, results have also been obtained concerning the existence and asymptotic stability of solutions to the initial-boundary value problem for the time-dependent case [24]; also, for the case of steady plane Poiseuille flow, i.e., the problem governed by the nonlinear boundary-value problem (1.11a,b), with $0 \le \alpha < 1$, results have also been obtained concerning the existence and asymptotic stability of solutions to the initial-boundary value problem for the time-dependent case [24]; for the case of steady plane Poiseuille flow, i.e., the problem governed by the nonlinear boundary-value problem (1.16), (1.17), it has been proven

[25] that under specific restrictions on the constitutive parameters ε, μ_0, μ_1, and α, the plate separation, and the (constant) pressure gradient, the solution vector

$$\mathbf{u}^{\#} = (u(x_{2;}\varepsilon, \mu_1), 0, 0) \tag{1.26}$$

is the unique solution of the equilibrium problem (in three dimensions) corresponding to (1.13–1.15), in the domain

$$\Omega_a = \{(x_1, x_2, x_3) \mid x_2 \in [-a, a], -\infty < x_1, x_3 < \infty\}, \tag{1.27}$$

which satisfies the regularity condition

$$\mathbf{u} - \mathbf{u}^{\#} \in H^4(\Omega_a). \tag{1.28}$$

In the present work we do not set for ourselves the task of establishing, within the context of the bipolar model (1.11a,b), with $0 \le \alpha < 1$, as broad a range of results for the problem of proper Couette flow as has been established, to date, for the problem of plane Poiseuille flow; rather we shall content ourselves with deriving the relevant nonlinear boundary-value problem, with solving, in closed form, that problem for the case in which both ε and μ_1 are zero, and then proving the existence of a unique solution that depends continuously on ε and μ_1 as these constitutive parameters tend to zero. Along the way we will compare our results to those predicted by the classical solution, as given by (1.3), will study the limit of the tangential velocity field as $\alpha \to 1^-$, and will compute relevant quantities such as the frictional couple exerted on the fluid inside a cylindrical surface of radius r by the fluid exterior to that surface. Our results are expected to be of some utility to experimental fluid dynamicists who are beginning to study the implications of non-Newtonian models such as those governed by (1.11a,b).

2. The nonlinear boundary-value problem. In Cartesian coordinates $(x_1, x_2, x_3) \equiv (x, y, z)$, the equations of an incompressible bipolar viscous fluid assume the form (1.13) where, without loss of generality, we will set $\rho = 1$. We transform our problem to cylindrical coordinates (r, θ, z) so that the velocity vector $\mathbf{u}$ has components v_r, v_θ, and v_z that are related to the components u_i, $i = 1, 2, 3$ by

$$\begin{aligned} u_1 &= v_r \cos\theta - v_\theta \sin\theta, \\ u_2 &= v_r \sin\theta + v_\theta \cos\theta, \\ u_3 &= v_z. \end{aligned} \tag{2.1}$$

We are going to look for solutions of (1.13), in cylindrical coordinates, of the form (1.8); for such a velocity field (2.1) becomes

$$\begin{aligned} u_1 &= -v(r)\sin\theta, \\ u_2 &= v(r)\cos\theta, \\ u_3 &= 0, \end{aligned} \tag{2.2}$$

or $\mathbf{u} = (-v(r)\sin\theta, v(r)\cos\theta, 0)$. Since the angular velocity $\dot{\theta} = \Omega(r) = \frac{v(r)}{r}$, we may also write $\mathbf{u} = (-\Omega(r)y, \Omega(r)x, 0)$. We recall the relations

$$\begin{aligned} \frac{\partial r}{\partial x} &= \cos\theta, & \frac{\partial r}{\partial y} &= \sin\theta, \\ \frac{\partial \theta}{\partial x} &= -\frac{\sin\theta}{r}, & \frac{\partial \theta}{\partial y} &= \frac{\cos\theta}{r}. \end{aligned} \tag{2.3}$$

Our task is now to express the quantities e_{ij}, $\frac{\partial}{\partial x_k} e_{ij}$, and Δe_{ij} in terms of r and θ; we begin by noting that, as a consequence of (2.2), (2.3),

$$\begin{aligned} \frac{\partial u_1}{\partial x} &= -\left(v'(r) - \frac{v(r)}{r}\right)\sin\theta\cos\theta, \\ \frac{\partial u_1}{\partial y} &= -v'(r)\sin^2\theta - \frac{v(r)}{r}\cos^2\theta, \\ \frac{\partial u_2}{\partial x} &= v'(r)\cos^2\theta + \frac{v(r)}{r}\sin^2\theta, \\ \frac{\partial u_2}{\partial y} &= \left(v'(r) - \frac{v(r)}{r}\right)\sin\theta\cos\theta = -\frac{\partial u_1}{\partial x}. \end{aligned} \tag{2.4}$$

For the sake of convenience we set

$$f(r) = v'(r) - \frac{v(r)}{r}. \tag{2.5}$$

Now, for $i = 1, 2$, we compute for the convective derivatives appearing on the left-hand side of (1.13):

$$\begin{aligned} \frac{\partial u_1}{\partial t} + u_j \frac{\partial u_1}{\partial x_j} &= -\frac{v^2(r)}{r}\cos\theta, \\ \frac{\partial u_2}{\partial t} + u_j \frac{\partial u_2}{\partial x_j} &= -\frac{v^2(r)}{r}\sin\theta. \end{aligned} \tag{2.6}$$

Also,

$$\begin{aligned} e_{11} &= \frac{\partial u_1}{\partial x} = -f(r)\sin\theta\cos\theta = -\tfrac{1}{2}f(r)\sin 2\theta, \\ e_{22} &= \frac{\partial u_2}{\partial y} = f(r)\sin\theta\cos\theta = \tfrac{1}{2}f(r)\sin 2\theta, \\ e_{12} = e_{21} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) = \tfrac{1}{2}f(r)\cos 2\theta, \end{aligned} \tag{2.7}$$

and $e_{ij} = 0$, otherwise. From (2.6) it follows that

$$|e|^2 = e_{ij}e_{ij} = \tfrac{1}{2}f^2(r). \tag{2.8}$$

We now set

$$h(r) = \left(\varepsilon + \tfrac{1}{2}f^2(r)\right)^{-\alpha/2} f(r) \tag{2.9}$$

and compute that

$$\begin{aligned}\frac{\partial}{\partial x}&[(\varepsilon+|e|^2)^{-\alpha/2}e_{11}]\\&=-\frac{1}{2}\frac{\partial}{\partial x}[(\varepsilon+|e|^2)^{-\alpha/2}f(r)\sin 2\theta]\\&=-\frac{1}{2}\frac{\partial}{\partial x}[h(r)\sin 2\theta]\\&=-\frac{1}{2}\left(h'(r)\sin 2\theta\cos\theta-\frac{2h(r)}{r}\cos 2\theta\sin\theta\right).\end{aligned}\tag{2.10}$$

An analogous computation produces

$$\frac{\partial}{\partial y}[(\varepsilon+|e|^2)^{-\alpha/2}e_{12}]=\frac{1}{2}\left(h'(r)\cos 2\theta\sin\theta-\frac{2h(r)}{r}\sin 2\theta\cos\theta\right)\tag{2.11}$$

and, therefore,

$$\frac{\partial}{\partial x}[(\varepsilon+|e|^2)^{-\alpha/2}e_{11}]+\frac{\partial}{\partial y}[(\varepsilon+|e|^2)^{-\alpha/2}e_{12}]=-\left(\frac{1}{2}h'(r)+\frac{h(r)}{r}\right)\sin\theta.\tag{2.12}$$

In a similar manner we have

$$\frac{\partial}{\partial x}[(\varepsilon+|e|^2)^{-\alpha/2}e_{21}]=\frac{1}{2}\left(h'(r)\cos 2\theta\cos\theta+\frac{2h(r)}{r}\sin 2\theta\sin\theta\right)\tag{2.13}$$

and

$$\frac{\partial}{\partial y}[(\varepsilon+|e|^2)^{-\alpha/2}e_{22}]=\frac{1}{2}\left(h'(r)\sin 2\theta\sin\theta+\frac{2h(r)}{r}\cos 2\theta\cos\theta\right)\tag{2.14}$$

so that

$$\frac{\partial}{\partial x}[(\varepsilon+|e|^2)^{-\alpha/2}e_{21}]+\frac{\partial}{\partial y}[(\varepsilon+|\mathbf{e}|^2)^{-\alpha/2}e_{22}]=\left(\frac{h'(r)}{2}+\frac{h(r)}{r}\right)\cos\theta.\tag{2.15}$$

Our next set of computations is directed at producing the components of the tensor

$$\tau_{ijk}=\frac{\partial}{\partial x_k}(e_{ij}).$$

Note that, by virtue of (1.11b), $t_{ijk}=2\mu_1\tau_{ijk}$. We have, as a consequence of (2.7),

$$\tau_{111}=\frac{\partial}{\partial x}e_{11}=-\frac{1}{2}f'(r)\frac{\partial r}{\partial x}\sin 2\theta-f(r)\cos 2\theta\frac{\partial\theta}{\partial x}$$

so that

$$\tau_{111}=-\tfrac{1}{2}f'(r)\sin 2\theta\cos\theta+\frac{f(r)}{r}\cos 2\theta\sin\theta\tag{2.16}$$

and, in a similar manner,

$$\tau_{112}=-\tfrac{1}{2}f'(r)\sin 2\theta\sin\theta-\frac{f(r)}{r}\cos 2\theta\cos\theta,\tag{2.17}$$

$$\tau_{221}=-\tau_{111},\quad \tau_{222}=-\tau_{112},\tag{2.18}$$

$$\tau_{121}=\tfrac{1}{2}f'(r)\cos 2\theta\cos\theta+\frac{f(r)}{r}\sin 2\theta\sin\theta,\tag{2.19}$$

$$\tau_{122}=\tfrac{1}{2}f'(r)\cos 2\theta\sin\theta-\frac{f(r)}{r}\sin 2\theta\cos\theta,\tag{2.20}$$

$$\tau_{211}=\tau_{121},\quad \tau_{212}=\tau_{122},\tag{2.21}$$

and $t_{ijk} = 0$, otherwise. Since the fluid is assumed to be confined between rigid cylinders of radii r_1 and r_2 $(> r_1)$,

$$\begin{cases} \text{for } r = r_2 : \nu_1 = \cos\theta\,, \quad \nu_2 = \sin\theta\,, \\ \text{for } r = r_1 : \nu_1 = -\cos\theta\,, \quad \nu_2 = -\sin\theta. \end{cases} \tag{2.22}$$

