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ASYMPTOTIC ANALYSIS OF THE GINZBURG-LANDAU MODEL OF SUPERCONDUCTIVITY: REDUCTION TO A FREE BOUNDARY MODEL

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Abstract. A detailed formal asymptotic analysis of the Ginzburg-Landau model of superconductivity is performed and it is found that the leading-order solution satisfies a vectorial version of the Stefan problem for the melting or solidification of a pure material. The first-order correction to this solution is found to contain terms analogous to those of surface tension and kinetic undercooling in the scalar Stefan model. However, the "surface energy" of a superconducting material is found to take both positive and negative values, defining type I and type II superconductors respectively, leading to the conclusion that the free boundary model is only appropriate for type I superconductors.

1. Introduction. We consider macroscopic models for materials that are able to change their phase from being normally conducting (normal) to superconducting. The transformation occurs across a curve $H_c(T)$, known as the critical magnetic field, in the plane of the applied temperature and the magnitude of the applied magnetic field (Fig. 1). The critical temperature T_c is the temperature at which the transition occurs in zero magnetic field. The properties characteristic of the superconducting state are those of perfect conductivity (implying that the electric field $\mathbf{E} = \mathbf{0}$ in the superconducting state), and perfect diamagnetism, that is, the expulsion of magnetic flux from any superconducting region (so that the magnetic field $\mathbf{H}$ is also zero in the superconducting state). This latter property is usually known as the Meissner effect.

Consider a superconducting material occupying a region Ω, placed in an external magnetic field, which is then increased so that at some point on the boundary of the sample the field becomes greater than the critical field. The field will then start to penetrate the material at this point, and the superconductor will start to become normal again. While this conversion is occurring the superconductor might consist of an expanding normal region separated from the remaining superconducting region by a smooth boundary Γ.

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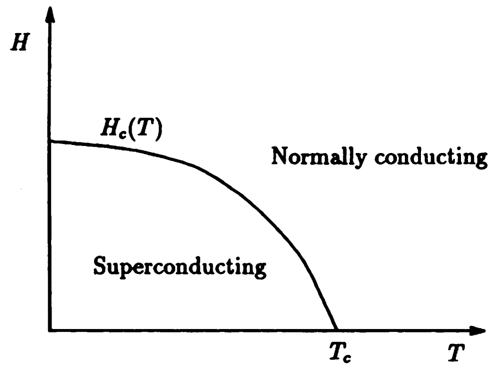

FIG. 1. The response of a superconductor in the presence of an applied magnetic field

The following (dimensionless) free-boundary model describing this situation was written down in [7]:

$$\frac{\partial \mathbf{H}}{\partial t} = -(\text{curl})^2\mathbf{H} = \nabla^2\mathbf{H} \quad \text{in the normal region}, \tag{1}$$

$$\mathbf{H} = \mathbf{0} \quad \text{in the superconducting region}, \tag{2}$$

$$|\mathbf{H}| = H_c \quad \text{on } \Gamma_N, \tag{3}$$

$$\text{curl } \mathbf{H} \wedge \mathbf{n} = -v_n\mathbf{H} \quad \text{on } \Gamma_N, \tag{4}$$

where $\mathbf{n}$ is the unit normal to the interface (taken to point towards the normal region), v_n is the normal velocity of the interface (positive if the superconducting region is expanding), and Γ_N denotes the interface approached from the normal region. This model is derived from Maxwell's equations, neglecting the displacement current. Latent heat and Joule heating effects are also neglected, so that the conversion is assumed to occur under isothermal conditions (in Sec. 4 we will remove this assumption and include thermal effects in the model). Equation (2) describes the Meissner effect, Eq. (3) follows from Fig. 1, and Eq. (4) follows from flux conservation across the interface Γ. Equations (1)–(4) must be supplemented by Maxwell's equations in the region external to the superconducting material (with the displacement current again neglected), initial conditions, and the usual conditions on $\mathbf{H}$ and $\mathbf{E}$ at an interface between two media on the fixed boundary $\partial\Omega$.

The model (1)–(4) is similar in form to a one-phase Stefan problem, albeit in a vectorial form, which is the simplest macroscopic model that could be written down for an evolving phase boundary in the classical theory of melting or solidification [8]. Indeed, the two-dimensional version of the above problem written down in [18] is exactly a one-phase Stefan problem.

The Stefan model is known to be well-posed just as long as neither superheating nor supercooling occurs [17], which in this case corresponds to the condition $H > H_c$ in the normal region. If this condition is violated, for example if the sample was originally in the normal state and we reduced the applied magnetic field through the critical field, then the model appears to be ill-posed and thus needs to be regularised [9]. In the Stefan problem a popular way of doing this is to include surface tension

(Gibbs-Thompson) and/or kinetic undercooling effects, so that the melting temperature depends on the curvature and/or normal velocity of the interface. An analogous regularisation of the model (1)–(4) was proposed in [19], in which it was suggested that the interface condition (3) should be modified to

$$|\mathbf{H}| = H_c - \frac{H_c}{2}\sigma\tilde{\kappa} \tag{5}$$

as Γ is approached from the normal region, where $\tilde{\kappa}$ is the mean curvature (positive if the centre of curvature lies in the superconducting region) and σ is the (dimensionless) "surface energy" of a normal/superconducting interface. The physical justification for the addition of such a term in the superconductivity model is not as clear as that for the solidification model.

An alternative regularisation of the Stefan model is the phase field model, in which the free boundary is smoothed out completely by the introduction of an order parameter, representing the fraction of solid material, which varies smoothly from -1 in the solid region to $+1$ in the liquid region [3]. It has been shown in [2] that in a formal asymptotic analysis of the phase field model as the thickness of this transition region tends to zero the leading-order solution satisfies the Stefan problem. Depending on the scaling of the parameters either surface tension and/or kinetic cooling terms can be made to appear in the interface temperature of this leading-order problem.

The aim of this paper is to perform a similar analysis of the superconductivity problem. In the next section we introduce the Ginzburg-Landau theory of superconductivity, in which the phase boundary is smoothed by the introduction of an order parameter as in the phase field model, and we examine the "surface energy" of a normal/superconducting transition layer. We then proceed in Sec. 3 to relate the Ginzburg-Landau model to the free-boundary model above by performing a formal asymptotic analysis as the interface thickness tends to zero. We will see that the leading-order solution satisfies the vectorial Stefan model (1)–(4). Furthermore, at first order we see the emergence of "kinetic undercooling" and "surface tension" effects in the magnitude of the magnetic field on the boundary. In the superconductivity model there does not appear to be a scaling of the parameters which will bring these terms into the leading-order solution, since unlike the phase field model there is no variable well depth and so we cannot increase the size of the surface energy. Since the stabilising terms appear only at first order we do not expect them to appreciably affect the outer solution until the normal velocity or curvature of the boundary becomes large. Thus we expect very intricate morphologies, even from the Ginzburg-Landau model. Moreover, whereas the surface energy of a solid/liquid interface is always positive we find that the surface energy of a normal/superconducting interface can take both positive and negative values, defining type I and type II superconductors respectively, and leading to qualitatively very different behaviour.

In Sec. 4 we relax the assumption that the conversion occurs under isothermal conditions and include thermal effects in both the free boundary and the Gizburg-Landau models. A formal asymptotic analysis of the Ginzburg-Landau model again leads to the free boundary model at leading order, but with the addition of an extra term, which arises because the number density of superconducting electrons is not constant in the superconducting phase, but is dependent on the temperature.

Finally, we mention that the analogy between models for solidification and models for superconductivity is explored in more detail in [7].

2. Ginzburg-Landau models. For a more complete introduction to the Ginzburg-Landau theory of superconductivity the reader is referred to [7, 10] and the references therein. Here we merely state the dimensionless, time-dependent Ginzburg-Landau equations as

$$-\alpha\xi^2\frac{\partial\Psi}{\partial t} - \frac{\alpha\xi i}{\lambda}\Psi\phi + \left(\xi\nabla\Psi - \frac{i}{\lambda}\mathbf{A}\right)^2\Psi = \Psi(|\Psi|^2 - 1), \tag{6}$$

$$-\lambda^2(\mathrm{curl})^2\mathbf{A} = \lambda^2\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla\phi\right) + \frac{i\xi\lambda}{2}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) + |\Psi|^2\mathbf{A}, \tag{7}$$

where Ψ is the (in this case complex) superconducting order parameter, and $\mathbf{A}$ and ϕ are the magnetic vector potential and the electric scalar potential, respectively, which are such that

$$\mathbf{H} = \mathrm{curl}\,\mathbf{A}, \qquad \mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi; \tag{8}$$

$\mathbf{A}$ is unique up to the addition of a gradient; once $\mathbf{A}$ is given, ϕ is unique up to the addition of a function of t. Here α, λ, and ξ are positive material constants (dependent on the temperature); λ is known as the penetration depth, ξ as the coherence length. The ratio of these length scales is $\kappa = \lambda/\xi$, called the Ginzburg-Landau parameter.

In the steady state, with $\phi = 0$, these equations result from minimising the Ginzburg-Landau formulation of the Gibbs free energy [15],

$$\int\left(-|\Psi|^2 + \frac{|\Psi|^4}{2} + |\mathbf{H}|^2 - 2\mathbf{H}\cdot\mathbf{H}_0 + \left|\xi\nabla\Psi - \frac{i}{\lambda}\mathbf{A}\Psi\right|^2\right)dV,$$

where $\mathbf{H}_0$ is the applied magnetic field. In the time-dependent case they can be obtained as a limiting case of the microscopic BCS equations [1, 16].

The thermodynamic critical field is defined as the magnetic field at which the free energy of the wholly superconducting state becomes less than that of the normal state. In these units it is given by $H_c = 1/\sqrt{2}$. We will see in Sec. 3 that the critical field of the introduction corresponds exactly to this thermodynamic critical field.

Eqs. (6)–(8) are gauge invariant in the sense that they are invariant under transformations of the type

$$\mathbf{A} \to \mathbf{A} + \nabla\omega, \qquad \phi \to \phi - \frac{\partial\omega}{\partial t}, \qquad \Psi \to \Psi e^{i\omega/(\xi\lambda)}.$$

We may write the equations in terms of real variables by introducing the new gauge invariant potentials

$$\mathbf{Q} = \mathbf{A} - \xi\lambda\nabla\chi, \qquad \mathbf{\Phi} = \phi + \xi\lambda\frac{\partial\chi}{\partial t}, \tag{9}$$

where $\Psi = fe^{i\chi}$. We then obtain coupled equations for just f and $\mathbf{Q}$:

$$-\alpha\xi^2\frac{\partial f}{\partial t} + \xi^2\nabla^2 f = f^3 - f + \frac{f|\mathbf{Q}|^2}{\lambda^2}, \tag{10}$$

$$\alpha f^2\Phi + \operatorname{div}(f^2\mathbf{Q}) = 0, \tag{11}$$

$$-\lambda^2(\operatorname{curl})^2\mathbf{Q} = \lambda^2\left(\frac{\partial\mathbf{Q}}{\partial t} + \nabla\Phi\right) + f^2\mathbf{Q}. \tag{12}$$

2.1. *Surface energy of a normal superconducting interface.* When we perform a formal asymptotic analysis of the Ginzburg-Landau equations in Sec. 3 we will assume that the solution comprises normal and superconducting domains separated by thin transition layers. We examine here the surface energy of a stationary, planar transition layer by considering the isothermal Ginzburg-Landau equations in one dimension. We will find in Sec. 3 that this is a local model for transition layers in general.