Therefore, the second set of boundary conditions in (1.15) becomes

$$\begin{aligned} t_{1jk}\nu_j\nu_k|_{r=r_2} &= 2\mu_1\tau_{1jk}\nu_j\nu_k|_{r=r_2} \\ &= 2\mu_1(\tau_{111}\nu_1^2 + \tau_{112}\nu_1\nu_2 + \tau_{121}\nu_2\nu_1 + \tau_{122}\nu_2^2) \\ &= -\mu_1 f'(r_2)\sin\theta = 0\,, \end{aligned} \tag{2.23}$$

$$t_{ijk}\nu_j\nu_k|_{r=r_1} = -\mu_1 f'(r_1)\sin\theta = 0\,, \tag{2.24}$$

and

$$\begin{aligned} t_{2jk}\nu_j\nu_k|_{r=r_2} &= 2\mu_1\tau_{2jk}\nu_j\nu_k|_{r=r_2} \\ &= 2\mu_1(\tau_{211}\nu_1^2 + \tau_{212}\nu_1\nu_2 + \tau_{221}\nu_2\nu_1 + \tau_{222}\nu_2^2) \\ &= \mu_1 f'(r_2)\cos\theta = 0\,, \end{aligned} \tag{2.25}$$

$$t_{2jk}\nu_j\nu_k|_{r=r_1} = \mu_1 f'(r_1)\cos\theta = 0. \tag{2.26}$$

The first set of boundary conditions in (1.15) is, of course, just the same as for the classical model, i.e.,

$$\frac{v(r_2)}{r_2} \equiv \Omega(r_2) = \Omega_2\,, \qquad \frac{v(r_1)}{r_1} \equiv \Omega(r_1) = \Omega_1. \tag{2.27}$$

Our next task is to express the terms Δe_{ij}, in cylindrical coordinates, for the special steady motion defined by (1.8). We begin by recalling a series of trigonometric identities, namely

$$\begin{aligned} \sin 2\theta\cos\theta + \cos 2\theta\sin\theta &= \sin 3\theta\,, \\ \sin 2\theta\cos\theta - \cos 2\theta\sin\theta &= \sin\theta\,, \end{aligned} \tag{2.28a}$$

$$\begin{aligned} \frac{d}{d\theta}(\sin 2\theta\cos\theta) &= \frac{1}{2}(3\cos 3\theta + \cos\theta)\,, \\ \frac{d}{d\theta}(\cos 2\theta\sin\theta) &= \frac{1}{2}(3\cos 3\theta - \cos\theta)\,, \end{aligned} \tag{2.28b}$$

$$\begin{aligned} \cos 2\theta\cos\theta - \sin 2\theta\sin\theta &= \cos 3\theta\,, \\ \cos 2\theta\cos\theta + \sin 2\theta\sin\theta &= \cos\theta\,, \end{aligned} \tag{2.28c}$$

and

$$\begin{aligned} \frac{d}{d\theta}(\cos 2\theta\cos\theta) &= -\frac{1}{2}(3\sin 3\theta + \sin\theta)\,, \\ \frac{d}{d\theta}(\sin 2\theta\sin\theta) &= \frac{1}{2}(3\sin 3\theta - \sin\theta). \end{aligned} \tag{2.28d}$$

We now compute, with the aid of (2.28a–d), that

$$\begin{aligned} \frac{\partial^2}{\partial x^2}e_{11} = &-\frac{1}{2}f'' r_x \sin 2\theta\cos\theta - \frac{1}{4}f'(3\cos 3\theta + \cos\theta)\theta_x \\ &+ \left(\frac{f}{r}\right)' r_x \cos 2\theta\sin\theta + \frac{1}{2}\frac{f}{r}(3\cos 3\theta - \cos\theta)\theta_x \end{aligned}$$

or

$$\frac{\partial^2}{\partial x^2}e_{11} = -\frac{1}{2}f'' \sin 2\theta \cos^2\theta = \frac{1}{4}\frac{f}{r}(3\cos 3\theta + \cos\theta)\sin\theta$$
$$+ \left(\frac{f}{r}\right)' \cos 2\theta \sin\theta \cos\theta - \frac{1}{2}\frac{f}{r^2}(3\cos 3\theta - \cos\theta)\sin\theta \tag{2.29}$$

and, in a like fashion,

$$\frac{\partial^2}{\partial y^2}e_{11} = -\frac{1}{2}f'' \sin 2\theta \sin^2\theta - \frac{1}{4}\frac{f'}{r}(3\sin 3\theta - \sin\theta)\cos\theta$$
$$- \left(\frac{f}{r}\right)' \cos 2\theta \sin\theta \cos\theta + \frac{1}{2}\frac{f}{r^2}(3\sin 3\theta + \sin\theta)\cos\theta \tag{2.30}$$

so that

$$\Delta e_{11} = \left(-\frac{1}{2}f'' - \frac{1}{2}\frac{f'}{r} + \frac{2f}{r^2}\right)\sin 2\theta. \tag{2.31}$$

As is easily verified, a series of calculations entirely analogous to those that led to (2.31) now yield the following:

$$\Delta e_{22} = -\Delta e_{11}, \tag{2.32}$$

$$\Delta e_{12} = \left(\frac{1}{2}f'' + \frac{1}{2}\frac{f'}{r} - \frac{2f}{r^2}\right)\cos 2\theta, \tag{2.33}$$

$$\Delta e_{21} = \Delta e_{12}, \tag{2.34}$$

and $\Delta e_{ij} = 0$, otherwise. If we now set

$$g(r) = -\frac{1}{2}f'' - \frac{1}{2}\frac{f'}{r} - \frac{2f}{r^2} \tag{2.35}$$

then (2.31–2.34) can be expressed as

$$\begin{aligned} \Delta e_{11} &= g(r)\sin 2\theta = -\Delta e_{22}, \\ \Delta e_{12} &= -g(r)\cos 2\theta = \Delta e_{21}, \\ \Delta e_{ij} &= 0 \text{ (otherwise).} \end{aligned} \tag{2.36}$$

Progressing in a more or less methodical fashion with the computation of the components of the vector $\frac{\partial}{\partial x_j}(\Delta e_{ij})$ we have, by virtue of (2.36), and (2.3):

$$\frac{\partial}{\partial x}(\Delta e_{11}) = g' \sin 2\theta \cos\theta - \frac{2g}{r}\cos 2\theta \sin\theta, \tag{2.37a}$$

$$\frac{\partial}{\partial y}(\Delta e_{11}) = g' \sin 2\theta \sin\theta + \frac{2g}{r}\cos 2\theta \cos\theta, \tag{2.37b}$$

$$\frac{\partial}{\partial x}(\Delta e_{12}) = -g' \cos 2\theta \cos\theta - \frac{2g}{r}\sin 2\theta \sin\theta, \tag{2.37c}$$

and

$$\frac{\partial}{\partial y}(\Delta e_{12}) = -g' \cos 2\theta \sin\theta + \frac{2g}{r}\sin 2\theta \cos\theta. \tag{2.37d}$$

Therefore, for $i = 1$:

$$\begin{aligned} \frac{\partial}{\partial x_j}(\Delta e_{1j}) &= \frac{\partial}{\partial x}\Delta e_{11} + \frac{\partial}{\partial y}\Delta e_{12} \\ &= \left(g' + \frac{2g}{r}\right)\sin\theta , \end{aligned} \tag{2.38}$$

while for $i = 2$:

$$\begin{aligned} \frac{\partial}{\partial x_j}(\Delta e_{2j}) &= \frac{\partial}{\partial x}\Delta e_{21} + \frac{\partial}{\partial y}\Delta e_{22} \\ &= -\left(g' + \frac{2g}{r}\right)\cos\theta . \end{aligned} \tag{2.39}$$

Finally, with $p = p(r, \theta)$:

$$\begin{aligned} \frac{\partial p}{\partial x} &= \frac{\partial p}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}\sin\theta , \\ \frac{\partial p}{\partial y} &= \frac{\partial p}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial p}{\partial \theta}\cos\theta . \end{aligned} \tag{2.40}$$

To synthesize the approximate set of equations, for the proper Couette flow of a bipolar viscous fluid, now we combine (2.6), (2.40), (2.12), (2.15), (2.38), and (2.39), where $f(r)$, $h(r)$, and $g(r)$ are given, respectively, by (2.5), (2.9), and (2.35), and we obtain the relations:

$$\begin{aligned} &\frac{\partial p}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}\sin\theta - \rho\frac{v^2(r)}{r}\cos\theta \\ &\quad = -2\mu_0\left(\frac{h'}{2} + \frac{h}{r}\right)\sin\theta + 2\mu_1\left(g' + \frac{2g}{r}\right)\sin\theta , \end{aligned} \tag{2.41}$$

$$\begin{aligned} &\frac{\partial p}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial p}{\partial \theta}\cos\theta - \rho\frac{v^2(r)}{v}\sin\theta \\ &\quad = 2\mu_0\left(\frac{h'}{2} + \frac{h}{r}\right)\cos\theta - 2\mu_1\left(g' + \frac{2g}{r}\right)\cos\theta . \end{aligned} \tag{2.42}$$

Multiplying (2.41) by $\cos\theta$, (2.42) by $\sin\theta$ and, adding, we find that

$$\frac{\partial p}{\partial r} - \rho\frac{v^2(r)}{r} = 0. \tag{2.43}$$

Multiplying (2.41) by $\cos\theta$, and (2.42) by $\sin\theta$, and, now, subtracting the resulting equations, we obtain

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = 2\mu_0\left(\frac{h'}{2} + \frac{h}{r}\right) + 2\mu_1\left(g' + \frac{2g}{r}\right) \tag{2.44}$$

where $p = p(r, \theta)$. Associated with (2.43), (2.44) are the boundary conditions (2.27), i.e.,

$$\mathrm{BC}_0(r_i) : \frac{v(r_i)}{r_i} \equiv \Omega(r_i) = \Omega_i , \quad i = 1, 2 \tag{2.45}$$

as well as those that follow from (2.24–2.26), namely, for $i = 1, 2$

$$\mathrm{BC}_1(r_i) : t_{1jk}\nu_j\nu_k|_{r=r_i} = -\mu_1 f'(r_i)\sin\theta = 0, \tag{2.46}$$

$$\mathrm{BC}_2(r_i) : t_{2jk}\nu_j\nu_k|_{r=r_i} = \mu_1 f'(r_i)\cos\theta = 0. \tag{2.47}$$

Multiplying (2.46) by $\sin\theta$, and subtracting the result from (2.47), multiplied through by $\cos\theta$, we easily obtain

$$\mu_1 f'(r_i) = 0, \quad i = 1, 2 \tag{2.48}$$

so that our final set of boundary conditions for the system (2.43), (2.44) is just (2.45) and (2.48). By simply noting that $h' + 2h/r = (r^2h)'/r^2$, with the similar result for $g(r)$, it is clear that (2.44) can be rewritten in the form

$$r\frac{\partial p}{\partial\theta} = \mu_0(r^2h)' + 2\mu_1(r^2g)'. \tag{2.49}$$

If $\mu_1 > 0$ and, as in the classical situation, we look for a solution for which $\frac{\partial p}{\partial\theta} = 0$, then our boundary-value problem may be summarized as follows: find $v = v(r)$ such that

$$\mu_0(r^2h)' + 2\mu_1(r^2g)' = 0, \quad 0 < r_1 \le r \le r_2, \tag{2.50}$$

$$v(r_i) = \Omega_i r_i, \quad i = 1, 2, \tag{2.51}$$

$$f'(r_i) = 0, \quad i = 1, 2, \tag{2.52}$$

where $f(r) = v'(r) - \frac{v(r)}{r}$, $h(r) = (\varepsilon + \frac{1}{2}f^2(r))^{-\alpha/2}f(r)$, $g(r) = -\frac{1}{2}f''(r) - \frac{1}{2}\frac{f'(r)}{r} + \frac{2f(r)}{r^2}$, and $0 \le \alpha < 1$. Once the tangential velocity distribution $v(r)$ has been determined by (2.51–2.52), and it will be shown in §4 that a uniquely defined solution exists, the pressure distribution $p = p(r)$ may be deduced by integration of (2.43).

It will turn out to be useful, with respect to the analysis that follows, to rewrite the boundary-value problem (2.50–2.52) in terms of the angular velocity $\Omega(r) = v(r)/r$; to this end we note the series of identities

$$f(r) = r\Omega'(r), \tag{2.53a}$$

$$h(r) = \left(\varepsilon + \frac{1}{2}(r\Omega'(r))^2\right)^{-\alpha/2} r\Omega'(r), \tag{2.53b}$$

$$g(r) = -\frac{1}{2}\left(r\Omega'''(r) + 3\Omega''(r) - \frac{3\Omega'(r)}{r}\right), \tag{2.53c}$$

from which it follows that (2.50–2.52) is equivalent to

$$-\left[\frac{r^3\Omega'(r)}{(\varepsilon + \frac{1}{2}(r\Omega'(r)^2)^{\alpha/2}}\right]' + \mu(r^3\Omega'''(r) + 3r^2\Omega''(r) - 3r\Omega'(r))' = 0, \tag{2.54}$$

$$\Omega(r_i) = \Omega_i, \quad i = 1, 2, \tag{2.55}$$

$$(r\Omega'(r))'(r_i) = 0, \quad i = 1, 2, \tag{2.56}$$

with $\mu = \mu_1/\mu_0$, and $p(r)$ determined by (2.43).

3. The bipolar viscous fluid with $\varepsilon = \mu_1 = 0$. In the special case where $\varepsilon = \mu_1 = 0$ the boundary-value problem (2.50–2.52) (equivalently, (2.54–2.56)) may be integrated in closed form and it is instructive to compute this solution for $0 < \alpha < 1$. For the mathematical analysis of existence, uniqueness, and continuous dependence, which is presented in the next section, it is more advantageous to work with (2.50–2.52); in the present situation it is better to work with (2.50–2.52). Thus, with $\varepsilon = \mu_1 = 0$ in (2.50) we have

$$h'_\alpha(r) + \frac{2}{r} h_\alpha(r) = 0 \tag{3.1}$$

with

$$h_\alpha(r) = 2^{\alpha/2} \cdot \frac{f_\alpha(r)}{|f_\alpha(r)|^\alpha} \tag{3.2}$$

and $f_\alpha(r) = v'_\alpha(r) - \frac{v_\alpha(r)}{r}$. Integrating (3.1), and noting (3.2), we have for some c_α

$$\frac{f_\alpha(r)}{|f_\alpha(r)|^\alpha} = \frac{c_\alpha}{r^2}. \tag{3.3}$$

We now make the assumption that for $0 < r_1 \le r \le r_2$,

$$f_\alpha(r) > 0 \Leftrightarrow r\Omega'_\alpha(r) > 0 \tag{3.4}$$

in which case $c_\alpha > 0$, and for $0 < \alpha < 1$,

$$f_\alpha(r) = c_\alpha^{1/(1-\alpha)} r^{2/(\alpha-1)} \equiv \lambda_\alpha r^{2/(\alpha-1)}, \quad \lambda_\alpha > 0. \tag{3.5}$$

From the definition of $f(r)$ it follows easily from (3.5) that, for $0 < r_1 \le r \le r_2$,

$$\left(\frac{v_\alpha(r)}{r}\right)' = \lambda_\alpha r^{(3-\alpha)/(\alpha-1)}, \quad 0 \le \alpha < 1. \tag{3.6}$$

Integration of (3.6) now yields (for some constant δ_α, and with $\eta_\alpha = \frac{1}{2}\lambda_\alpha(\alpha - 1)$)

$$v_\alpha(r) = \eta_\alpha r^{(\alpha+1)/(\alpha-1)} + \delta_\alpha r, \quad 0 < \alpha < 1 \tag{3.7}$$

for $0 < r_1 \le r \le r_2$, as the expression for the tangential velocity field, while for the angular velocity we have, of course,

$$\Omega_\alpha(r) = \eta_\alpha r^{2/(\alpha-1)} + \delta_\alpha, \quad 0 \le \alpha < 1. \tag{3.8}$$

Applying the boundary conditions (2.51) and (2.52) we easily find that

$$\eta_\alpha = \frac{\Omega_1 - \Omega_2}{r_1^{2/(\alpha-1)} - r_2^{2/(\alpha-1)}}, \quad 0 \le \alpha < 1, \tag{3.9a}$$

$$\delta_\alpha = \frac{\Omega_1 r_2^{2/(\alpha-1)} - \Omega_2 r_1^{2/(\alpha-1)}}{r_2^{2/(\alpha-1)} - r_1^{2/(\alpha-1)}}, \quad 0 \le \alpha < 1, \tag{3.9b}$$

and, therefore, for $0 \le \alpha < 1$,

$$v_\alpha(r) = \left(\frac{\Omega_1 r_2^{2/(\alpha-1)} - \Omega_2 r_1^{2/(\alpha-1)}}{r_2^{2/(\alpha-1)} - r_1^{2/(\alpha-1)}}\right) r + \left(\frac{\Omega_1 - \Omega_2}{r_1^{2/(\alpha-1)} - r_2^{2/(\alpha-1)}}\right) r^{(\alpha+1)/(\alpha-1)}. \tag{3.10}$$

Remarks.

1. For $\alpha = 0$ it is easy to check that (3.10) reduces to the classical result as described by (1.3).
2. It is interesting to note the result for $\lim_{\alpha \to 1^-} v_\alpha(r)$ as given by (3.10). First of all we may write (since $\alpha - 1 = -|\alpha - 1|$, $0 < \alpha < 1$) that

$$\eta_\alpha r^{(\alpha+1)/(\alpha-1)} = \frac{r_1(\Omega_1 - \Omega_2)}{\left(\frac{r}{r_1}\right)^{(\alpha+1)/|\alpha-1|}\left(1 - \left(\frac{r_1}{r_2}\right)^{(\alpha+1)/|\alpha-1|}\frac{r_1}{r_2}\right)}$$

where $r_2 > r > r_1$; therefore $\eta_\alpha r^{(\alpha+1)/(\alpha-1)} \to 0$ as $\alpha \to 1^-$. Also

$$\delta_\alpha = \Omega_2 \left[\frac{\left(\frac{\Omega_1}{\Omega_2}\right)\left(\frac{r_1}{r_2}\right)^{2/|\alpha-1|} - 1}{\left(\frac{r_1}{r_2}\right)^{2/|\alpha-1|} - 1}\right]$$

so that $\delta_\alpha \to \Omega_2$ as $\alpha \to 1^-$. Thus for $\Omega_2 > \Omega_1$, $v_\alpha(r) \to \Omega_2 r$ as $\alpha \to 1^-$, which is a rigid-body rotation in which the tangential stresses are everywhere zero.