We take the field $\mathbf{H}$ to be directed along the z-axis and the magnetic vector potential $\mathbf{A}$ to be directed along the y-axis. We make the assumption that all functions are dependent on x only. Then $H_3 = dA_2/dx$, or simply $H = dA/dx$. Equation (7) now implies $\nabla\chi = 0$. in which case Ψ may be taken to be real, so that $\Psi = f$, $\mathbf{A} = \mathbf{Q}$. We work on the length scale of the penetration depth by rescaling x and Q and λ to obtain

$$\frac{1}{\kappa^2}f'' = f^3 - f + Q^2 f, \tag{13}$$

$$Q'' = f^2 Q, \tag{14}$$

where $\kappa = \lambda/\xi$ is the Ginzburg-Landau parameter and $' = d/dx$. We note that Eqs. (13), (14) form a Hamiltonian system, with the Hamiltonian given by

$$\mathscr{H} = \frac{f^4}{2} + f^2 - Q^2 f^2 - \frac{(f')^2}{\kappa^2} - (Q')^2.$$

Hence

$$\frac{(f')^2}{\kappa^2} + (Q')^2 = \frac{f^4}{2} - f^2 + f^2 Q^2 + \text{const.} \tag{15}$$

In order that we have a local model for the transition region between normal and superconducting parts of a material we need to apply the boundary conditions

$$Q \to 0,\ \ f \to 1 \quad \text{as } x \to -\infty, \tag{16}$$

$$Q' \to H_0,\ \ f \to 0 \quad \text{as } x \to \infty, \tag{17}$$

where the field on the normal side of the region is equal to H_0. The equations admit a solution if and only if $H_0 = H_c$. To see this we note that the boundary conditions (16) imply that the constant in (15) is in this case equal to 1/2. Hence, as $x \to \infty$, $Q' \to 1/\sqrt{2}$, providing f decays sufficiently quickly that $fQ \to 0$. Since $H_c = 1/\sqrt{2}$ in these units we see that in order for a normal/superconducting transition layer to exist the limiting value of the field in the normal region as the domain boundary is approached must be equal to H_c. A rigorous demonstration of this result is given in [6], where the existence and uniqueness of the solution when $H_0 = H_c$ is proved, and it is shown that the solution necessarily satisfies

$$0 < f < 1, \quad Q > 0, \quad f' < 0, \quad Q' > 0$$

and that f decays exponentially and $Q \sim x/\sqrt{2} + c + O(e^{-Kx^2})$ as $x \to \infty$.

Let us now examine the surface energy associated with a plane boundary between normal and superconducting phases, which is defined in [15] to be the excess of the Gibbs free energy of such a transition region over the Gibbs free energy of the normal or superconducting phases at the critical field, i.e.,

$$\sigma = \int_{-\infty}^{\infty} (\mathscr{G}_{sH} - \mathscr{G}_{nH})\, dx.$$

Writing the free energy densities in terms of f and Q, the solution to (13)–(17), we find

$$\sigma = \lambda \int_{-\infty}^{\infty} \left((1 - f^2)^2 + \frac{2(f')^2}{\kappa^2} + 2f^2Q^2 - 2Q'(\sqrt{2} - Q') \right) dx.$$

By (15) we have

$$\sigma = 4\lambda \int_{-\infty}^{\infty} \left(\frac{(f')^2}{\kappa^2} - Q' \left(\frac{1}{\sqrt{2}} - Q' \right) \right) dx.$$

Now

$$\begin{aligned} \int_{-\infty}^{X} \{f^2Q^2 - Q'(1/\sqrt{2} - Q')\}\, dx &= \int_{-\infty}^{X} \{Q''Q - Q'/\sqrt{2} + (Q')^2\}\, dx \\ &= Q(X)\{Q'(X) - 1/\sqrt{2}\} \end{aligned}$$

by (14). Hence the integral tends to zero as $X \to \infty$, and so

$$\sigma = 4\lambda \int_{-\infty}^{\infty} \left(\frac{(f')^2}{\kappa^2} - f^2Q^2 \right) dx. \tag{18}$$

It is found numerically that $\sigma > 0$ for $\kappa < 1/\sqrt{2}$ and $\sigma < 0$ for $\kappa > 1/\sqrt{2}$. We now prove

PROPOSITION. For $\kappa < 1/\sqrt{2}$, $\kappa = 1/\sqrt{2}$, $\kappa > 1/\sqrt{2}$ we have respectively $\sigma > 0$, $\sigma = 0$, $\sigma < 0$.

Proof. We consider the case $\kappa < 1/\sqrt{2}$. Define functions F and G by

$$F(x) = f^2 - 1 + \sqrt{2}Q', \tag{19}$$

$$G(x) = \kappa^{-1}f' + fQ. \tag{20}$$

We aim to show that $F < 0$, $G < 0$. Differentiating (19), (20) we find

$$F' = \sqrt{2}f(\sqrt{2}f' + fQ), \tag{21}$$

$$G' = \kappa f(f^2 - 1 + \kappa^{-1}Q') + \kappa QG. \tag{22}$$

Hence

$$F' > \sqrt{2} f G, \tag{23}$$

$$G' > \kappa f F + \kappa Q G, \tag{24}$$

since $f > 0$, $f' < 0$. Since, by (15),

$$2G(\kappa^{-1} f' - fQ) = F(f^2 - 1 - \sqrt{2} Q'), \tag{25}$$

and $f' < 0$, $0 < f < 1$, $Q' > 0$, we have

$$\begin{aligned} F > 0 &\Leftrightarrow G > 0, \\ F = 0 &\Leftrightarrow G = 0, \\ F < 0 &\Leftrightarrow G < 0. \end{aligned}$$

Suppose, for a contradiction, that there is a point x_0 such that $G(x_0) \geq 0$. Then by (23), (24), and (25) we have $F(x_0) \geq 0$, $F'(x_0) > 0$, $G'(x_0) > 0$. Suppose now that there is a first point x_1 greater than x_0 such that $F'(x_1) = 0$. Then (23) implies $G(x_1) < 0$, whence (25) implies $F(x_1) < 0$, which leads to a contradiction of the minimality of x_1. Therefore, there is no such point x_1 and we have $F'(x) > 0$ for all $x > x_0$. Hence $F(x) > F(x_0) \geq 0$ for all $x > x_0$. This contradicts the fact that $F(x) \to 0$ as $x \to \infty$, since $f \to 0$ and $Q' \to 1/\sqrt{2}$. Hence there does not exist x_0 such that $G(x_0) \geq 0$. Therefore,

$$\begin{aligned} G(x) < 0 \quad &\text{for all } x, \\ F(x) < 0 \quad &\text{for all } x. \end{aligned}$$

Since

$$\sigma = 4\lambda \int_{-\infty}^{\infty} \{G(\kappa^{-1} f' - fQ)\}\, dx,$$

the result follows.

The case $\kappa > 1/\sqrt{2}$ is exactly similar. For $\kappa = 1/\sqrt{2}$ a similar proof shows that $F \equiv 0$, $G \equiv 0$, and so $\sigma = 0$.

We have the following definition.

Definition. Superconductors with $\kappa < 1/\sqrt{2}$ are known as type I superconductors. Superconductors with $\kappa > 1/\sqrt{2}$ are known as type II superconductors.

Because of the above result we expect very different behaviour from type I and type II superconductors. Whereas a type I superconductor will aim to minimise the surface area of normal/superconducting interfaces, a type II superconductor will aim to maximise it, leading to a quite different morphology. We will see in the next section how the surface energy influences an asymptotic solution of the Ginzburg-Landau equations.

3. Asymptotic solution of the Ginzburg-Landau model: Reduction to a free-boundary model. We consider in this section the formal asymptotic limit of the Ginzburg-Landau equations as λ and $\xi \to 0$. The analysis is similar to that performed by Caginalp on the phase field model in [2].

We assume that the material comprises normal and superconducting domains separated by thin transition layers. This assumption, as will we see, is only valid for type I superconductors. A local analysis of such a transition layer will reveal that, as claimed in the previous section, f and $|\mathbf{Q}|$ satisfy the stationary, planar transition layer equations (13)–(17) to leading order. The leading-order outer solution will be found to satisfy the vectorial Stefan problem.

Consideration of the first-order terms in the outer solution will reveal the emergence of "surface tension" and "kinetic undercooling" terms, as in the modified Stefan model.

We have the following result.

THEOREM. In the formal asymptotic limit of the Ginzburg-Landau model as $\lambda, \xi \to 0$, with α and $\kappa = \lambda/\xi$ fixed, one obtains the vectorial Stefan model (1)–(4) at leading order.

A complete determination of the solution will involve initial and fixed boundary conditions. However, they will be left unspecified since our primary interest is rather the conditions on the free boundary between the two regions.

The Ginzburg-Landau equations (10)–(12), together with the relation (8), are

$$-\frac{\alpha\lambda^2}{\kappa^2}\frac{\partial f}{\partial t} + \frac{\lambda^2}{\kappa^2}\nabla^2 f = f^3 - f + \frac{f|\mathbf{Q}|^2}{\lambda^2}, \tag{26}$$

$$\alpha f^2\Phi + \operatorname{div}(f^2\mathbf{Q}) = 0, \tag{27}$$

$$-\lambda^2 \operatorname{curl} \mathbf{H} = \lambda^2\frac{\partial \mathbf{Q}}{\partial t} + \lambda^2\nabla\Phi + f^2\mathbf{Q}, \tag{28}$$

$$\mathbf{H} = \operatorname{curl} \mathbf{Q}. \tag{29}$$

We define $\Gamma(t)$ by

$$\Gamma(t) = \{\mathbf{r}\colon f(\mathbf{r}, t) = \eta\}, \tag{30}$$

where η is to be specified later, but certainly $0 < \eta < 1$. At leading order Γ will be the "interface" of the outer solution. The choice of η will not affect the interface conditions at leading order, and so any value of η will serve to prove the proposition. However, when we go on to consider the first-order correction to the leading-order solution the choice of η becomes relevant, and we wish to choose η to make the calculations as simple as possible. We note that there is no obvious choice for η as in the phase field, when symmetry suggests choosing $\eta = 0$.

Outer expansions. Away from the transition region we formally expand all functions in powers of λ to obtain the outer expansions, denoted by the subscript o, as

$$f_o(\mathbf{r}, t, \lambda) = f_o^{(0)}(\mathbf{r}, t) + \lambda f_o^{(1)}(\mathbf{r}, t) + \cdots, \tag{31}$$

$$\mathbf{\Phi}_o(\mathbf{r}, t, \lambda) = \mathbf{\Phi}_o^{(0)}(\mathbf{r}, t) + \lambda \mathbf{\Phi}_o^{(1)}(\mathbf{r}, t) + \cdots, \tag{32}$$

$$\mathbf{Q}_o(\mathbf{r}, t, \lambda) = \mathbf{Q}_0^{(0)}(\mathbf{r}, t) + \lambda \mathbf{Q}_o^{(1)}(\mathbf{r}, t) + \cdots, \tag{33}$$

$$\mathbf{H}_o(\mathbf{r}, t, \lambda) = \mathbf{H}_o^{(0)}(\mathbf{r}, t) + \lambda \mathbf{H}_o^{(1)}(\mathbf{r}, t) + \cdots, \tag{34}$$

$$\Gamma(t, \lambda) = \Gamma^{(0)}(t) + \lambda\Gamma^{(1)}(t) + \cdots. \tag{35}$$

We note that the expansions (31)–(34) may be discontinuous across $\Gamma^{(0)}(t)$, but will be smooth otherwise. Substituting (31)–(34) into (26)–(29) and equating powers of

λ yields at leading order

$$f_o^{(0)}|\mathbf{Q}_o^{(0)}|^2 = 0, \tag{36}$$

$$\alpha(f_o^{(0)})^2\mathbf{\Phi}_o^{(0)} + \operatorname{div}((f_o^{(0)})^2\mathbf{Q}_o^{(0)}) = 0, \tag{37}$$

$$(f_o^{(0)})^2\mathbf{Q}_o^{(0)} = \mathbf{0}, \tag{38}$$

$$\mathbf{H}_o^{(0)} = \operatorname{curl} \mathbf{Q}_o^{(0)}. \tag{39}$$

We see by (36) that either $f_o^{(0)} = 0$, or $\mathbf{Q}_o^{(0)} = \mathbf{0}$, corresponding to normal and superconducting regions respectively. We consider these cases separately.