3. It is a relatively easy task to compute the tangential stress on an element of the surface of a cylinder of radius r, $r_1 \le r \le r_2$, if the tangential velocity distribution is prescribed by (3.10). We note that, as a consequence of (1.10) and (3.10),

$$e_{r\theta} = \frac{1}{\alpha - 1}\eta_\alpha r^{2/(\alpha-1)}, \quad r_1 \le r \le r_2 \tag{3.11}$$

and

$$|e|^2 = \frac{1}{2}\left(v'(r) - \frac{v(r)}{r}\right)^2 = \frac{1}{2}(2e_{r\theta})^2 = 2e_{r\theta}^2. \tag{3.12}$$

Therefore, with $\varepsilon = \mu_1 = 0$, $0 \le \alpha < 1$,

$$\begin{aligned} t_{r\theta} &= 2\mu_0[2e_{r\theta}^2]^{-\alpha/2} e_{r\theta} = 2^{1-\frac{\alpha}{2}}\mu_0 e_{r\theta}^{1-\alpha} \\ &= 2^{1-\frac{\alpha}{2}}\mu_0\left(\frac{\eta_\alpha}{\alpha - 1}\right)^{1-\alpha} r^{-2} \\ &= 2^{1-\frac{\alpha}{2}}\mu_0\left(\frac{|\eta_\alpha|}{1-\alpha}\right)^{1-\alpha} r^{-2} \end{aligned}$$

since $\eta_\alpha < 0$ for $\Omega_2 > \Omega_1$ and $r_2 > r_1$. Substituting for η_α in the above expression for $t_{r\theta}$ and simplifying, we obtain, for $0 \le \alpha < 1$,

$$t_{r\theta} = 2^{1-\frac{\alpha}{2}}\mu_0\left[\frac{\Omega_2 - \Omega_1}{(1-\alpha)(r_1^{2/(\alpha-1)} - r_2^{2/(\alpha-1)})}\right]^{1-\alpha} r^{-2}. \tag{3.13}$$

For the frictional couple exerted on the fluid inside a cylindrical surface of radius r by the fluid outside, $r_1 < r < r_2$, we then have

$$2\pi r^2 t_{r\theta} = 2^{1-\frac{\alpha}{2}}\pi\mu_0\left[\frac{\Omega_2 - \Omega_1}{(1-\alpha)(r_1^{2/(\alpha-1)} - r_2^{2/(\alpha-1)})}\right]^{1-\alpha} \tag{3.14}$$

which is, of course, in agreement with the classical result (1.2) for $\alpha = 0$.

4. Existence, uniqueness, and continuous dependence. In this section we will establish the existence of a unique solution to the boundary-value problem (2.54–2.56), in an appropriate class of functions, and then show that the solution depends continuously on the parameters ε and μ as both $\varepsilon \to 0^+$ and $\mu \to 0^+$; in a sense that will be made precise below, this analysis will justify our study of the case in which $\varepsilon = \mu_1 = 0$, i.e., the situation in which (2.54–2.56) reduces to the boundary-value problem

$$\left[\frac{r^3\Omega'(r)}{(r\Omega'(r))^{\alpha}}\right]' = 0, \qquad r_1 \le r \le r_2, \tag{4.1a}$$

$$\Omega(r_i) = \Omega_i \tag{4.1b}$$

whose solution is

$$\Omega_\alpha(r) = \left(\frac{\Omega_1 - \Omega_2}{r_1^{2/(\alpha-1)} - r_2^{2/(\alpha-1)}}\right) r^{2/(\alpha-1)} + \left(\frac{\Omega_1 r_2^{2/(\alpha-1)} - \Omega_2 r_1^{2/(\alpha-1)}}{r_2^{2/(\alpha-1)} - r_1^{2/(\alpha-1)}}\right) \tag{4.2}$$

for $r_1 \le r \le r_2$, $0 \le \alpha < 1$. When $\alpha = \mu_1 = 0$, (2.54–2.56) reduces to the boundary-value problem based on the Stokes' constitutive law, namely,

$$(r^3\Omega'(r))' = 0, \qquad r_1 \le r \le r_2, \tag{4.3a}$$

$$\Omega(r_i) = \Omega_i, \qquad i = 1, 2 \tag{4.3b}$$

whose unique solution, viz.,

$$\Omega_0(r) = \frac{1}{r^2}\left(\frac{\Omega_1 - \Omega_2}{r_1^{-2} - r_2^{-2}}\right) + \left(\frac{\Omega_1 r_1^2 - \Omega_2 r_2^2}{r_1^2 - r_2^2}\right) \tag{4.4}$$

may be obtained either from (1.3) or from (4.2), upon setting $\alpha = 0$. We note that $\Omega_0(r)$ satisfies

$$(r^3\Omega_0'''(r) + 3r^2\Omega_0''(r) - 3r\Omega_0'(r))' \equiv 0 \tag{4.5}$$

on (r_1, r_2) but that

$$(r\Omega_0'(r))' = \frac{4}{r^3}\left(\frac{\Omega_1 - \Omega_2}{r_1^{-2} - r_2^{-2}}\right) \ne 0. \tag{4.6}$$

Finally, when $\mu_1 = 0$, the boundary-value problem (2.54–2.56) reduces to

$$\left[\frac{r^3\Omega'(r)}{\varepsilon + \frac{1}{2}(r\Omega'(r))^2)^{\alpha/2}}\right]' = 0, \qquad r_1 \le r \le r_2, \tag{4.7a}$$

$$\Omega(r_i) = \Omega_i, \qquad i = 1, 2 \tag{4.7b}$$

whose solution (granted that one exists and is uniquely determined) will be denoted as $\Omega_{\varepsilon,\alpha}(r)$. In fact, if we denote the solution of the boundary-value problem (2.54–2.56) as $\Omega(r; \varepsilon, \mu, \alpha)$—again, granted that one exists and is uniquely determined—then we clearly have the identifications:

$$\begin{aligned}\Omega_{\alpha,\varepsilon}(r) &= \Omega(r;\varepsilon,0,\alpha),\\ \Omega_0(r) &= \Omega(r;\varepsilon,0,0),\\ \Omega_\alpha(r) &= \Omega(r;0,0,\alpha) \equiv \Omega_{\alpha,0}(r).\end{aligned} \tag{4.8}$$

Our first theorem concerns the existence and uniqueness result for the system (2.54–2.56):

THEOREM 4.1. For $\mu_1 > 0$ ($\Leftrightarrow \mu > 0$) the boundary-value problem (2.54–2.56) has a unique solution $\Omega(r;\varepsilon,\mu,\alpha)$, in $H_0^2(r_1,r_2)$, for all α such that $0 \le \alpha < 1$, and for all $\varepsilon \ge 0$.

Proof. The case $\alpha = 0$ is trivial; when $\alpha = 0$, (2.54–2.56) reduces to the linear boundary-value problem

$$-(r^3\Omega'(r))' + \mu(r^3\Omega'''(r) + 3r^3\Omega''(r) - 3r\Omega'(r))' = 0 \tag{4.9a}$$

for $r_1 \le r \le r_2$, with

$$\Omega(r_i) = \Omega, \quad i = 1, 2, \tag{4.9b}$$

$$(r\Omega'(r))'(r_i) = 0, \quad i = 1, 2, \tag{4.9c}$$

which has by the standard theory for linear boundary-value problems [24], a unique solution in $H^2(r_1, r_2)$; actually, it is easy to show that the unique solution of (4.9a,b,c) lies in $C^\infty(r_1, r_2)$.

Now, consider the case where $0 < \alpha < 1$. We denote by $\tilde{\Omega}(r)$ the unique classical solution of the boundary-value problem

$$(r^3\tilde{\Omega}'''(r) + 3r^2\tilde{\Omega}''(r) - 3r\tilde{\Omega}'(r))' = 0 \tag{4.10a}$$

for $r_1 \le r \le r_2$, with, once again

$$\Omega(r_i) = \Omega_i, \quad i = 1, 2, \tag{4.10b}$$

$$(r\tilde{\Omega}'(r))'(r_i) = 0, \quad i = 1, 2. \tag{4.10c}$$

We note that $\tilde{\Omega}(r) \in C^\infty(r_1, r_2)$ and that although, as a consequence of (4.5), the classical solution $\Omega_0(r)$ of (4.3a,b), satisfies (4.10a), $\tilde{\Omega}(r) \ne \Omega_0(r)$ by virtue of (4.6). If we set

$$u(r) = \Omega(r;\varepsilon,\mu,\alpha) - \tilde{\Omega}(r) \tag{4.11}$$

then, as a consequence of (2.54)–(2.56), coupled with (4.10a,b,c), we easily find that, for $r_1 \le r \le r_2$, $u(r)$ satisfies

$$\begin{aligned}&-\left[\frac{r^3(u'(r) + \tilde{\Omega}(r))}{\varepsilon + \frac{1}{2}(ru'(r) + r\tilde{\Omega}'(r))^2)^{\alpha/2}}\right]' \\ &\quad + \mu(r^3u'''(r) + 3r^2u''(r) - 3ru'(r))' = 0,\end{aligned} \tag{4.12a}$$

$$u(r_i) = 0, \quad i = 1, 2, \tag{4.12b}$$

$$(ru'(r))'(r_i) = 0, \quad i = 1, 2. \tag{4.12c}$$

Thus, in order to show that (2.54)–(2.56) possesses a unique solution in $H_0^2(r_1, r_2)$, it is sufficient to prove that (4.12a,b,c) has a unique solution in $H_0^2(r, r_2)$.

Let $H = H_0^{3/2+\sigma}(r_1, r_2)$ with $0 < \sigma < 1/2$ and denote by W_M the closed ball of radius $M > 0$ in $H_0^2(r_1, r_2)$. As a consequence of standard embedding results [24, 25], we know that W_M is compactly embedded in H for any $\sigma < 1/2$. For the sake of convenience, we define the linear map L by

$$(Lu)(r) = \mu(r^3 u'''(r) + 3r^2 u''(r) - 3ru'(r))' \tag{4.12d}$$

and for a fixed, but arbitrary, $h \in V$ we consider the linear boundary-value problem

$$(Lu)(r) = \left[\frac{r^3(h'(r) + \tilde{\Omega}'(r))}{\varepsilon + \frac{1}{2}(rh'(r) + r\tilde{\Omega}'(r))^2)^{\alpha/2}}\right]', \quad r_1 \le r \le r_2, \tag{4.14a}$$

$$u(r_i) = (ru'(r))'(r_i) = 0, \quad i = 1, 2. \tag{4.14b}$$

With

$$a(p, q) \equiv \int_{r_1}^{r_2} p(r)(Lq)(r)\,dr \tag{4.15}$$

we have

$$\begin{aligned}
a(u, u) &= \mu \int_{r_1}^{r_2} (r^3 u'''(r) + 3r^2 u''(r) - 3ru'(r))' u(r)\,dr \\
&= -\mu \int_{r_1}^{r_2} (r^3 u'''(r) + 3r^2 u''(r) - 3ru'(r)) u'(r)\,dr \\
&= -\mu \int_{r_1}^{r_2} [r(r(ru')')' - 4ru'(r)] u'(r)\,dr \\
&= \mu \int_{r_1}^{r_2} r[(ru')'^2 + 4u'^2(r)]\,dr \\
&= \mu \int_{r_1}^{r_2} r[r^2 u''(r)^2 + 2ru''(r)u'(r) + 5u'^2(r)]\,dr
\end{aligned}$$

and, therefore

$$\begin{aligned}
a(u, u) &\ge \frac{3}{4}\mu \int_{r_1}^{r_2} [r^3 u''(r)^2 + ru'^2(r)]\,dr \\
&\ge \alpha_0 \|u\|^2_{H_0^2(r_1, r_2)}
\end{aligned} \tag{4.16}$$

where $\alpha_0 = \frac{3}{4}\mu \min(r_1^3, r_1)$. By the Lax-Milgram Lemma [24] we may conclude that the boundary-value problem (4.14a,b) has a unique solution $u \in H_0^2(r_1, r_2)$ that satisfies

$$\begin{aligned}\|u\|_{H_0^2(r_1,r_2)} &\leq \frac{1}{\alpha_0}\left\|\frac{r^3(h'(r)+\widetilde{\Omega}'(r))}{\varepsilon+\frac{1}{2}(rh'(r)+r\widetilde{\Omega}'(r))^2)^{\alpha/2}}\right\|_{L^2(r_1,r_2)} \\ &\leq \frac{1}{\alpha_0}\|2^{\alpha/2}r^{3-\alpha}|h'(r)+\widetilde{\Omega}'(r)|^{1-\alpha}\|_{L^2(r_1,r_2)} \\ &= \frac{1}{\alpha_0}\cdot 2^{\alpha/2}\left[\int_{r_1}^{r_2} r^{2(3-\alpha)/\alpha}\,dr\right]^{\alpha/2} \\ &\qquad \times \|h'(r)+\widetilde{\Omega}'(r)\|_{L^2(r_1,r_2)}^{1-\alpha}\end{aligned}$$

where we have used the Hölder inequality. Thus

$$\|u\|_{H_0^2(r_1,r_2)} \leq C\|h'(r)+\widetilde{\Omega}'(r)\|_{L^2(r_1,r_2)}^{1-\alpha} \tag{4.17}$$

with $C=\frac{1}{\alpha_0}2^{\alpha/2}[\int_{r_1}^{r_2} r^{\frac{2(3-\alpha)}{\alpha}}\,dr]^{\alpha/2}$. We now apply Young's inequality, i.e.,

$$|a|\cdot|b| \leq \sigma|a|^p + \sigma^{-\frac{1}{p-1}}|b|^{p'}, \qquad \frac{1}{p}+\frac{1}{p'}=1$$

to (4.17), with $p=1/(1-\alpha)$ and $\sigma=\frac{1}{4}$, and we obtain the estimate

$$\begin{aligned}\|u\|_{H_0^2(r_1,r_2)} &\leq \tfrac{1}{4}\|h'(r)+\widetilde{\Omega}'(r)\|_{L^2(r_1,r_2)}+\widetilde{C} \\ &\leq \tfrac{1}{4}(\|h\|_{H_0^1(r_1,r_2)}+\|\widetilde{\Omega}\|_{H_0^1(r_1,r_2)})+\widetilde{C}\end{aligned} \tag{4.18}$$

with $\widetilde{C}=\widetilde{C}(\alpha;r_1,r_2)$ independent of u.

We now define the mapping $T: h \to u$ where for each fixed $h \in V$, u is the unique solution of the boundary-value problem (4.14a,b). By virtue of the estimate (4.18), $\exists M>0$ sufficiently large such that $T: W_M \to W_M$; we want to show that the map T is continuous. For $h_1, h_2 \in W_M$, we set $u_1=Th_1$ and $u_2=Th_2$. Then

$$\begin{aligned}&\mu(r^3(u_1-u_2)'''+3r^2(u_1-u_2)''-3r(u_1-u_2)')' \\ &\quad = \left[\frac{r^3(h_1'(r)+\widetilde{\Omega}'(r))}{z_1(r)}\right]'-\left[\frac{r^3(h_2'(r)+\widetilde{\Omega}'(r))}{z_2(r)}\right]'\end{aligned} \tag{4.19}$$

with

$$z_i(r)=\left(\varepsilon+\tfrac{1}{2}(rh_i'(r)+r\widetilde{\Omega}'(r))^2\right)^{\alpha/2}, \quad i=1,2. \tag{4.20}$$

Multiplying (4.19) by u_1-u_2, integrating the result over (r_1, r_2), and then integrating by parts, we obtain

$$\begin{aligned}&\mu\int_{r_1}^{r_2}[r^3(u_1-u_2)'''+3r^2(u_1-u_2)''-3r(u_1-u_2)']'(u_1-u_2)\,dr \\ &\quad = -\int_{r_1}^{r_2}\frac{r^3(h_1'(r)+\widetilde{\Omega}'(r))}{z_1(r)}(u_1-u_2)'\,dr+\int_{r_1}^{r_2}\frac{r^3(h_2'(r)+\widetilde{\Omega}'(r))}{z_2(r)}(u_1-u_2)'\,dr.\end{aligned} \tag{4.21}$$

Now, from (4.21), the definition of the bilinear form a in (4.15), and the estimate (4.16), we have

$$\begin{aligned}\alpha_0\|u_1-u_2\|^2_{H_0^2(r_1,r_2)} &\le \int_{r_1}^{r_2}\frac{r^3(h_2'-h_1')}{z_2(r)}(u_1-u_2)'\,dr\\ &+\int_{r_1}^{r_2}\frac{r^3(u_1-u_2)'(h_1'+\widetilde{\Omega}')[z_1(r)-z_2(r)]}{z_1(r)\cdot z_2(r)}\,dr\end{aligned}\tag{4.22}$$

where we have added (and subtracted) the integral $\int_{r_1}^{r_2}\frac{r^3(h_1'+\widetilde{\Omega}')}{z_2}(u_2-u_1)'\,dr$. For the case where $\varepsilon>0$ we have, clearly, that $z_i(r)\ge\varepsilon^{\alpha/2}$ for $i=1,2$ and, since $h_1, h_2, u_1, u_2\in W_M$, $\exists N>0$ such that

$$|u_i|\le N,\quad |h_i|\le N,\quad |u_i'|\le N,\quad |h_i'|\le N,\qquad i=1,2.\tag{4.23}$$

Using this information now in (4.22) we find the estimate

$$\begin{aligned}\alpha_0\|u_1-u_2\|_{H_0^2(r_1,r_2)} &\le \frac{2N}{\varepsilon^{\alpha/2}}r_2^3\int_{r_1}^{r_2}|h_2'(r)-h_1'(r)|\,dr\\ &+\frac{4N^2r_2^3}{\varepsilon^{\alpha}}\int_{r_1}^{r_2}|z_1(r)-z_2(r)|\,dr.\end{aligned}\tag{4.24}$$

Noting that with $\eta(\zeta)=(\varepsilon+\frac{1}{2}\zeta^2)^{\alpha/2}$, for $\zeta\in R^1$, the derivative $\eta'(\zeta)=\frac{\alpha}{2}\zeta(\varepsilon+\frac{1}{2}\zeta^2)^{-\alpha/2}$ is bounded on compact subsets of R^1, and taking account of (4.23), and the definition of $z_i(r)$, $i=1,2$, i.e., (4.20), we have, for some $C_1>0$

$$|z_1(r)-z_2(r)|\le C_1|(rh_1'(r)+r\widetilde{\Omega}'(r))-(rh_2'(r)+r\widetilde{\Omega}'(r))|$$

or

$$|z_1(r)-z_2(r)|\le C_1r|h_1'(r)-h_2'(r)|.\tag{4.25}$$

Combining (4.25) with (4.24) we, therefore, deduce the existence of a $\widetilde{C}>0$, independent of u_i, $i=1,2$, such that

$$\|u_1-u_2\|_{H_0^2(r_1,r_2)}\le\widetilde{C}\|h_1-h_2\|_{H_0^1(r_1,r_2)},\tag{4.26}$$

thus establishing the continuity of the map $T: W_M\to W_M$ for the case in which $\varepsilon>0$. On the other hand, if $\varepsilon=0$ then by (4.16) and (4.21) we have

$$\begin{aligned}&\alpha_0\|u_1-u_2\|_{H_0^2(r_1,r_2)}\\ &\quad\le 2^{\alpha/2}\int_{r_1}^{r_2}r^{3-\alpha}(u_1'-u_2')\left[\frac{h_2'+\widetilde{\Omega}'}{|h_2'+\widetilde{\Omega}'|^\alpha}-\frac{h_1'+\widetilde{\Omega}'}{|h_1'+\widetilde{\Omega}'|^\alpha}\right]dr.\end{aligned}\tag{4.27}$$

However, for arbitrary $a, b\in R^1$, and $0\le\alpha<1$, it is an easy exercise to verify the elementary inequality

$$\left|\frac{a}{|a|^\alpha}-\frac{b}{|b|^\alpha}\right|\le 2^\alpha|a-b|^{1-\alpha}.\tag{4.28}$$

Combining, in this case, (4.23), (4.27), and (4.28), we find, with the help of the Hölder inequality, that

$$\begin{aligned}\alpha_0\|u_1 - u_2\|^2_{H_0^2(r_1, r_2)} &\le C_2 \int_{r_1}^{r_2} |h_1' - h_2'|^{1-\alpha}\, dr \\ &\le C_3 \left[\int_{r_1}^{r_2} |h_1' - h_2'|\, dr\right]^{(1-\alpha)/2}\end{aligned}$$

for some $C_2, C_3 > 0$. Therefore, with $C_4 = C_3/\alpha_0$

$$\|u_1 - u_2\|_{H_0^2(r_1, r_2)} \le C_4 \|h_1 - h_2\|^{(1-\alpha)/2}_{H_0^1(r_1, r_2)} \tag{4.29}$$

and the continuity of $T : W_M \to W_M$ again follows, this time for the case $\varepsilon = 0$.