Normal region. With $f_o^{(0)} \equiv 0$, $\mathbf{Q}_o^{(0)} \not\equiv \mathbf{0}$ we equate powers of λ at the next order in (26)–(28) to give

$$f_o^{(1)}|\mathbf{Q}_o^{(0)}|^2 = 0, \tag{40}$$

$$\alpha(f_o^{(1)})^2\mathbf{\Phi}_o^{(0)} + \operatorname{div}((f_o^{(1)})^2\mathbf{Q}_o^{(0)}) = 0, \tag{41}$$

$$-\operatorname{curl} \mathbf{H}_o^{(0)} = \frac{\partial \mathbf{Q}_o^{(0)}}{\partial t} + \nabla\mathbf{\Phi}_o^{(0)} + (f_o^{(1)})^2\mathbf{Q}_o^{(0)}. \tag{42}$$

By (40) we have that $f_o^{(1)} \equiv 0$. Taking the curl of Eq. (42) and using Eq. (39) we have

$$-(\operatorname{curl})^2\mathbf{H}_o^{(0)} = \frac{\partial \mathbf{H}_o^{(0)}}{\partial t}. \tag{43}$$

Noting that

$$\operatorname{div} \mathbf{H}_o^{(0)} = \operatorname{div}(\operatorname{curl} \mathbf{Q}_o^{(0)}) = 0,$$

we see that

$$\nabla^2\mathbf{H}_o^{(0)} = \frac{\partial \mathbf{H}_o^{(0)}}{\partial t}. \tag{44}$$

In fact, if we continue in this way, we find

$$\nabla^2\mathbf{H}_o = \frac{\partial \mathbf{H}_o}{\partial t} + O(\lambda^n) \quad \text{in the normal region} \tag{45}$$

for any n.

Superconducting region. With $f_o^{(0)} \not\equiv 0$, we have

$$\mathbf{Q}_o^{(0)} \equiv \mathbf{0}, \qquad \mathbf{H}_o^{(0)} \equiv \mathbf{0}, \qquad \mathbf{\Phi}_o^{(0)} \equiv 0. \tag{46}$$

Equating powers of λ at the next order in each equation we have

$$0 = (f_o^{(0)})^3 - f_o^{(0)} + f_o^{(0)}|\mathbf{Q}_o^{(1)}|^2, \tag{47}$$

$$\alpha(f_o^{(0)})^2\mathbf{\Phi}_o^{(1)} + \operatorname{div}((f_o^{(0)})^2\mathbf{Q}_o^{(1)}) = 0, \tag{48}$$

$$\mathbf{0} = (f_o^{(0)})^2\mathbf{Q}_o^{(1)}, \tag{49}$$

$$\mathbf{H}_o^{(1)} = \operatorname{curl} \mathbf{Q}_o^{(1)}. \tag{50}$$

Therefore,

$$\mathbf{Q}_o^{(1)} \equiv \mathbf{0}, \quad \mathbf{H}_o^{(1)} \equiv \mathbf{0}, \quad \mathbf{\Phi}_o^{(1)} \equiv 0, \quad f_o^{(0)} \equiv 1. \tag{51}$$

In fact, if we continue in this way, we find

$$\mathbf{H} = O(\lambda^n) \quad \text{in the superconducting region} \tag{52}$$

for any n.

Inner expansions. Let $\Gamma(t)$ be given by the surface

$$\mathbf{r} = (x, y, z) = \mathbf{R}(s_1(x, y, z), s_2(x, y, z), t),$$

i.e., $\mathbf{R}(s_1, s_2, t)$ is such that

$$f(\mathbf{R}(s_1, s_2, t), t) = \eta.$$

We parameterise the surface $\mathbf{R}$ such that s_1 and s_2 are the principal directions. We define new variables ρ and τ by the equations

$$\mathbf{r} = \mathbf{R}(s_1, s_2, t) + \lambda\rho\mathbf{n},$$
$$t = \tau.$$

We then have a new local orthogonal coordinate system (s_1, s_2, ρ, τ), with scaling factors

$$h_1 = (\mathbf{r}_{s_1} \cdot \mathbf{r}_{s_1})^{1/2} = E^{1/2}(1 + \lambda\rho\tilde{\kappa}_1), \tag{53}$$

$$h_2 = (\mathbf{r}_{s_2} \cdot \mathbf{r}_{s_2})^{1/2} = G^{1/2}(1 + \lambda\rho\tilde{\kappa}_2), \tag{54}$$

$$h_3 = (\mathbf{r}_\rho \cdot \mathbf{r}_\rho)^{1/2} = \lambda, \tag{55}$$

where $\tilde{\kappa}_1$, $\tilde{\kappa}_2$ are the principal curvatures in the s_1, s_2 directions, respectively, positive if the centre of curvature lies in the superconducting region, $E = (\frac{\partial \mathbf{R}}{\partial s_1} \cdot \frac{\partial \mathbf{R}}{\partial s_1})$, and $G = (\frac{\partial \mathbf{R}}{\partial s_2} \cdot \frac{\partial \mathbf{R}}{\partial s_2})$. We can now use the general formulae for curl, div, $\mathbf{v} \cdot \nabla$, etc. in curvilinear coordinates, which are listed in the appendix for ease of reference. Only the expression for $\mathbf{v} \cdot \nabla$ is nonstandard and is calculated in [4]. We set

$$\mathbf{Q}_i = Q_{i,1}\mathbf{e}_1 + Q_{i,2}\mathbf{e}_2 + Q_{i,3}\mathbf{n}, \tag{56}$$

$$\mathbf{H}_i = H_{i,1}\mathbf{e}_1 + H_{i,2}\mathbf{e}_2 + H_{i,3}\mathbf{n}, \tag{57}$$

where $\mathbf{e}_i = \mathbf{R}_{s_i}/|\mathbf{R}_{s_i}|$. Noting that $\partial/\partial t$ becomes $\partial/\partial\tau - \mathbf{v}\cdot\nabla$ in the new coordinates, where $\mathbf{v}$ is the velocity of the interface Γ, Eqs. (26)–(29) become

$$\begin{aligned} &\frac{\alpha\lambda v_n}{\kappa^2}\frac{\partial f_i}{\partial \rho} + \frac{1}{\kappa^2}\frac{\partial^2 f_i}{\partial \rho^2} + \frac{\lambda(\tilde{\kappa}_1 + \tilde{\kappa}_2)}{\kappa^2}\frac{\partial f_i}{\partial \rho} + O(\lambda^2) \\ &\quad = f_i^3 - f_i + \frac{f_i}{\lambda^2}(Q_{i,1}^2 + Q_{i,2}^2 + Q_{i,3}^2), \end{aligned} \tag{58}$$

$$\begin{aligned} \lambda\alpha f_i^2\Phi_i &+ \frac{1}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial[G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)f_i^2 Q_{i,1}]}{\partial s_1}\right) \\ &+ \frac{1}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial[E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)f_i^2 Q_{i,2}]}{\partial s_2}\right) \\ &+ \frac{1}{\lambda(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial[(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)f_i^2 Q_{i,3}]}{\partial \rho}\right) = 0, \end{aligned} \tag{59}$$

$$\begin{aligned}
&\lambda\frac{\partial H_{i,2}}{\partial\rho}+\frac{\lambda^2\tilde{\kappa}_2 H_{i,2}}{1+\lambda\rho\tilde{\kappa}_2}-\frac{\lambda^2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial H_{i,3}}{\partial s_2}\\
&\quad=-v_n\lambda\frac{\partial Q_{i,1}}{\partial\rho}+\lambda^2\frac{\partial Q_{i,1}}{\partial\tau}-\frac{\lambda^2 v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial Q_{i,1}}{\partial s_1}\\
&\qquad-\frac{\lambda^2 v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial Q_{i,1}}{\partial s_2}+\frac{\lambda^2 v_1\tilde{\kappa}_1 Q_{i,3}}{(1+\lambda\rho\tilde{\kappa}_1)}\\
&\qquad-\frac{\lambda^2}{2(EG)^{1/2}}\left(\frac{v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial G}{\partial s_2}-\frac{v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial E}{\partial s_1}\right)Q_{i,2}\\
&\qquad+\frac{\lambda^2}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial\Phi_i}{\partial s_1}+f_i^2 Q_{i,1},
\end{aligned}\tag{60}$$

$$\begin{aligned}
&-\lambda\frac{\partial H_{i,1}}{\partial\rho}-\frac{\lambda^2\tilde{\kappa}_1 H_{i,1}}{1+\lambda\rho\tilde{\kappa}_1}+\frac{\lambda^2}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial H_{i,3}}{\partial s_1}\\
&\quad=-v_n\lambda\frac{\partial Q_{i,2}}{\partial\rho}+\lambda^2\frac{\partial Q_{i,2}}{\partial\tau}-\frac{\lambda^2 v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial Q_{i,2}}{\partial s_1}\\
&\qquad-\frac{\lambda^2 v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial Q_{i,2}}{\partial s_2}+\frac{\lambda^2 v_2\tilde{\kappa}_2 Q_{i,3}}{(1+\lambda\rho\tilde{\kappa}_2)}\\
&\qquad+\frac{\lambda^2}{2(EG)^{1/2}}\left(\frac{v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial G}{\partial s_2}-\frac{v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial E}{\partial s_1}\right)Q_{i,1}\\
&\qquad+\frac{\lambda^2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial\Phi_i}{\partial s_2}+f_i^2 Q_{i,2},
\end{aligned}\tag{61}$$

$$\begin{aligned}
&\frac{\lambda^2}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial[E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)H_{i,1}]}{\partial s_2}\right)\\
&\quad-\frac{\lambda^2}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial[G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)H_{i,2}]}{\partial s_1}\right)\\
&=-v_n\lambda\frac{\partial Q_{i,3}}{\partial\rho}+\lambda^2\frac{\partial Q_{i,3}}{\partial\tau}-\frac{\lambda^2 v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial Q_{i,3}}{\partial s_1}-\frac{\lambda^2 v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial Q_{i,3}}{\partial s_2}\\
&\quad-\frac{\lambda^2 v_1\tilde{\kappa}_1 Q_{i,1}}{(1+\lambda\rho\tilde{\kappa}_1)}-\frac{\lambda^2 v_2\tilde{\kappa}_2 Q_{i,2}}{(1+\lambda\rho\tilde{\kappa}_2)}+\lambda\frac{\partial\Phi_i}{\partial\rho}+f_i^2 Q_{i,3},
\end{aligned}\tag{62}$$

$$H_{i,1} = -\frac{1}{\lambda}\frac{\partial Q_{i,2}}{\partial \rho} - \frac{\tilde{\kappa}_2 Q_{i,2}}{1+\lambda\rho\tilde{\kappa}_2} + \frac{1}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial Q_{i,3}}{\partial s_2}, \tag{63}$$

$$H_{i,2} = \frac{1}{\lambda}\frac{\partial Q_{i,1}}{\partial \rho} + \frac{\tilde{\kappa}_1 Q_{i,1}}{1+\lambda\rho\tilde{\kappa}_1} - \frac{1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial Q_{i,3}}{\partial s_1}, \tag{64}$$

$$\begin{aligned} H_{i,3} = {} & \frac{1}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial[G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)Q_{i,2}]}{\partial s_1}\right) \\ & - \frac{1}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial[E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)Q_{i,1}]}{\partial s_2}\right), \end{aligned} \tag{65}$$

where $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_n\mathbf{n}$. We formally expand all functions in the inner variables in powers of λ to obtain the inner expansions as

$$f_i(s_1, s_2, \rho, \tau, \lambda) = f_i^{(0)}(s_1, s_2, \rho, \tau) + \lambda f_i^{(1)}(s_1, s_2, \rho, \tau) + \cdots, \tag{66}$$

$$\mathbf{\Phi}_i(s_1, s_2, \rho, \tau, \lambda) = \mathbf{\Phi}_i^{(0)}(s_1, s_2, \rho, \tau) + \lambda\mathbf{\Phi}_i^{(0)}(s_1, s_2, \rho, \tau) + \cdots, \tag{67}$$

$$\mathbf{Q}_i(s_1, s_2, \rho, \tau, \lambda) = \mathbf{Q}_i^{(0)}(s_1, s_2, \rho, \tau) + \lambda\mathbf{Q}_i^{(1)}(s_1, s_2, \rho, \tau) + \cdots, \tag{68}$$

$$\mathbf{H}_i(s_1, s_2, \rho, \tau, \lambda) = \mathbf{H}_i^{(0)}(s_1, s_2, \rho, \tau) + \lambda\mathbf{H}_i^{(1)}(s_1, s_2, \rho, \tau) + \cdots. \tag{69}$$

We also expand

$$\mathbf{R}(s_1, s_2, \tau, \lambda) = \mathbf{R}^{(0)}(s_1, s_2, \tau) + \lambda\mathbf{R}^{(1)}(s_1, s_2, \tau) + \cdots,$$

which gives

$$E(s_1, s_2, \tau, \lambda) = E^{(0)}(s_1, s_2, \tau) + \lambda E^{(1)}(s_1, s_2, \tau) + \cdots,$$

etc.