As a direct consequence of the Schauder fixed-point theorem we may now conclude that there exists, for $M > 0$ sufficiently large, a unique $u \in W_M$ such that $Tu = u$; this establishes, of course, the existence of a unique solution u of the boundary-value problem (4.12a,b,c), for $\mu > 0$, and, hence, for the original boundary-value problem (2.54)–(2.56). □

We now turn our attention to the boundary-value problem (4.7a,b) which is, of course, what the boundary-value problem (2.54)–(2.56) formally reduces to if we set $\mu = 0$; our basic result may be stated as follows:

THEOREM 4.3. For $\varepsilon \ge 0$, the boundary-value problem (4.7a,b) has a unique solution in $H^1(r_1, r_2)$.

Proof. For $\varepsilon = 0$ the unique solution of (4.7a,b), $\Omega_\alpha(r) = \Omega_{\alpha,0}(r)$, is given explicitly by (4.2); so we turn to the case in which $\varepsilon > 0$. We will first establish the uniqueness of solutions to (4.7a,b) under the assumption that solutions do exist in $H^1(r_1, r_2)$. So, suppose that $\omega_1, \omega_2 \in H^1(r_1, r_2)$ are solutions of (4.7a,b). Then

$$-\left[\frac{r^3\omega_1'(r)}{(\varepsilon + \frac{1}{2}(r\omega_1'(r))^2)^{\alpha/2}}\right]' + \left[\frac{r^3\omega_2'(r)}{(\varepsilon + \frac{1}{2}(r\omega_2'(r))^2)^{\alpha/2}}\right]' = 0. \tag{4.30}$$

We note that $\omega_1(r_i) = \Omega_i$, $\omega_2(r_i) = \Omega_i$, $i = 1, 2$. Multiplying (4.30) by $\omega_1 - \omega_2$, and integrating by parts, we obtain

$$\int_{r_1}^{r_2} r^2(\omega_1' - \omega_2')\left[\frac{r\omega_1'(r)}{(\varepsilon + \frac{1}{2}(r\omega_1'(r))^2)^{\alpha/2}} - \frac{r\omega_2'(r)}{(\varepsilon + \frac{1}{2}(r\omega_2'(r))^2)^{\alpha/2}}\right] dr = 0. \tag{4.31}$$

Now for any $a, b \in R^1$, $\varepsilon \ge 0$, and $0 \le \alpha < 1$, we have the elementary inequality

$$(a - b)\left(\frac{a}{(\varepsilon + \frac{1}{2}a^2)^{\alpha/2}} - \frac{b}{\left(\varepsilon + \frac{1}{2}b^2\right)^{\alpha/2}}\right) \ge 0 \tag{4.32}$$

and, therefore, as a direct consequence of (4.31) we have $\omega_1' = \omega_2'$ a.e. on $[r_1, r_2]$. However, $\omega_1(r_i) = \omega_2(r_i)$, $i = 1, 2$; so $\omega_1 = \omega_2$ a.e. on (r_1, r_2) and it follows that solutions of (4.7a,b) in $H^1(r_1, r_2)$ are unique if they exist. □

We now want to establish the existence of solutions to (4.7a,b), for $\varepsilon > 0$, in $H^1(r_1, r_2)$; in the course of doing this we will also prove that the unique solution of (2.54)–(2.56) converges, as $\mu \to 0^+$, to the unique solution of (4.7a,b), in the norm of $C^{1+\sigma}$, for $0 < \sigma < 1/2$, thus establishing the continuous dependence of solutions of (2.54)–(2.56) on μ as $\mu \to 0^+$. The continuous dependence result just cited will be formally delineated in Theorem 3 once the present proof has been completed.

As in the proof of Theorem 1, i.e., (4.11), we set $u(r) = \Omega(r; \varepsilon, \mu, \alpha) - \widetilde{\Omega}(r)$ with $\Omega(r; \varepsilon, \mu, \alpha)$ the unique solution of (2.54)–(2.56), for $\mu > 0$, and $\widetilde{\Omega}(r)$ the unique solution of (4.10a,b,c). We also set

$$\begin{aligned} s(r) &= u'(r), \\ W_0(r) &= r\widetilde{\Omega}'(r), \\ W(r) &= r\Omega'(r; \varepsilon, \mu, \alpha), \\ z(r) &= \varepsilon + \tfrac{1}{2}W^2(r). \end{aligned} \tag{4.33}$$

Clearly, both $u(r)$ and $s(r)$ depend on μ, but we will refrain, for the time being, from writing $u_\mu(r)$ or $s_\mu(r)$. Using the notation in (4.11), (4.33), and the fact that $\widetilde{\Omega}(r)$ satisfies (4.10a), we may rewrite (2.54) in the form

$$-\left[\frac{r^2 W(r)}{Z(r)^{\alpha/2}}\right]' + \mu(r^3 s''(r) + 3r^2 s'(r) - 3rs(r)) = 0. \tag{4.34}$$

Integrating (4.34) over (r_1, r), for $r \le r_2$, we find that

$$-\frac{r^2 W(r)}{Z(r)^{\alpha/2}} + \mu(r^3 s''(r) + 3r^2 s'(r) - 3rs(r)) = A_\mu \tag{4.35}$$

where

$$A_\mu = -\frac{r_1^2 W(r_1)}{Z(r_1)^{\alpha/2}} + \mu(r_1^3 s''(r_1) + 3r_1^2 s'(r_1) - 3r_1 s(r_1)). \tag{4.36}$$

By virtue of (2.55) and (4.10b),

$$\begin{aligned} \int_{r_1}^{r_2} s(r)\,dr &= u(r_2) - u(r_1) \\ &= (\Omega(r_2) - \widetilde{\Omega}(r_2)) - (\Omega(r_1) - \widetilde{\Omega}(r_1)) \\ &= 0. \end{aligned} \tag{4.37}$$

Therefore, if we multiply (4.35) by $s(r)$, integrate over (r_1, r_2), and then integrate by parts, we obtain

$$\begin{aligned} \int_{r_1}^{r_2} \frac{rW^2(r)}{Z(r)^{\alpha/2}}\,dr - \int_{r_1}^{r_2} \frac{rW(r)W_0(r)}{Z(r)^{\alpha/2}}\,dr \\ + \mu \int_{r_1}^{r_2} r[(rs(r))'^2 + 4s^2(r)]\,dr = 0 \end{aligned} \tag{4.38}$$

where we have used (4.37), the obvious relation $rs(r) = W(r) - W_0(r)$, and the fact that as $(rs(r))'(r_i) = 0$, for $i = 1, 2$,

$$\begin{aligned}\int_{r_1}^{r_2} (r^3 s''(r) + 3r^2 s'(r) - 3rs(r))s(r)\,dr \\ = \int_{r_1}^{r_2} [r(r(rs(r))')' - 4rs(r)]s(r)\,dr \\ = -\int_{r_1}^{r_2} r[(rs(r))'^2 + 4s^2(r)]\,dr.\end{aligned} \tag{4.39}$$

Now, for $\varepsilon > 0$, we set

$$E_\varepsilon = \{r \mid W^2(r) > 2\varepsilon\}. \tag{4.40}$$

Then $\forall r \in E_\varepsilon$,

$$\frac{rW^2(r)}{Z(r)^{\alpha/2}} = \frac{rW^2(r)}{\left(\varepsilon + \frac{1}{2}W^2(r)\right)^{\alpha/2}} > r_1 |W(r)|^{2-\alpha} \tag{4.41}$$

while $\forall r \in \{[r_1, r_2]/E_\varepsilon\} \equiv E_\varepsilon^c$ we have $|W(r)|^{2-\alpha} \le (2\varepsilon)^{(2-\alpha)/2}$. Therefore,

$$\begin{aligned}\int_{r_1}^{r_2} |W(r)|^{2-\alpha}\,dr &= \int_{E_\varepsilon} |W(r)|^{2-\alpha}\,dr + \int_{E_\varepsilon^c} |W(r)|^{2-\alpha}\,dr \\ &\le \frac{1}{r_1}\int_{r_1}^{r_2} \frac{rW^2(r)}{Z(r)^{\alpha/2}}\,dr + (2\varepsilon)^{(2-\alpha)/2}\,\mathrm{meas}(E_\varepsilon^c)\end{aligned} \tag{4.42}$$

where $\mathrm{meas}(E_\varepsilon^c) \le r_2 - r_1$. Combining (4.38) with (4.42) we obtain the estimate

$$\begin{aligned}\int_{r_1}^{r_2} |W(r)|^{2-\alpha}\,dr + \frac{\mu}{r_1}\int_{r_1}^{r_2} r[(rs(r))'^2 + 4s^2(r)]\,dr \\ \le \frac{1}{r_1}\int_{r_1}^{r_2} \frac{rW(r)W_0(r)}{Z(r)^{\alpha/2}}\,dr + (2\varepsilon)^{(2-\alpha)/2}(r_2 - r_1).\end{aligned} \tag{4.43}$$