At this point we make a note of the matching conditions that will be used to relate the inner and outer solutions. These are as follows:

$$F_i^{(0)}(s_1, s_2, \rho, \tau) \sim F_o^{(0)}(\mathbf{R}_N^{(0)}, t) + o(1), \tag{70}$$

$$F_i^{(1)}(s_1, s_2, \rho, \tau) \sim \rho\frac{\partial F_o^{(0)}}{\partial n}(\mathbf{R}_N^{(0)}, t) + F_o^{(1)}(\mathbf{R}_N^{(0)}, t) + (\mathbf{R}^{(1)}\cdot\nabla F_o^{(0)})(\mathbf{R}_N^{(0)}, t) + o(1), \tag{71}$$

where F represents any of the functions under consideration. These conditions are based on the matching principle [24]

$$(m \text{ term inner})(n \text{ term outer}) = (n \text{ term outer})\ (m \text{ term inner}),$$

and they are derived in [4].

Substituting the expansions (66)–(69) into Eqs. (58)–(65) and equating powers of λ we find at leading order

$$f_i^{(0)}((Q_{i,1}^{(0)})^2 + (Q_{i,2}^{(0)})^2 + (Q_{i,3}^{(0)})^2) = 0. \tag{72}$$

Since $f_i^{(0)}$ cannot be identically zero if it is to match with the outer solution we have

$$\mathbf{Q}_i^{(0)} = \mathbf{0}. \tag{73}$$

Equating powers of λ at the next order in (58)–(65) we have

$$\frac{1}{\kappa^2}\frac{\partial^2 f_i^{(0)}}{\partial \rho^2} = (f_i^{(0)})^3 - f_i^{(0)} + f_i^{(0)}\{(Q_{i,1}^{(1)})^2 + (Q_{i,2}^{(1)})^2 + (Q_{i,3}^{(1)})^2\}, \tag{74}$$

$$0 = \alpha(f_i^{(0)})^2\Phi_i^{(0)} + \frac{\partial((f_i^{(0)})^2 Q_{i,3}^{(1)})}{\partial \rho} \tag{75}$$

$$\frac{\partial H_{i,2}^{(0)}}{\partial \rho} = (f_i^{(0)})^2 Q_{i,1}^{(1)}, \tag{76}$$

$$-\frac{\partial H_{i,1}^{(0)}}{\partial \rho} = (f_i^{(0)})^2 Q_{i,2}^{(1)}, \tag{77}$$

$$0 = (f_i^{(0)})^2 Q_{i,3}^{(1)} + \frac{\partial \Phi_i^{(0)}}{\partial \rho}, \tag{78}$$

$$H_{i,1}^{(0)} = -\frac{\partial Q_{i,2}^{(1)}}{\partial \rho}, \tag{79}$$

$$H_{i,2}^{(0)} = \frac{\partial Q_{i,1}^{(1)}}{\partial \rho}, \tag{80}$$

$$H_{i,3}^{(0)} = 0. \tag{81}$$

The matching condition (70) together with the outer expansions implies the following boundary conditions on the inner variables:

$$f_i^{(0)} \to 1, \ \mathbf{Q}_i^{(1)} \to 0, \ \mathbf{H}_i^{(0)} \to 0, \ \Phi_i^{(0)} \to 0 \quad \text{as } \rho \to -\infty, \tag{82}$$

$$f_i^{(0)} \to 0 \quad \text{as } \rho \to \infty, \tag{83}$$

$$f_i^{(0)}(s_1, s_2, 0, \tau) = \eta. \tag{84}$$

We see that the choice of η simply fixes the translate of the leading-order inner solution by specifying $f_i^{(0)}(s_1, s_2, 0, \tau)$. Our aim is to determine the values of $\mathbf{H}_i$, $\mathbf{Q}_i$, and Φ_i as $\rho \to \infty$. Using the matching condition (70) again we have

$$H_{o,3}^{(0)}(\mathbf{R}_N^{(0)}, t) = \lim_{\rho\to\infty} H_{i,3}^{(0)}.$$

Hence we see immediately by (81) that we have

$$H_{o,3}^{(0)}(\mathbf{R}_N^{(0)}, t) = 0. \tag{85}$$

By (75) and (78) we have

$$\frac{\partial^2 \Phi_i^{(0)}}{\partial \rho^2} = \alpha(f_i^{(0)})^2\Phi_i^{(0)}. \tag{86}$$

Equation (78) implies $\partial\Phi_i^{(0)}/\partial\rho \to 0$ as $\rho \to \infty$. We have $\Phi_i^{(0)} \to 0$ as $\rho \to -\infty$. Since Eq. (86) implies that $\Phi_i^{(0)}$ is convex we therefore have

$$\Phi_i^{(0)} \equiv 0.$$

Now by Eq. (78)

$$Q_{i,3}^{(1)} \equiv 0.$$

We multiply (77) by $-H_{i,1}^{(0)}$, (76) by $H_{i,2}^{(0)}$, (74) by $\partial f_i^{(0)}/\partial\rho$, add and integrate to give

$$\frac{1}{\kappa^2}\left(\frac{\partial f_i^{(0)}}{\partial\rho}\right) + (H_{i,1}^{(0)})^2 + (H_{i,2}^{(0)})^2 = \frac{((f_i^{(0)})^2-1)^2}{2} + (f_i^{(0)})^2\{(Q_{i,1}^{(1)})^2 + (Q_{i,2}^{(1)})^2\},$$

where we have used the fact that $f_i^{(0)} \to 1$, $Q_{i,1}^{(1)} \to 0$, $Q_{i,2}^{(1)} \to 0$ as $\rho \to -\infty$. Letting ρ tend to infinity we have

$$\lim_{\rho\to\infty}\{(H_{i,1}^{(0)})^2 + (H_{i,2}^{(0)})^2\}^{1/2} = \lim_{\rho\to\infty}|\mathbf{H}_i^{(0)}| = 1/\sqrt{2},$$

since $f_i^{(0)} \to 0$, $(f_i^{(0)})^2\{(Q_{i,1}^{(1)})^2+(Q_{i,2}^{(1)})^2\} \to 0$ as $\rho \to \infty$. Using matching condition (70) again we have

$$\{(H_{o,1}^{(0)})^2 + (H_{o,2}^{(0)})^2\}^{1/2}(\mathbf{R}_N^{(0)}, t) = |\mathbf{H}_o^{(0)}|(\mathbf{R}_N^{(0)}, t) = 1/\sqrt{2} = H_c. \tag{87}$$

We note that the solution to the leading-order equations is not determined uniquely by the boundary conditions (82)–(84) since we have for $f_i^{(0)}$, $Q_{i,1}^{(1)}$, $Q_{i,2}^{(1)}$, three second-order equations with only five boundary conditions. However, we see that

$$\frac{\partial}{\partial\rho}\left[\frac{\partial Q_{i,1}^{(1)}}{\partial\rho}Q_{i,2}^{(1)} - \frac{\partial Q_{i,2}^{(1)}}{\partial\rho}Q_{i,1}^{(1)}\right] = \frac{\partial^2 Q_{i,1}^{(1)}}{\partial\rho^2}Q_{i,2}^{(1)} - \frac{\partial^2 Q_{i,2}^{(1)}}{\partial\rho}Q_{i,1}^{(1)} = 0,$$

by (76) and (77). Therefore

$$\frac{\partial Q_{i,1}^{(1)}}{\partial\rho}Q_{i,2}^{(1)} - \frac{\partial Q_{i,2}^{(1)}}{\partial\rho}Q_{i,1}^{(1)} = \text{const.} = 0,$$

by the boundary conditions as $\rho \to -\infty$. Hence

$$Q_{i,1}^{(1)} = C(s_1, s_2, \tau)Q_{i,2}^{(1)}, \tag{88}$$

where C is an unknown function of s_1, s_2, τ which will be determined by the outer solution in the normal region. Now if we let

$$Q_i^{(1)} = |\mathbf{Q}_i^{(1)}| = \sqrt{1+C^2}\,Q_{i,2}^{(1)},$$

we have

$$\frac{1}{\kappa^2}\frac{\partial^2 f_i^{(0)}}{\partial\rho^2} = (f_i^{(0)})^3 - f_i^{(0)} + f_i^{(0)}(Q_i^{(1)})^2, \tag{89}$$

$$\frac{\partial^2 Q_i^{(1)}}{\partial\rho^2} = (f_i^{(0)})^2 Q_i^{(0)}. \tag{90}$$

Therefore $f_i^{(0)}$ and $Q_i^{(1)}$ satisfy Eqs. (13), (14) with boundary conditions (16), (17). Hence there is a unique solution for $f_i^{(0)}$ and $Q_i^{(1)}$. Moreover, we have that $f_i^{(0)}$ decays exponentially as $\rho \to \infty$. Since no term can grow exponentially if it is to match with the outer region we conclude that any term involving $f_i^{(0)}$ as a numerator will tend to zero as $\rho \to \infty$. We also have

$$Q_i^{(1)} \sim \frac{\rho}{\sqrt{2}} + c + O(e^{-K\rho^2}) \quad \text{as } \rho \to \infty,$$

where c and K are constant. Here we make our choice of η, which we take to be such that $c = 0$. Thus we see that it is not f but $|\mathbf{Q}|$ that gives the natural choice for η. The simplicity that this choice of η induces will become apparent when we consider the interface conditions at first order. Since $Q_{i,1}^{(1)}$, $Q_{i,2}^{(1)}$ are multiples of $Q_i^{(1)}$ we therefore have

$$Q_{i,1}^{(1)} \sim \rho H_{o,2}^{(0)}(\mathbf{R}_N^{(0)}, t) + O(e^{-K\rho^2}) \quad \text{as } \rho \to \infty, \tag{91}$$

$$Q_{i,2}^{(1)} \sim -\rho H_{o,1}^{(0)}(\mathbf{R}_N^{(0)}, t) + O(e^{-K\rho^2}) \quad \text{as } \rho \to \infty, \tag{92}$$

where we have made use of the matching condition (71).

Equating powers of λ at the next order in Eqs. (58), (60), (61), (63), and (64) we find

$$\begin{aligned}\frac{\alpha v_n^{(0)} + \tilde{\kappa}_1^{(0)} + \tilde{\kappa}_2^{(0)}}{\kappa^2}\frac{\partial f_i^{(0)}}{\partial \rho} + \frac{1}{\kappa^2}\frac{\partial^2 f_i^{(1)}}{\partial \rho^2}& \\ = 3(f_i^{(0)})^2 f_i^{(1)} - f_i^{(1)} + f_i^{(1)}((Q_{i,1}^{(1)})^2 + (Q_{i,2}^{(1)})^2) &+ 2f_i^{(0)}(Q_{i,1}^{(1)}Q_{i,1}^{(2)} + Q_{i,2}^{(1)}Q_{i,2}^{(2)}),\end{aligned} \tag{93}$$

$$\frac{\partial H_{i,2}^{(1)}}{\partial \rho} + \tilde{\kappa}_2^{(0)} H_{i,2}^{(0)} = -v_n^{(0)}\frac{\partial Q_{i,1}^{(1)}}{\partial \rho} + 2f_i^{(0)} f_i^{(1)} Q_{i,1}^{(1)} + (f_i^{(0)})^2 Q_{i,1}^{(2)}, \tag{94}$$

$$-\frac{\partial H_{i,1}^{(1)}}{\partial \rho} - \tilde{\kappa}_1^{(0)} H_{i,1}^{(0)} = -v_n^{(0)}\frac{\partial Q_{i,2}^{(1)}}{\partial \rho} + 2f_i^{(0)} f_i^{(1)} Q_{i,2}^{(1)} + (f_i^{(0)})^2 Q_{i,2}^{(2)}, \tag{95}$$

$$H_{i,1}^{(1)} = -\frac{\partial Q_{i,2}^{(2)}}{\partial \rho} - \tilde{\kappa}_2^{(0)} Q_{i,2}^{(1)}, \tag{96}$$

$$H_{i,2}^{(1)} = \frac{\partial Q_{i,1}^{(2)}}{\partial \rho} + \tilde{\kappa}_1^{(0)} Q_{i,1}^{(1)}. \tag{97}$$

Letting $\rho \to \infty$ in Eqs. (94) and (95) we have

$$\lim_{\rho\to\infty}\left[\frac{\partial H_{i,1}^{(1)}}{\partial \rho} + \tilde{\kappa}_1^{(0)} H_{i,1}^{(0)}\right] = -v_n^{(0)} \lim_{\rho\to\infty} H_{i,1}^{(0)},$$

$$\lim_{\rho\to\infty}\left[\frac{\partial H_{i,2}^{(1)}}{\partial \rho} + \tilde{\kappa}_2^{(0)} H_{i,2}^{(0)}\right] = -v_n^{(0)} \lim_{\rho\to\infty} H_{i,2}^{(0)},$$

since the terms involving $f_i^{(0)}$ tend to zero. Using the matching conditions (70) and (71) we have

$$\frac{\partial H_{o,1}^{(0)}}{\partial n} + \tilde{\kappa}_1^{(0)} H_{o,1}^{(0)} = -v_n^{(0)} H_{o,1}^{(0)} \quad \text{on } \Gamma_N^{(0)}, \tag{98}$$

$$\frac{\partial H_{o,2}^{(0)}}{\partial n} + \tilde{\kappa}_2^{(0)} H_{o,2}^{(0)} = -v_n^{(0)} H_{o,2}^{(0)} \quad \text{on } \Gamma_N^{(0)}. \tag{99}$$

At this stage we have derived all interface conditions to lowest order. If we were interested in a complete determination of the solution we could now solve the outer problem (44) with the boundary conditions (85), (87), (98), (99) to determine $\mathbf{H}_o^{(0)}$ and $\Gamma^{(0)}$. We show in Lemma 3 that the conditions (85), (98), and (99) are equivalent to the condition (4) when $v_n \neq 0$. The proposition is then proved.

We continue with the inner problem in order to find the first-order correction to the magnitude of the magnetic field on the free boundary.

Multiplying (94) by $H_{i,2}^{(0)} = \partial Q_{i,1}^{(1)}/\partial\rho$, (95) by $-H_{i,1}^{(0)} = \partial Q_{i,2}^{(1)}/\partial\rho$, (93) by $\partial f_i^{(0)}/\partial\rho$, (74) by $\partial f_i^{(1)}/\partial\rho$, (76) by $H_{i,2}^{(1)} = \partial Q_{i,1}^{(2)}/\partial\rho + \tilde{\kappa}_1^{(0)} Q_{i,1}^{(1)}$, (77) by $-H_{i,1}^{(1)} = \partial Q_{i,2}^{(2)}/\partial\rho + \tilde{\kappa}_2^{(0)} Q_{i,2}^{(1)}$, adding and integrating gives

$$\begin{aligned}
&\frac{1}{\kappa^2}\frac{\partial f_i^{(0)}}{\partial\rho}\frac{\partial f_i^{(1)}}{\partial\rho} + H_{i,1}^{(0)}H_{i,1}^{(1)} + H_{i,2}^{(0)}H_{i,2}^{(1)} + \frac{(\alpha v_n^{(0)} + \tilde{\kappa}_1^{(0)} + \tilde{\kappa}_2^{(0)})}{\kappa^2}\int_{-\infty}^{\rho}\left(\frac{\partial f_i^{(0)}}{\partial\rho}\right)^2 d\rho \\
&\quad + \tilde{\kappa}_1^{(0)}\int_{-\infty}^{\rho}\left(\frac{\partial Q_{i,2}^{(1)}}{\partial\rho}\right)^2 d\rho + \tilde{\kappa}_2^{(0)}\int_{-\infty}^{\rho}\left(\frac{\partial Q_{i,1}^{(1)}}{\partial\rho}\right)^2 d\rho \\
&= (f_i^{(0)})^3 f_i^{(1)} - f_i^{(0)} f_i^{(1)} + (f_i^{(0)})^2 (Q_{i,1}^{(1)}Q_{i,1}^{(2)} + Q_{i,2}^{(1)}Q_{i,2}^{(2)}) \\
&\quad + f_i^{(0)} f_i^{(1)}\{(Q_{i,1}^{(1)})^2 + (Q_{i,2}^{(1)})^2\} + \tilde{\kappa}_1^{(0)}\int_{-\infty}^{\rho}(f_i^{(0)}Q_{i,1}^{(1)})^2\, d\rho \\
&\quad + \tilde{\kappa}_2^{(0)}\int_{-\infty}^{\rho}(f_i^{(0)}Q_{i,2}^{(1)})^2\, d\rho - v_n^{(0)}\int_{-\infty}^{\rho}\left\{\left(\frac{\partial Q_{i,1}^{(1)}}{\partial\rho}\right)^2 + \left(\frac{\partial Q_{i,2}^{(1)}}{\partial\rho}\right)^2\right\} d\rho.
\end{aligned}$$

Letting $\rho \to \infty$ in this equation is equivalent to a solvability condition for the first-order terms.

Now

$$\begin{aligned}
\int_{-\infty}^{\rho}\left(\frac{\partial Q_{i,k}^{(1)}}{\partial\rho}\right)^2 d\rho &= Q_{i,k}^{(1)}\frac{\partial Q_{i,k}^{(1)}}{\partial\rho} - \int_{-\infty}^{\rho} Q_{i,k}^{(1)}\frac{\partial^2 Q_{i,k}^{(1)}}{\partial\rho^2}\, d\rho, \\
&= Q_{i,k}^{(1)}\frac{\partial Q_{i,k}^{(1)}}{\partial\rho} - \int_{-\infty}^{\rho}(f_i^{(0)}Q_{i,k}^{(1)})^2\, d\rho, \quad k = 1, 2
\end{aligned}$$

on integration by parts, and using (76), (77). Hence

$$
\begin{aligned}
&\frac{1}{\kappa^2}\frac{\partial f_i^{(0)}}{\partial\rho}\frac{\partial f_i^{(1)}}{\partial\rho}+H_{i,1}^{(0)}H_{i,1}^{(1)}+H_{i,2}^{(0)}H_{i,2}^{(1)}+\frac{\alpha v_n^{(0)}+\tilde\kappa_1^{(0)}+\tilde\kappa_2^{(0)}}{\kappa^2}\int_{-\infty}^{\rho}\left(\frac{\partial f_i^{(0)}}{\partial\rho}\right)^2 d\rho \\
&\qquad+(v_n^{(0)}+\tilde\kappa_1^{(0)})Q_{i,2}^{(1)}\frac{\partial Q_{i,2}^{(1)}}{\partial\rho}+(v_n^{(0)}+\tilde\kappa_2^{(0)})Q_{i,1}^{(1)}\frac{\partial Q_{i,1}^{(1)}}{\partial\rho} \\
&\quad=(f_i^{(0)})^3f_i^{(1)}-f_i^{(0)}f_i^{(1)}+(f_i^{(0)})^2(Q_{i,1}^{(1)}Q_{i,1}^{(2)}+Q_{i,2}^{(1)}Q_{i,2}^{(2)}) \\
&\qquad+f_i^{(0)}f_i^{(1)}\{(Q_{i,1}^{(1)})^2+(Q_{i,2}^{(1)})^2\} \\
&\qquad+(v_n^{(0)}+\tilde\kappa_1^{(0)}+\tilde\kappa_2^{(0)})\int_{-\infty}^{\rho}(f_i^{(0)})^2\{(Q_{i,1}^{(1)})^2+(Q_{i,2}^{(1)})^2\}\,d\rho.
\end{aligned}
\tag{100}
$$

It is at this point that the reason for our choice of η becomes apparent. We have

$$
\begin{aligned}
(v_n^{(0)}+\tilde\kappa_2^{(0)})Q_{i,1}^{(1)}\frac{\partial Q_{i,1}^{(1)}}{\partial\rho}&\sim(v_n^{(0)}+\tilde\kappa_2^{(0)})\rho(H_{o,2}^{(0)}(\mathbf{R}_N^{(0)},t))^2+o(1)\quad\text{by (91)}\\
&=-\rho H_{o,2}^{(0)}(\mathbf{R}_N^{(0)},t)\frac{\partial H_{o,2}^{(0)}}{\partial n}(\mathbf{R}_N^{(0)},t)+o(1)\quad\text{by (98).}
\end{aligned}
$$

Similarly,

$$(v_n^{(0)}+\tilde\kappa_1^{(0)})Q_{i,2}^{(1)}\frac{\partial Q_{i,2}^{(1)}}{\partial\rho}\sim-\rho H_{o,1}^{(0)}(\mathbf{R}_N^{(0)},t)\frac{\partial H_{o,1}^{(0)}}{\partial n}(\mathbf{R}_N^{(0)},t)+o(1).$$

Hence, letting $\rho\to\infty$ in (100) we have

$$
\begin{aligned}
&H_{o,1}^{(0)}(\mathbf{R}_N^{(0)},t)\lim_{\rho\to\infty}\left\{H_{i,1}^{(1)}-\rho\frac{\partial H_{o,1}^{(0)}}{\partial n}(\mathbf{R}_N^{(0)},t)\right\}\\
&\qquad+H_{o,2}^{(0)}(\mathbf{R}_N^{(0)},t)\lim_{\rho\to\infty}\left\{H_{i,2}^{(1)}-\rho\frac{\partial H_{o,2}^{(0)}}{\partial n}(\mathbf{R}_N^{(0)},t)\right\}\\
&\qquad\qquad=-v_n^{(0)}(\alpha\delta-\gamma)-(\tilde\kappa_1^{(0)}+\tilde\kappa_2^{(0)})(\delta-\gamma),
\end{aligned}
$$

where

$$\delta=\frac{1}{\kappa^2}\int_{-\infty}^{\infty}\left(\frac{\partial f_i^{(0)}}{\partial\rho}\right)^2 d\rho,\qquad \gamma=\int_{-\infty}^{\infty}(f_i^{(0)})^2\{(Q_{i,1}^{(1)})^2+(Q_{i,2}^{(1)})^2\}\,d\rho.$$

Using the matching condition (71) we have

$$
\begin{aligned}
&H_{o,1}^{(0)}(\mathbf{R}_N^{(0)},t)\{H_{0,1}^{(1)}(\mathbf{R}_N^{(0)},t)+(\mathbf{R}^{(1)}\cdot\nabla H_{o,1}^{(0)})(\mathbf{R}_N^{(0)},t)\}\\
&\qquad+H_{o,2}^{(0)}(\mathbf{R}_N^{(0)},t)\{H_{o,2}^{(1)}(\mathbf{R}_N^{(0)},t)+(\mathbf{R}^{(1)}\cdot\nabla H_{o,2}^{(0)})(\mathbf{R}_N^{(0)},t)\}\\
&\qquad\qquad=-v_n^{(0)}(\alpha\delta-\gamma)-(\tilde\kappa_1^{(0)}+\tilde\kappa_2^{(0)})(\delta-\gamma).
\end{aligned}
\tag{101}
$$

Note that σ, the surface energy defined in Sec. 2, is given by $\sigma=4\lambda(\delta-\gamma)$.