By virtue of the Hölder inequality, and the definition of $Z(r)$, i.e., (4.33), we have the following estimate for the integral on the right-hand side of (4.43):

$$\begin{aligned}&\left|\int_{r_1}^{r_2} \frac{rW(r)W_0(r)}{Z(r)^{\alpha/2}}\,dr\right| \\ &\le 2^{\alpha/2}\int_{r_1}^{r_2} r|W(r)|^{1-\alpha}|W_0(r)|\,dr \\ &\le 2^{\alpha/2}\left[\int_{r_1}^{r_2} |W(r)|^{2-\alpha}\,dr\right]^{(1-\alpha)/(2-\alpha)} \cdot \left[\int_{r_1}^{r_2} |rW_0(r)|^{2-\alpha}\,dr\right]^{1/(2-\alpha)}.\end{aligned} \tag{4.44}$$

Applying the Young inequality to (4.44), with $p = \frac{2-\alpha}{1-\alpha}$ and $\sigma = \frac{r_1}{2^{1+\alpha/2}}$, we now find that

$$\frac{1}{r_1}\left|\int_{r_1}^{r_2} \frac{rW(r)W_0(r)}{Z(r)^{\alpha/2}}\,dr\right| \le \frac{1}{2}\int_{r_1}^{r_2} |W(r)|^{2-\alpha}\,dr + C_* \tag{4.45}$$

with C_* independent of both μ and ε, where we have used the fact that $W_0(\cdot) \in L^\infty(r_1, r_2)$. Combining (4.43) with (4.45) we are led to the estimate

$$\begin{aligned}\int_{r_1}^{r_2} |W(r)|^{2-\alpha}\,dr &+ \frac{2\mu}{r_1}\int_{r_1}^{r_2} r[rs(r))'^2 + 4s^2(r)]\,dr \\ &\le 2(2\varepsilon)^{(2-\alpha)/2}(r_2 - r_1) + 2C_*.\end{aligned} \tag{4.46}$$

For our next set of estimates, we multiply (4.35) by $(rs(r))''$, integrate over (r_1, r_2), and then integrate by parts; since

$$A_\mu \int_{r_1}^{r_2} (rs(r))''\,dr = A_\mu (rs(r))'\Big|_{r_1}^{r_2} = 0 \tag{4.47}$$

and

$$\begin{aligned}&\int_{r_1}^{r_2} (r^3 s''(r) + 3r^2 s'(r) - 3rs(r))(rs(r))''\,dr \\ &\quad = \int_{r_1}^{r_2} [r^2(rs(r))''^2 + 3(rs(r))'^2 + r^2 s'(r)(rs(r))'']\,dr\end{aligned} \tag{4.48}$$

we obtain

$$\begin{aligned}&\int_{r_1}^{r_2} \left[\frac{r^2 W(r)}{Z(r)^{\alpha/2}}\right]' (rs(r))'\,dr + \mu\int_{r_1}^{r_2} r^2 (rs(r))'^2\,dr \\ &\quad + 3\mu\int_{r_1}^{r_2} (rs(r))'^2\,dr + \mu\int_{r_1}^{r_2} r^2 s'(r)(rs(r))''\,dr = 0.\end{aligned} \tag{4.49}$$

We now note that

$$\begin{aligned}&\int_{r_1}^{r_2} \left[\frac{r^2 W(r)}{Z(r)^{\alpha/2}}\right]' (rs(r))'\,dr \\ &\quad = \int_{r_1}^{r_2} \left(\frac{2rW(r)}{Z(r)^{\alpha/2}} + r^2\left[\frac{W(r)}{Z(r)^{\alpha/2}}\right]'\right)(rs(r))'\,dr \\ &\quad = \int_{r_1}^{r_2} r^2 \left[\frac{W(r)}{Z(r)^{\alpha/2}}\right]' W'(r)\,dr - \int_{r_1}^{r_2} r^2\left[\frac{W(r)}{Z(r)^{\alpha/2}}\right]' W_0'(r)\,dr \\ &\quad\quad + \int_{r_1}^{r_2} \frac{2rW(r)}{Z(r)^{\alpha/2}}(rs(r))'\,dr \\ &\quad = \int_{r_1}^{r_2} r^2 \left[\frac{W(r)}{Z(r)^{\alpha/2}}\right]' W'(r)\,dr + \int_{r_1}^{r_2} \frac{W(r)}{Z(r)^{\alpha/2}}(r^2 W_0'(r))'\,dr \\ &\quad\quad + \int_{r_1}^{r_2} \frac{2rW(r)}{Z(r)^{\alpha/2}}(rs(r))'\,dr\end{aligned}$$

where we have used the fact that $rs(r) = W(r) - W_0(r)$ as well as the boundary conditions $W_0'(r_i) = (r\widetilde{\Omega}'(r))'(r_i) = 0$, $i = 1, 2$. For the first integral on the right-hand side of the last equation in (4.50), we compute that

$$\begin{aligned}\int_{r_1}^{r_2} r^2 \left[\frac{W(r)}{Z(r)^{\alpha/2}}\right]' W'(r)\,dr \\ = \int_{r_1}^{r_2} r^2 \frac{W'^2(r)}{Z(r)^{\alpha/2}} \left\{\frac{\varepsilon + \frac{1}{2}(1-\alpha)W^2(r)}{1+\frac{1}{2}W^2(r)}\right\} dr.\end{aligned} \tag{4.51}$$

However, $\forall\alpha$ such that $0 \le \alpha < 1$,

$$1-\alpha \le \frac{\varepsilon + \frac{1}{2}(1-\alpha)\eta}{\varepsilon + \frac{1}{2}\eta} \le 1, \quad \forall\eta \ge 0. \tag{4.52}$$

Therefore, combining (4.49)–(4.52) we easily obtain the estimate

$$\begin{aligned}(1-\alpha)\int_{r_1}^{r_2} r^2 \frac{W'^2(r)}{Z(r)^{\alpha/2}}\,dr + \mu\int_{r_1}^{r_2} r^2 (rs(r))''^2\,dr \\ + 3\mu\int_{r_1}^{r_2} (rs(r))'^2\,dr \le -\mu\int_{r_1}^{r_2} r^2 s'(r)(rs(r))''\,dr \\ - \int_{r_1}^{r_2} \frac{W(r)}{Z(r)^{\alpha/2}}(r^2 W_0'(r))'\,dr - \int_{r_1}^{r_2} \frac{2rW(r)}{Z(r)^{\alpha/2}}(rs(r))'\,dr.\end{aligned} \tag{4.53}$$

Using the Young inequality, and the estimate (4.46), we now note the following series of estimates for the first integral on the right-hand side of (4.53):

$$\begin{aligned}&\left|-\mu\int_{r_1}^{r_2} r^2 s'(r)(rs(r))''\,dr\right| \\ &\le \mu\int_{r_1}^{r_2} |rs'(r)|\cdot|r(rs(r))''|\,dr \\ &\le \mu\left[\int_{r_1}^{r_2} r^2 s'^2(r)\,dr\right]^{1/2}\left[\int_{r_1}^{r_2} r^2 (rs(r))''^2\,dr\right]^{1/2} \\ &\le \frac{1}{2}\mu\int_{r_1}^{r_2} r^2 (rs(r))''^2\,dr + 2\mu\int_{r_1}^{r_2} r^2 s'^2(r)\,dr \\ &\le \frac{1}{2}\mu\int_{r_1}^{r_2} r^2 (rs(r))''^2\,dr + 2\mu\int_{r_1}^{r_2} [(rs(r))' - s(r)]^2\,dr \\ &\le \frac{1}{2}\mu\int_{r_1}^{r_2} r^2 (rs(r))''^2\,dr + 4\mu\int_{r_1}^{r_2} ((rs(r))'^2 - s^2(r))\,dr \\ &\le \frac{1}{2}\mu\int_{r_1}^{r_2} r^2 (rs(r))''^2\,dr + \frac{4\mu}{r_1}\int_{r_1}^{r_2} r[(rs(r))'^2 - s^2(r)]\,dr.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|-\mu\int_{r_1}^{r_2} r^2 s'(r)(rs(r))''\,dr\right| \\ \le \frac{1}{2}\mu\int_{r_1}^{r_2} r^2 (rs(r))''^2\,dr + 4(r_2 - r_1)(2\varepsilon)^{(2-\alpha)/2} + 4C_*.\end{aligned} \tag{4.54}$$

For the second integral on the right-hand side of (4.53), we have the estimates

$$\begin{aligned}\left|\int_{r_1}^{r_2} \frac{W(r)}{Z(r)^{\alpha/2}}(r^2 W_0'(r))'\,dr\right| &\le 2^{\alpha/2}\int_{r_1}^{r_2} |W(r)|^{1-\alpha}|(r^2 W_0'(r))'|\,dr \\ &\le \frac{1}{2}\int_{r_1}^{r_2} |W(r)|^{2-\alpha}\,dr + C_1\end{aligned} \tag{4.55}$$

where we have, once again, used the Hölder and Young inequalities, and C_1 is independent of both μ and ε. Finally, for the last integral on the right-hand side of (4.53), we compute that

$$\begin{aligned}\left|\int_{r_1}^{r_2} \frac{2rW(r)}{Z(r)^{\alpha/2}}\cdot(rs(r))'\,dr\right| \\ \le \left|\int_{r_1}^{r_2} \frac{2rW(r)}{Z(r)^{\alpha/2}}\cdot W'(r)\,dr\right| + \left|\int_{r_1}^{r_2} \frac{2rW(r)}{Z(r)^{\alpha/2}}\cdot W_0'(r)\,dr\right|,\end{aligned} \tag{4.56}$$

since $rs(r) = W(r) - W_0(r)$. However,

$$\begin{aligned}\left|\int_{r_1}^{r_2} \frac{2rW(r)}{Z(r)^{\alpha/2}} W_0'(r)\,dr\right| &\le 2^{\frac{\alpha}{2}+1}\int_{r_1}^{r_2} |W(r)|^{1-\alpha}|rW_0'(r)|\,dr \\ &\le \frac{1}{2}\int_{r_1}^{r_2} |W(r)|^{2-\alpha}\,dr + C_2\end{aligned} \tag{4.57}$$

with C_2 independent of μ and ε, while

$$\begin{aligned}\left|\int_{r_1}^{r_2} \frac{2rW(r)}{Z(r)^{\alpha/2}} W'(r)\,dr\right| &\le 2\int_{r_1}^{r_2} \left|\frac{W(r)}{Z(r)^{\alpha/4}}\right|\cdot\left|\frac{rW'(r)}{Z(r)^{\alpha/4}}\right|\,dr \\ &\le 2\left[\int_{r_1}^{r_2} \frac{W^2(r)}{Z(r)^{\alpha/2}}\,dr\right]^{1/2}\left[\int_{r_1}^{r_2} \frac{r^2 W'^2(r)}{Z(r)^{\alpha/2}}\,dr\right]^{1/2} \\ &\le \frac{1}{2}(1-\alpha)\int_{r_1}^{r_2} \frac{r^2 W'^2(r)}{Z(r)^{\alpha/2}}\,dr + \frac{8}{1-\alpha}\int_{r_1}^{r_2} \frac{W^2(r)}{Z(r)^{\alpha/2}}\,dr \\ &\le \frac{1}{2}(1-\alpha)\int_{r_1}^{r_2} \frac{r^2 W'^2(r)}{Z(r)^{\alpha/2}}\,dr + \frac{8}{1-\alpha}\int_{r_1}^{r_2} |W|^{2-\alpha}\,dr.\end{aligned} \tag{4.58}$$