We calculate the value of $|\mathbf{H}_o|$ on the boundary Γ_N. We have

$$\begin{aligned}
|\mathbf{H}_o(\mathbf{R}_N, t)|^2 &= (H_{o,1}(\mathbf{R}_N, t))^2 + (H_{o,2}(\mathbf{R}_N, t))^2 + (H_{o,3}(\mathbf{R}_N, t))^2 \\
&= (H^{(0)}_{o,1}(\mathbf{R}^{(0)} + \lambda\mathbf{R}^{(1)} + \cdots, t) + \lambda H^{(1)}_{o,1}(\mathbf{R}^{(0)} + \lambda\mathbf{R}^{(1)} + \cdots, t))^2 \\
&\quad + (H^{(0)}_{o,2}(\mathbf{R}^{(0)} + \lambda\mathbf{R}^{(1)} + \cdots, t) + \lambda H^{(1)}_{o,2}(\mathbf{R}^{(0)} + \lambda\mathbf{R}^{(1)} + \cdots, t))^2 + O(\lambda^2) \\
&= (H^{(0)}_{o,1}(\mathbf{R}^{(0)}_N, t))^2 + (H^{(0)}_{o,2}(\mathbf{R}^{(0)}_N, t))^2 \\
&\quad + 2\lambda H^{(0)}_{o,1}(\mathbf{R}^{(0)}_N, t)[H^{(1)}_{o,1}(\mathbf{R}^{(0)}_N, t) + (\mathbf{R}^{(1)} \cdot \nabla H^{(0)}_{o,1})(\mathbf{R}^{(0)}_N, t)] \\
&\quad + 2\lambda H^{(0)}_{o,1}(\mathbf{R}^{(0)}_N, t)[H^{(1)}_{o,1}(\mathbf{R}^{(0)}_N, t) + (\mathbf{R}^{(1)} \cdot \nabla H^{(0)}_{o,1})(\mathbf{R}^{(0)}_N, t)] + \cdots \\
&= \tfrac{1}{2} - 2\lambda\{v^{(0)}_n(\alpha\delta - \gamma) + (\tilde{\kappa}^{(0)}_1 + \tilde{\kappa}^{(0)}_2)(\delta - \gamma)\} + O(\lambda^2)
\end{aligned}$$

by (87) and (101). Hence

$$\begin{aligned}
|\mathbf{H}_o(\mathbf{R}_N, t)| &= \frac{1}{\sqrt{2}} - \sqrt{2}\lambda\{v^{(0)}_n(\alpha\delta - \gamma) + (\tilde{\kappa}^{(0)}_1 + \tilde{\kappa}^{(0)}_2)(\delta - \gamma)\} + O(\lambda^2) \\
&= H_c - 2H_c\lambda\{v^{(0)}_n(\alpha\delta - \gamma) + (\tilde{\kappa}^{(0)}_1 + \tilde{\kappa}^{(0)}_2)(\delta - \gamma)\} + O(\lambda^2) \qquad (102) \\
&= H_c - 2H_c\lambda v^{(0)}_n(\alpha\delta - \gamma) - \frac{H_c}{2}(\tilde{\kappa}^{(0)}_1 + \tilde{\kappa}^{(0)}_2)\sigma + O(\lambda^2),
\end{aligned}$$

where $H_c = 1/\sqrt{2}$, and σ is the surface energy.

We see from (102) that, as in the phase field model, we have the emergence of "surface tension" and "kinetic undercooling" terms in the magnitude of the magnetic field at the interface. For type I superconductors, where $\sigma > 0$, the "surface tension" term will have a stabilising effect as in the phase field model, and the Ginzburg-Landau model will be a "regularization" of the free boundary model. For type II superconductors, where $\sigma < 0$, this term will be destabilising. Hence for type II superconductors, even in situations where the free boundary model is well-posed, it might not accurately represent the solution of the Ginzburg-Landau model. A common observation in type II superconductors is that the normal regions break up into "cores" (or "vortices"), each of size comparable to the coherence length [5, 14]. In such a situation the assumption of normal and superconducting regions large compared to the penetration depth , crucial to the analysis of Sec. 3, is obviously not valid.

The rôle of the "kinetic undercooling" term depends also on the value of α. For $\alpha\delta - \gamma > 0$ it will be stabilising; for $\alpha\delta - \gamma < 0$ it will be destabilising. Thus we expect that α will also be an important parameter in determining the form of the solution to the Ginzburg-Landau equations.

We should not take the analogy with the phase field model too far, since in the phase field model a scaling of the parameters can be found in which these terms appear at leading order, whereas the "surface tension" and "kinetic undercooling" terms above appear only at first order, and so will not appreciably affect the solution unless the normal velocity or mean curvature of the boundary are of order $1/\lambda$. Thus, although the Ginzburg-Landau model seems to be a regularisation of the vectorial Stefan model, because the surface energy is very small we expect very intricate morphologies even from solutions to the Ginzburg-Landau equations. Numerical simulations [13, 20] and experimental observations [12, 22, 23] seem to agree with this. There does not appear to be a scaling of the parameters in which the stabilising terms appear at leading order.

We complete this section by proving that the boundary conditions (85), (98), and (99) are indeed equivalent to the condition (4), when $v_n \neq 0$.

LEMMA. The boundary conditions

$$H_{o,3} = 0 \quad \text{on } \Gamma_N, \tag{103}$$

$$\frac{\partial H_{o,1}}{\partial n} + \tilde{\kappa}_1 H_{o,1} = -v_n H_{o,1} \quad \text{on } \Gamma_N, \tag{104}$$

$$\frac{\partial H_{o,2}}{\partial n} + \tilde{\kappa}_2 H_{o,2} = -v_n H_{o,2} \quad \text{on } \Gamma_N \tag{105}$$

can be written as the single boundary condition

$$(\operatorname{curl} \mathbf{H}_o) \wedge \mathbf{n} = -v_n \mathbf{H} \quad \text{on } \Gamma_N. \tag{106}$$

Proof. Near the boundary we transform coordinates to (s_1, s_2, n), where

$$\mathbf{r} = (x, y, z) = \mathbf{R}(s_1, s_2, \tau) = n\mathbf{n}(s_1, s_2, \tau).$$

As before (s_1, s_2, n) form a local orthogonal curvilinear coordinate system, with scaling factors

$$\begin{aligned} h_1 &= E^{1/2}[1 + n\tilde{\kappa}_1], \\ h_2 &= G^{1/2}[1 + n\tilde{\kappa}_2], \\ h_3 &= 1. \end{aligned}$$

In these coordinates $\operatorname{curl} \mathbf{H}$ is given by

$$\begin{aligned} \operatorname{curl} \mathbf{H} = {} & \frac{1}{G^{1/2}(1 + n\tilde{\kappa}_2)} \left[\frac{\partial H_3}{\partial s_2} - \frac{\partial}{\partial n}(G^{1/2}(1 + n\tilde{\kappa}_2)H_2) \right] \mathbf{e}_1 \\ & - \frac{1}{E^{1/2}(1 + n\tilde{\kappa}_1)} \left[\frac{\partial H_3}{\partial s_1} - \frac{\partial}{\partial n}(E^{1/2}(1 + n\tilde{\kappa}_1)H_1) \right] \mathbf{e}_2 \\ & - \frac{1}{(EG)^{1/2}(1 + n\tilde{\kappa}_1)(1 + n\tilde{\kappa}_2)} \left[\frac{\partial}{\partial s_1}(G^{1/2}(1 + n\tilde{\kappa}_2)H_2) \right. \\ & \left. - \frac{\partial}{\partial s_2}(E^{1/2}(1 + n\tilde{\kappa}_1)H_1) \right] \mathbf{n}. \end{aligned}$$

Hence

$$\begin{aligned} (\operatorname{curl} \mathbf{H}) \wedge \mathbf{n} = {} & - \frac{1}{E^{1/2}(1 + n\tilde{\kappa}_1)} \left[\frac{\partial H_3}{\partial s_1} - \frac{\partial}{\partial n}(E^{1/2}(1 + n\tilde{\kappa}_1)H_1) \right] \mathbf{e}_1 \\ & - \frac{1}{G^{1/2}(1 + n\tilde{\kappa}_2)} \left[\frac{\partial H_3}{\partial s_2} - \frac{\partial}{\partial n}(G^{1/2}(1 + n\tilde{\kappa}_2)H_2) \right] \mathbf{e}_2. \end{aligned}$$

On the interface $n = 0$ we have $H_3 = 0$; hence $\partial H_3/\partial s_1 = 0$, $\partial H_3/\partial s_2 = 0$. Thus

$$(\mathrm{curl}\,\mathbf{H}) \wedge \mathbf{n} = \left(\frac{\partial H_1}{\partial n} + \tilde{\kappa}_1 H_1\right)\mathbf{e}_1 + \left(\frac{\partial H_2}{\partial n} + \tilde{\kappa}_2 H_2\right)\mathbf{e}_2 \quad \text{on } \Gamma_N.$$

The boundary conditions (103)–(105) can then be seen to be equivalent to (106) for $v_n \neq 0$.

4. Thermal effects.

4.1. *The vectorial Stefan model.* We can include thermal effects in the model (1)–(4) by allowing H_c in (3) to depend on the temperature T, and appending equations for T on either side of the free boundary, together with Stefan-type conditions on the free boundary itself. This has been done in one space dimension in [11], and in general in [7]. Heat is generated via Ohmic heating in the normal region. The dimensionless temperature T from T_c scaled with T_c satisfies

$$\nabla^2 T = \beta\frac{\partial T}{\partial t} - \gamma|\mathrm{curl}\,\mathbf{H}|^2 \quad \text{in } \Omega_N, \tag{107}$$

$$\nabla^2 T = \beta\frac{\partial T}{\partial t} \quad \text{in } \Omega_S, \tag{108}$$

$$[T]_S^N = 0, \tag{109}$$

$$\left[\frac{\partial T}{\partial n}\right]_S^N = -\hat{L}(T)v_n, \tag{110}$$

where

$$\beta = \frac{\rho c}{\mu\zeta k}, \qquad \gamma = \frac{H_e^2}{\zeta k T_c} \tag{111}$$

are dimensionless parameters measuring the ratios of thermal to electromagnetic time scales and Ohmic heating to thermal conduction, respectively, and $kT_c\mu_s\zeta\hat{L}(T)$ is the latent heat. Here ρ is the density, c the specific heat, k the thermal conductivity, μ the permeability, and ζ the electrical conductivity of the normal region; all are assumed constant. Finally, H_e is the value of the magnetic field used to nondimensionalize $\mathbf{H}$ which will be given explicitly below.

4.2. *Anisothermal Ginzburg-Landau equations.* When the temperature is allowed to vary we must employ a different non-dimensionalization of the Ginzburg-Landau equations. The scalings used are given in [7] and result in the equations

$$-\alpha\xi^2\frac{\partial f}{\partial t} + \xi^2\nabla^2 f = Tf + f^3 + \frac{f|\mathbf{Q}|^2}{\lambda^2} \quad \text{in } \Omega, \tag{112}$$

$$\alpha f^2\Phi + \mathrm{div}(f^2\mathbf{Q}) = 0, \tag{113}$$

$$-\lambda^2(\mathrm{curl})^2\mathbf{Q} = \lambda^2\left(\frac{\partial \mathbf{Q}}{\partial t} + \nabla\Phi\right) + f^2\mathbf{Q}. \tag{114}$$

We have assumed that the temperature is close to the critical temperature, so that the equations have been linearised in T. The analysis that follows is equally applicable to the more general case in which $Tf + f^3$ is replaced by $a(T)f + b(T)f^3$.

Linearising the dimensional critical magnetic field, $\widetilde{H}_c$, in T for temperatures close to T_c yields $\widetilde{H}_c = -hT$, say. In the above nondimensionalization $\mathbf{H}$ has been scaled with $\sqrt{2}h$, so that $H_e = \sqrt{2}h$ in (111) and the dimensionless critical field is given by $H_c = -T/\sqrt{2}$.

We must also now have an equation to determine T in the form of a heat balance equation as in the phase field model. Thermodynamic arguments imply that there is a release of latent heat on the transition from normally conducting to superconducting in the presence of a magnetic field. Following the phase field model, we take the rate of release of latent heat to be proportional to the rate of change of the number density of superconducting electrons. We must also include a term in the heat balance equation to account for the Ohmic heating due to the normal current. Thus the equation we require is

$$\nabla^2 T = \beta \frac{\partial T}{\partial t} - L \frac{\partial (f^2)}{\partial t} - \gamma |\mathbf{j}_N|^2 \tag{115}$$

$$= \beta \frac{\partial T}{\partial t} - L \frac{\partial (f^2)}{\partial t} - \gamma \left| \frac{\partial \mathbf{Q}}{\partial t} + \nabla \Phi \right|^2, \tag{116}$$

where β and γ are given by (111).

We note that in deriving (112)–(114), [16] assumed that Joule losses were small. It is not clear whether the equations would have the same form when we take Joule losses into account via (116), but we assume that this is the case. In particular, we are assuming that the relaxation of the coefficients a and b occurs on a much shorter time scale than that of the diffusion of temperature or magnetic field, and can therefore be taken to be instantaneous.

4.3. *Asymptotic solution of the Ginzburg-Landau equations under anisothermal conditions.* We now try to relate the Ginzburg-Landau model (112)–(116) to the free-boundary model of Sec. 4.1, allowing the temperature to vary in time and space. As in the previous section a complete determination of the solution will involve initial and fixed boundary conditions. However, they will be left unspecified since our primary interest is rather the free-boundary conditions. Much of the analysis is very similar to the isothermal case, so we merely sketch it here.

As before, we define $\Gamma(t)$ by

$$\Gamma(t) = \{\mathbf{r} \text{ such that } f(\mathbf{r}, t) = \eta\}, \tag{117}$$

where $0 < \eta < 1$.

Outer expansions. Away from the transition region we formally expand all functions in powers of λ as before. Substituting the expansions into (112)–(116) and equating powers of λ we find at leading order that

$$f_o^{(0)} \equiv 0, \tag{118}$$

$$-(\mathrm{curl})^2 \mathbf{H}_o^{(0)} = \nabla^2 \mathbf{H}_o^{(0)} = \frac{\partial \mathbf{H}_o^{(0)}}{\partial t}, \tag{119}$$

$$\nabla^2 T_o^{(0)} = \beta \frac{\partial T_o^{(0)}}{\partial t} - \gamma |\mathrm{curl}\ \mathbf{H}_o^{(0)}|^2 \tag{120}$$

in the normal region, and

$$\mathbf{Q}_o^{(0)} \equiv \mathbf{0}, \tag{121}$$

$$\mathbf{H}_o^{(0)} \equiv \mathbf{0}, \tag{122}$$

$$\Phi_o^{(0)} \equiv 0, \tag{123}$$

$$(f_o^{(0)})^2 = -T_o^{(0)}, \tag{124}$$

$$\nabla^2 T_o^{(0)} = (\beta + L)\frac{\partial T_o^{(0)}}{\partial t} \tag{125}$$

in the superconducting region.

Inner expansions. We note firstly that since Eqs. (113), (114) are identical to Eqs. (27)–(29) the boundary condition

$$\operatorname{curl} \mathbf{H}_o^{(0)} \wedge \mathbf{n}^{(0)} = -v_n^{(0)} \mathbf{H}_o^{(0)} \quad \text{on } \Gamma_N^{(0)} \tag{126}$$

will hold exactly as in the previous section.

We define the inner variables by

$$\mathbf{r} = \mathbf{R}(s_1, s_2) + \lambda\rho\mathbf{n},$$
$$t = \tau,$$

where the interface $\Gamma(t)$ is given by the surface

$$\mathbf{r} = (x, y, z) = \mathbf{R}(s_1(x, y, z), s_2(x, y, z), t),$$

and we write

$$\mathbf{Q}_i = Q_{i,1}\mathbf{e}_1 + Q_{i,2}\mathbf{e}_2 + Q_{i,3}\mathbf{n}, \tag{127}$$

$$\mathbf{H}_i = H_{i,1}\mathbf{e}_1 + H_{i,2}\mathbf{e}_2 + H_{i,3}\mathbf{n} \tag{128}$$

as before. Equations (113), (114) transform to inner variables exactly as Eqs. (59)–(65). Equations (112) and (116) become

$$\frac{\alpha\lambda v_n}{\kappa^2}\frac{\partial f_i}{\partial \rho} + \frac{1}{\kappa^2}\frac{\partial^2 f_i}{\partial \rho^2} + \frac{\lambda(\tilde{\kappa}_1 + \tilde{\kappa}_2)}{\kappa^2}\frac{\partial f_i}{\partial \rho} + O(\lambda^2)$$
$$= T_i f_i + f_i^3 + \frac{f_i}{\lambda^2}(Q_{i,1}^2 + Q_{i,2}^2 + Q_{i,3}^2), \tag{129}$$

$$\begin{aligned}\frac{1}{\lambda^2}\frac{\partial^2 T_i}{\partial \rho^2} + \frac{\tilde{\kappa}_1 + \tilde{\kappa}_2}{\lambda}\frac{\partial T_i}{\partial \rho} = &-\frac{v_n\beta}{\lambda}\frac{\partial T_i}{\partial \rho} + \frac{v_n L}{\lambda}\frac{\partial (f_i^2)}{\partial \rho} \\ &- \left(-\frac{v_n}{\lambda}\frac{\partial Q_{i,1}}{\partial \rho} + \frac{\partial Q_{i,1}}{\partial \tau} - \frac{v_1}{E^{1/2}}\frac{\partial Q_{i,1}}{\partial s_1} - \frac{v_2}{G^{1/2}}\frac{\partial Q_{i,1}}{\partial s_2} + \frac{1}{E^{1/2}}\frac{\partial \Phi_i}{\partial s_1}\right)^2 \\ &- \left(-\frac{v_n}{\lambda}\frac{\partial Q_{i,2}}{\partial \rho} + \frac{\partial Q_{i,2}}{\partial \tau} - \frac{v_1}{E^{1/2}}\frac{\partial Q_{i,2}}{\partial s_1} - \frac{v_2}{G^{1/2}}\frac{\partial Q_{i,2}}{\partial s_2} + \frac{1}{G^{1/2}}\frac{\partial \Phi_i}{\partial s_2}\right)^2 \\ &- \left(-\frac{v_n}{\lambda}\frac{\partial Q_{i,3}}{\partial \rho} + \frac{\partial Q_{i,3}}{\partial \tau} - \frac{v_1}{E^{1/2}}\frac{\partial Q_{i,3}}{\partial s_1} - \frac{v_2}{G^{1/2}}\frac{\partial Q_{i,3}}{\partial s_2} + \frac{1}{\lambda}\frac{\partial \Phi_i}{\partial \rho}\right)^2 + O(1),\end{aligned} \tag{130}$$

where $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_n\mathbf{n}$. We formally expand all functions in the inner variables in powers of λ to obtain the inner expansions as before. Substituting the expansions into the equations and equating powers of λ we find again that at leading order

$$\mathbf{Q}_i^{(0)} \equiv \mathbf{0}. \tag{131}$$

Equation coefficients of λ at the next order we find

$$\begin{aligned}\frac{1}{\kappa^2}\frac{\partial^2 f_i^{(0)}}{\partial \rho^2} &= T_i^{(0)} f_i^{(0)} + (f_i^{(0)})^3 \\ &\quad + f_i^{(0)}\{(Q_{i,1}^{(1)})^2 + (Q_{i,2}^{(1)})^2 + (Q_{i,3}^{(1)})^2\},\end{aligned} \tag{132}$$

$$\frac{\partial((f_i^{(0)})^2 Q_{i,3}^{(1)})}{\partial \rho} = -\alpha (f_i^{(0)})^2 \Phi_i^{(0)}, \tag{133}$$

$$\frac{\partial H_{i,2}^{(0)}}{\partial \rho} = (f_i^{(0)})^2 Q_{i,1}^{(1)}, \tag{134}$$

$$-\frac{\partial H_{i,1}^{(0)}}{\partial \rho} = (f_i^{(0)})^2 Q_{i,2}^{(1)}, \tag{135}$$

$$0 = (f_i^{(0)})^2 Q_{i,3}^{(1)} + \frac{\partial \Phi_i^{(0)}}{\partial \rho}, \tag{136}$$

$$H_{i,1}^{(0)} = -\frac{\partial Q_{i,2}^{(1)}}{\partial \rho}, \tag{137}$$

$$H_{i,2}^{(0)} = \frac{\partial Q_{i,1}^{(1)}}{\partial \rho}, \tag{138}$$

$$H_{i,3}^{(0)} = 0, \tag{139}$$

$$\frac{\partial^2 T_i^{(0)}}{\partial \rho^2} = 0. \tag{140}$$

The outer expansions imply the boundary conditions

$$(f_i^{(0)})^2 \to -T_o^{(0)}(\mathbf{R}_S^{(0)}, t), \quad \mathbf{Q}_i^{(1)} \to 0, \quad \mathbf{H}_i^{(0)} \to 0, \quad \Phi_i^{(0)} \to 0 \quad \text{as } \rho \to -\infty, \tag{141}$$

$$f_i^{(0)} \to 0 \quad \text{as } \rho \to \infty, \tag{142}$$

$$f_i^{(0)}(s_1, s_2, 0, \tau) = \eta. \tag{143}$$

Our aim is to determine the values of $\mathbf{H}_i$, $\mathbf{Q}_i$, and Φ_i as $\rho \to \infty$. Exactly as in the isothermal case we find $\Phi_i^{(0)} \equiv 0$, $Q_{i,3}^{(1)} \equiv 0$.