Combining (4.53)–(4.58) with (4.46) we obtain an estimate of the form

$$\begin{aligned}\int_{r_1}^{r_2} \frac{r^2 W'^2(r)}{Z(r)^{\alpha/2}}\,dr + \frac{\mu}{1-\alpha}\int_{r_1}^{r_2} r^2 (rs(r))''^2\,dr \\ + \frac{6\mu}{1-\alpha}\int_{r_1}^{r_2} (rs(r))'^2\,dr \\ \le C_3(1 + (r_2 - r_1)(2\varepsilon)^{(2-\alpha)/2})\end{aligned} \tag{4.59}$$

with C_3 independent of both μ and ε. We now set

$$\psi(W) = \int_0^W \frac{d\zeta}{(\varepsilon + \frac{1}{2}\zeta^2)^{\alpha/4}}. \tag{4.60}$$

Then, as a consequence of estimate (4.59) we obtain

$$\begin{aligned}\int_{r_1}^{r_2} r^2 \left[\frac{d}{dr}\psi(W(r))\right]^2 dr \\ = \int_{r_1}^{r_2} \frac{r^2 W'^2(r)}{(\varepsilon + \frac{1}{2}W^2(r))^{\alpha/2}}\,dr \\ \le C_3(1 + (r_2 - r_1)(2\varepsilon)^{(2-\alpha)/2}).\end{aligned} \tag{4.61}$$

Since $\psi(\cdot)$ is an even function and

$$\frac{1}{(\varepsilon + \frac{1}{2}\zeta^2)^{\alpha/4}} \le 2^{\alpha/4}|\zeta|^{-\alpha/2}$$

we have, for $0 \le \alpha < 1$,

$$\begin{aligned}|\psi(W)| &= \psi(|W|) \\ &\le 2^{\alpha/4}(1 - \alpha/2)^{-1}|W|^{1-\alpha/2} \\ &\le 4|W|^{1-\alpha/2}.\end{aligned} \tag{4.62}$$

Employing the estimate (4.46), in conjunction with the bound (4.62), we see that $\exists\ \psi_0 > 0$ (const.) such that

$$\int_{r_1}^{r^2} \psi^2(W(r))\,dr < \psi_0 \tag{4.63}$$

with ψ_0 independent of μ. Now $\forall f(\cdot) \in H^1(r_1, r_2)$, and $\forall \sigma > 0$, $\exists\ C_\sigma > 0$ such that

$$\max_{[r_1, r_2]} |f(r)| \le \sigma \left(\int_{r_1}^{r_2} f'^2(r)\,dr\right)^{1/2} + C_\sigma \left(\int_{r_1}^{r_2} f^2(r)\,dr\right)^{1/2} \tag{4.64}$$

(see, e.g., Lions [15, Lemma 5.1]); applying (4.64) with $f(r) = \psi(W(r))$, and employing both (4.64) and (4.63), we determine that for some $C_4 > 0$, C_4 independent of μ, we have

$$\max_{[r_1, r_2]} |\psi(W(r))| \le C_4. \tag{4.65}$$

In view of the definition (4.60), for $|\rho| > (2\varepsilon)^{1/2}$,

$$|\psi(\rho)| = \int_0^{|\rho|} \frac{d\zeta}{(\varepsilon + \frac{1}{2}\zeta^2)^{\alpha/4}} \geq (1-\alpha/2)^{-1}[|\rho|^{1-\alpha/2} - \varepsilon^{1-\alpha/2}]$$

or

$$|\rho|^{1-\alpha/2} \leq (1-\alpha/2)|\psi(\rho)| + \varepsilon^{1-\alpha/2}, \quad \text{for } |\rho| > (2\varepsilon)^{1/2}. \tag{4.66}$$

Combining (4.66) with (4.65), we easily deduce the existence of a constant $C > 0$, independent of μ, such that

$$\max_{[r_1, r_2]} |W(r)| \leq C. \tag{4.67}$$

If we now use (4.61) in conjunction with (4.64), and no longer suppress the dependence of W, u, or s on μ, it follows that $\exists\ \widetilde{C} > 0$, independent of μ, such that

$$\|\psi(W_\mu(\cdot))\|_{H^1(r_1, r_2)} \leq \widetilde{C}. \tag{4.68}$$

As a consequence of the uniform bound (4.68), if $\{\mu_n\}$, $\mu_n > 0$ for each integer n, is a sequence such that $\mu_n \to o^+$, as $n \to \infty$, there exists a subsequence $\{\mu_{n_k}\}$, and a function $\psi^0 \in H^1(r_1, r_2)$, such that as $n_k \to \infty$

$$\psi(W_{\mu_{n_k}}) \to \psi^0 \text{ in } H^1(r_1, r_2). \tag{4.69}$$

From standard embedding results we deduce from (4.69) that we also have, as $n_k \to \infty$,

$$\psi(W_{\mu_{n_k}}) \to \psi^0 \text{ in } C^{0,\sigma}, \quad 0 < \sigma < \tfrac{1}{2}. \tag{4.70}$$

Because (i.e., (4.60)) $\psi'(W) = (\varepsilon + \frac{1}{2}W^2)^{-\alpha/4}$, for $|W| \leq C$ (as is the case, by (4.67)) we have

$$k_1 < |\psi'(W)| < k_2 \quad \text{for some } k_1, k_2, \ 0 < k_1 < k_2. \tag{4.71}$$

Also, since ψ is monotone, ψ is invertible; thus, by (4.70), for $n_k \to \infty$,

$$W_{\mu_{n_k}} \to \psi^{-1}(\psi^0) \quad \text{in } C^{0,\sigma}(r_1, r_2), \ 0 < \sigma < 1/2. \tag{4.72}$$

Using the definition of $W(r)$, i.e., (4.33), and (4.72), we now find that, as $n_k \to \infty$,

$$\Omega'(\cdot; \varepsilon, \mu_{n_k}, \alpha) \to \frac{1}{r}\psi^{-1}(\psi^0) \quad \text{in } C^{0,\sigma}(r_1, r_2), \tag{4.73}$$

$$\Omega(\cdot; \varepsilon, \mu_{n_k}, \alpha) \to \widehat{\Omega}(\cdot) \quad \text{in } C^{1,\sigma}(r_1, r_2), \tag{4.74}$$

$0 < \sigma < 1/2$, where for $r \in [r_1, r_2]$

$$\widehat{\Omega}(r) = \Omega_1 + \int_{r_1}^{r} \frac{1}{\zeta}\psi^{-1}(\psi_0(\zeta))\, d\zeta. \tag{4.75}$$

By virtue of (4.46) and (4.59), we see that for some constant $C > 0$, independent of μ,

$$\mu\|s(\cdot)\|^2_{H^2(r_1, r_2)} \leq C \tag{4.76}$$

so that

$$\begin{aligned} \mu s_{\mu_{n_k}} &\to 0, \\ \mu s'_{\mu_{n_k}} &\to 0, \\ \mu s''_{\mu_{n_k}} &\to 0 \end{aligned} \tag{4.77}$$

in $L^2(r_1, r_2)$ as $n_k \to \infty$. Employing (4.74) and (4.77) it is easy to show that $\widehat{\Omega}(\cdot)$ is a solution of (4.7a,b); however, we have already shown that (Theorem 2) the solution of the boundary-value problem (4.7a,b) is, for $\varepsilon \geq 0$, uniquely defined in $H^1(r_1, r_2)$ if it exists. Therefore, we have established the following.

THEOREM 4.4. Since $\mu \to 0^+$, the unique solution of the boundary-value problem (2.54–2.56) $\Omega(\cdot\,; \varepsilon, \mu, \alpha)$ converges in $C^{1,\sigma}(r_1, r_2)$, $0 < \sigma < 1/2$, to the unique solution $\Omega_{\varepsilon,\alpha}(\cdot)$ of (4.76a,b) where $\Omega_{\varepsilon,\alpha}(\cdot) \equiv \widehat{\Omega}(\cdot)$ as given by (4.75).

The result in Theorem 3, above, also serves to establish the continuous dependence, in $C^{1,\sigma}(r_1, r_2)$, $0 < \sigma < 1/2$, of solutions of the boundary-value problem (2.54)–(2.56) with respect to perturbations in μ, for fixed $\varepsilon \geq 0$, and fixed α, $0 \leq \alpha < 1$. By examining carefully the estimates that led us to the uniform bound (4.76), i.e., (4.46) and (4.49), it is an easy exercise to show that solutions of (2.54)–(2.56) also depend continuously on ε, as $\varepsilon \to 0^+$, in the norm of $C^{2+\sigma}(r_1, r_2)$, for $0 < \sigma < 1/2$.

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