Integrating (140) we have

$$T_i^{(0)} = A\rho + B.$$

Since $T_i^{(0)}$ must be bounded as $\rho \to \pm\infty$ if the temperature on the interface is bounded and we are to match with the outer region we must have $A = 0$. Then

$$T_i^{(0)} = B = T_o^{(0)}(\mathbf{R}_N^{(0)}, t) = T_o^{(0)}(\mathbf{R}_S^{(0)}, t). \tag{144}$$

We multiply (135) by $-H_{i,1}^{(0)}$, (134) by $H_{i,2}^{(0)}$, (132) by $\partial f_i^{(0)}/\partial\rho$, add and integrate to give

$$\frac{1}{\kappa^2}\left(\frac{\partial f_i^{(0)}}{\partial\rho}\right)+(H_{i,1}^{(0)})^2+(H_{i,2}^{(0)})^2=\frac{((f_i^{(0)})^2+B)^2}{2}+(f_i^{(0)})^2\{(Q_{i,1}^{(1)})^2+(Q_{i,2}^{(1)})^2\},$$

where we have used the fact that $(f_i^{(0)})^2\to -B$, $Q_{i,1}^{(1)}\to 0$, $Q_{i,2}^{(1)}\to 0$ as $\rho\to-\infty$. Letting ρ tend to infinity we have

$$\lim_{\rho\to\infty}\{(H_{i,1}^{(0)})^2+(H_{i,2}^{(0)})^2\}^{1/2}=\lim_{\rho\to\infty}|\mathbf{H}_i^{(0)}|=\frac{|B|}{\sqrt{2}}.$$

Using matching condition (70) we have

$$\begin{aligned}|\mathbf{H}_o^{(0)}|(\mathbf{R}_N^{(0)},t)&=\frac{|T_o^{(0)}(\mathbf{R}^{(0)},t))|}{\sqrt{2}}\\&=H_c(T_o^{(0)}(\mathbf{R}^{(0)},t)).\end{aligned}\tag{145}$$

Equating powers of λ at the next order in Eq. (130) yields

$$\frac{\partial^2 T_i^{(1)}}{\partial\rho^2}=Lv_n^{(0)}\frac{\partial(f_i^{(0)})^2}{\partial\rho}.\tag{146}$$

Integrating over $(-\infty,\infty)$ gives

$$\begin{aligned}\left[\frac{\partial T_i^{(1)}}{\partial\rho}\right]_{-\infty}^{\infty}&=Lv_n^{(0)}[(f_i^{(0)})^2]_{-\infty}^{\infty}\\&=Lv_n^{(0)}B.\end{aligned}$$

Matching with the outer solution implies

$$\left[\frac{\partial T_o^{(0)}}{\partial n}\right]_S^N=v_n^{(0)}LT_o^{(0)}(\mathbf{R}^{(0)},t).\tag{147}$$

We can now solve the outer problem (119), (120), (122), (125) with the interface conditions (126), (144), (145), (147).

We note that the dimensional latent heat $\hat{l}$ is given by [21]

$$\hat{l}(\widetilde{T})=-\mu\widetilde{T}\widetilde{H}_c\frac{d\widetilde{H}_c}{d\widetilde{T}},$$

where $\widetilde{T}$ is the absolute temperature, and $\widetilde{H}_c$ is the dimensional thermodynamic critical field. Hence, on non-dimensionalizing $\hat{l}=kT_c\mu\zeta\widehat{L}$, as in Sec. 4.1, and linearising in T we find

$$\widehat{L}(T)=-LT,$$

where $L=h^2/(\zeta kT_c)$. Thus we see that (147) is in agreement with (110), for temperatures close to T_c. The leading-order outer problem in our asymptotic expansion is thus the free-boundary model written down in Sec. 4.1, except for the extra term $L\partial T/\partial t$ appearing in Eq. (125) which does not appear in Eq. (108). This term is a source term and is due to the fact that the number of superconducting electrons, and hence the rate of release of latent heat, is proportional to T near T_c. Thus a change in temperature in the superconducting region will produce a release or absorption of latent heat. This effect was not taken into consideration in Sec. 4.1, since it was assumed there that the density of superconducting electrons was constant in each phase.

5. Conclusion. We have performed a detailed formal asymptotic analysis of the Ginzburg-Landau model of superconductivity, under the assumption that the solution comprises normal and superconducting domains separated by thin transition layers. The leading-order outer solution was found to satisfy a vectorial version of the Stefan model for the melting or solidification of a pure material. Calculation of the first-order correction to the magnitude of the magnetic field on the interface of this leading-order solution revealed terms analogous to those describing Gibbs-Thompson (surface tension) and kinetic undercooling effects in the classical scalar Stefan model. However, it was shown in Sec. 2 that the "surface energy" associated with a normal/superconducting interface can take both positive and negative values, describing type I and type II superconductors, respectively. Thus the correction terms will only be stabilizing for type I superconductors. For such materials we expect the Ginzburg-Landau model to be a regularization of the free-boundary model in situations in which the latter is ill-posed. Even in this case, because the stabilizing terms only appear at first order, they will not appreciably affect the solution until the interface curvature or velocity becomes very large, and hence we expect very intricate morphologies even from solutions of the Ginzburg-Landau equations. Numerical simulations and experimental results seem to support this conjecture.

For type II superconductors the correction terms are destabilizing, and thus even in situations in which the free boundary model is well-posed, it may not accurately represent the solution of the Ginzburg-Landau equations. A common observation for type II superconductors is that the normal regions form "cores" of size comparable to the thickness of a domain boundary. In such a situation the asymptotic analysis of Sec. 3, and therefore the free-boundary model, is not valid.

Finally, we included thermal effects in both the free-boundary model and the Ginzburg-Landau model. A formal asymptotic analysis of the Ginzburg-Landau model for temperatures close to the transition temperature again led to the free-boundary model at leading order, but with the addition of a source term in the heat balance equation in the superconducting region. Such a term arises because the number of superconducting electrons, and hence the rate of release of latent heat, is proportional to $T - T_c$ for temperatures close to T_c. Thus a change in temperature in the superconducting region will produce a release or absorption of latent heat. This effect was not taken into consideration in the free-boundary model, where it was assumed that the number density of superconducting electrons was constant in each phase.

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Appendix. Operators in curvilinear coordinates. We list here for ease of reference expressions for the Laplacian, curl, divergence, and gradient of functions in orthogonal curvilinear coordinates (x_1, x_2, x_3) with scaling factors h_1, h_2, h_3. Let $\mathbf{e}_i$ be the unit vector in the x_i direction and let $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$. Then

$$\nabla F = \frac{1}{h_1}\frac{\partial F}{\partial x_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial F}{\partial x_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial F}{\partial x_3}\mathbf{e}_3,$$

$$\operatorname{div} \mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_1)}{\partial x_1} + \frac{\partial(h_1h_3F_2)}{\partial x_2} + \frac{\partial(h_1h_2F_3)}{\partial x_3}\right],$$

$$\operatorname{curl} \mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix},$$

$$\nabla^2 F = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial F_1}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial F_2}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial F_3}{\partial x_3}\right)\right].$$

We also list the expression for $\mathbf{v}\cdot\nabla$ in our inner coordinates, which was calculated in [4]:

$$\begin{aligned}
\mathbf{e}_1 \cdot [(\mathbf{v}\cdot\nabla)\mathbf{F}] &= \left(\frac{v_1}{h_1}\frac{\partial}{\partial s_1} + \frac{v_2}{h_2}\frac{\partial}{\partial s_2} + \frac{v_n}{h_3}\frac{\partial}{\partial n}\right)F_1 \\
&\quad - \frac{F_2}{2(EG)^{1/2}}\left(-\frac{v_1}{h_1}\frac{\partial E}{\partial s_2} + \frac{v_2}{h_2}\frac{\partial G}{\partial s_1}\right) + \frac{F_3 v_1 E^{1/2}\tilde{\kappa}_1}{h_1}, \\
\mathbf{e}_2 \cdot [(\mathbf{v}\cdot\nabla)\mathbf{F}] &= \left(\frac{v_1}{h_1}\frac{\partial}{\partial s_1} + \frac{v_2}{h_2}\frac{\partial}{\partial s_2} + \frac{v_n}{h_3}\frac{\partial}{\partial n}\right)F_2 \\
&\quad - \frac{F_1}{2(EG)^{1/2}}\left(\frac{v_1}{h_1}\frac{\partial E}{\partial s_2} - \frac{v_2}{h_2}\frac{\partial G}{\partial s_1}\right) + \frac{F_3 v_1 G^{1/2}\tilde{\kappa}_2}{h_1}, \\
\mathbf{n} \cdot [(\mathbf{v}\cdot\nabla)\mathbf{F}] &= \left(\frac{v_1}{h_1}\frac{\partial}{\partial s_1} + \frac{v_2}{h_2}\frac{\partial}{\partial s_2} + \frac{v_n}{h_3}\frac{\partial}{\partial n}\right)F_3 \\
&\quad - F_1\frac{v_1}{h_1}E^{1/2}\tilde{\kappa}_1 - F_2\frac{v_2}{h_2}G^{1/2}\tilde{\kappa}_2.
\end{aligned}$$

REFERENCES

[1] J. Bardeen, L. N. Cooper, and J. R. Schreiffer, Phys. Rev. **108**, 1175 (1957)

[2] G. Caginalp, *Stefan and Hele-Shaw type models as asymptotic limits of the phase field equations*, Phys. Rev. **A39**, 5887 (1989)

[3] G. Caginalp, *The dynamics of a conserved phase field system: Stefan-like, Hele-Shaw and Cahn-Hilliard models as asymptotic limits*, IMA J. Appl. Math. **44**, 77–94 (1990)

[4] S. J. Chapman, *Thesis*, Oxford University, 1991

[5] S. J. Chapman, *Nucleation of superconductivity in decreasing fields*, I & II, Europ. J. Appl. Math. **5**, 449–494 (1994)

[6] S. J. Chapman, S. D. Howison, J. B. McLeod, and J. R. Ockendon, *Normal-superconducting transitions in Ginzburg-Landau theory*, Proc. Roy. Soc. Edin. **119A**, 117–124 (1991)

[7] S. J. Chapman, S. D. Howison, and J. R. Ockendon, *Macroscopic models of superconductivity*, SIAM Review **34**, No. 4, 529–560 (1992)

[8] J. C. Crank, *Free and Moving Boundary Problems*, Oxford, 1984

[9] A. B. Crowley and J. R. Ockendon, *Modelling mushy regions*, Appl. Sci. Res. **44**, 1–7 (1987)

[10] Q. Du, M. D. Gunzburger, and S. Peterson, *Analysis and approximation of the Ginzburg-Landau model of superconductivity*, SIAM Review **34**, 54–81 (1992)

[11] A. J. W. Duijvestijn, *On the transition from superconducting to normal phase, accounting for latent heat and eddy currents*, IBM J. Research & Development **3**, 2, 132–139 (1959)

[12] T. E. Faber, *The intermediate state in superconducting plates*, Proc. Roy. Soc. **A248**, 461–481 (1958)

[13] H. Frahm, S. Ullah, and A. T. Dorsey, *Flux dynamics and the growth of the superconducting phase*, Phys. Rev. Lett. **66**, 23, 3067–3070 (1991)

[14] U. Essmann and H. Träuble, *The direct observation of individual flux lines in type* II *superconductors*, Phys. Lett. **A24**, 526 (1967)

[15] V. L. Ginzburg and L. D. Landau, *On the theory of superconductivity*, Zh. Èksper. Teoret. Fiz. **20**, 1064 (1950)

[16] L. P. Gor′kov and G. M. Éliashberg, *Generalisation of the Ginzburg-Landau equations for non-stationary problems in the case of alloys with paramagnetic impurities*, Soviet Phys. J.E.T.P. **27**, 328 (1968)

[17] S. D. Howison, A. A. Lacey, and J. R. Ockendon, *Singularity development in moving boundary problems*, Quart. J. Mech. Appl. Math. **38**, 343–360 (1985)

[18] J. B. Keller, *Propagation of a magnetic field into a superconductor*, Phys. Rev. **111**, 1497 (1958)

[19] C. G. Kuper, Philos. Mag. **42**, 961 (1951)

[20] F. Liu, M. Mondello, and N. Goldenfeld, *Kinetics of the superconducting transition*, Phys. Rev. Lett. **66**, 23, 3071–3074 (1991)

[21] A. C. Rose-Innes and E. H. Rhoderick, *Introduction to superconductivity*, Pergamon, 1978

[22] H. Träuble and U. Essmann, *Ein hochauflösendes Verfahren zur Untersuchung magnetischer Strukturen von Supraleitern*, Phys. Stat. Solids **18**, 813–828 (1966)

[23] H. Träuble and U. Essmann, *Die Beobachtung magnetischer Strukturen von Supraleitern zweiter Art*, Phys. Stat. Solids **20**, 95–111 (1967)

[24] M. Van Dyke, *Perturbation Methods in Fluid Mechanics*, The Parabolic Press, 1975