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ANALYSIS OF A RIGID FERROMAGNETIC BODY UNDER APPLIED MAGNETIC FIELDS

By

DEBORAH BRANDON (*Department of Mathematics, Carnegie Mellon University, Pittsburgh, PA*)

AND

ROBERT C. ROGERS (*Dept. of Mathematics and ICAM, Virginia Polytechnic Institute and State University, Blacksburg, Virginia*)

1. Introduction. Consider the following experiment. We take a large plate of thickness $2L$, composed of a rigid, uniaxial ferromagnetic material with easy axis perpendicular to the plate. The plate is subjected to a strong applied field parallel to the easy direction and thus becomes magnetically saturated parallel to this field. Suppose now that the applied field is slowly reduced in strength. Experimental observation of Bitter patterns and domain structures suggests that the following scenario is typical of the behavior of many materials.

1. At some critical value of the applied field, one observes "edge effects" as spike-like domains of reverse magnetization begin to appear at the surfaces of the plate.
2. As the applied field is reduced further, the entire plate will exhibit a highly oscillatory domain structure.
3. As the applied field is reversed in direction, the domains parallel to the applied field increase in size while the domains in the opposite direction shrink.
4. When the reversed field is sufficiently large, all domains disappear, leaving the specimen saturated parallel to the field.

This behavior has been modeled empirically using such devices as Preisach models. But there has been much less success in developing models based on fundamental material properties that allow one to deduce the behavior described above. In particular, while the most widely accepted model of ferromagnetic materials, micromagnetics, has been used to describe such phenomena as domain structures in materials with zero applied fields (see Miranker and Willner [10]), no one has been able to use this model to describe the global behavior of a ferromagnetic body as the applied field is varied and the various solutions lose and gain stability. The goal of this paper is to at least partially remedy this

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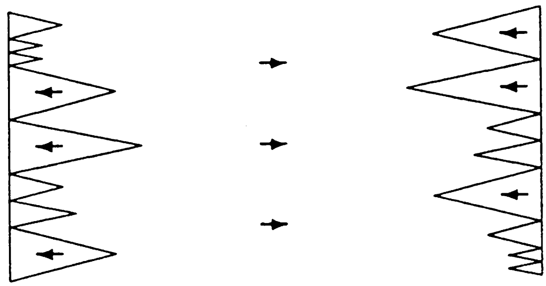

FIG. 1. Edge effects at the boundary of a specimen

situation using a relatively new model that employs the techniques of Young-measures and uses a nonlocal model for the exchange energy.

Young-measures and related techniques have now been established as important tools in the study of highly oscillatory structures in phase transitions. In ferromagnetism, these methods have been successfully employed to study the relationship between crystallography and domain structure [6] and magnetostriction [7, 13]. Young-measures essentially "average out" magnetic domains by modeling them in a probabilistic fashion.

The use of a nonlocal model for the exchange energy was originally proposed in [12] for the purpose of studying hysteresis using Young-measures. Traditional mathematical models of ferromagnetism (micromagnetics and domain theory) predict a large (and hence difficult to compute) number of oscillations in the magnetization, but do not allow one to go to the Young-measure limit. The nonlocal model was shown not to suffer from the so-called "coercivity paradox" [3] and a discrete version of the model was used to predict a rich class of hysteresis loops [8]. In this paper we show that the nonlocal model predicts the behavior described above, including magnetic saturation, the onset of edge effects, the spread of magnetic domains throughout the specimen, and the reversal of domains.

The term "edge effects" is used here to refer to the patterns of spike-like domains that appear at the edges of ferromagnetic specimens (see Fig. 1). James and Kinderlehrer [6] showed that in cubic crystals in the absence of an applied magnetic field and with exchange energy neglected, edge effects arise naturally in constructing a classical magnetization that minimizes the field energy. In this paper we address the situation in uniaxial crystals. In [3] we showed that if a sufficiently large magnetic field is applied in the "easy" direction of magnetization of a homogeneous uniaxial ferromagnetic body, the body becomes "magnetically saturated", i.e., the global minimizer of the energy is a uniform classical magnetization parallel to the applied field. As the applied field is decreased and reversed, this saturated state loses stability. In this paper we show that it loses stability to a solution that has measure-valued oscillatory edges.

The rest of this paper is organized as follows. In Sec. 2 we define some of the basic concepts from ferromagnetism with particular attention to applications of Young-measures. In Sec. 3 we develop a one-dimensional version of the energy minimization problem for an infinite plate of finite thickness and derive necessary conditions for the existence of minimizers. In Sec. 4 we find a family of stationary points that include saturated states,

states that exhibit edge effects, and purely measure-valued states. In Sec. 5 we show that these solutions are minimizers of the energy. In Sec. 6 we make some concluding remarks. In particular, we compare the minimizers we obtain for one-dimensional problems with functions that are periodic in directions parallel to the face of the body.

2. Definitions. In classical mathematical problems in the theory of static, rigid, ferromagnetic materials, one considers a body Ω in a given *applied magnetic field* $\mathbf{h}_0$ and seeks a *magnetization field* $\mathbf{m}\colon \Omega \to \mathbb{R}^3$ such that the magnetostatic energy

$$\mathcal{E}(\mathbf{m}) = \frac{1}{2}\int_{\mathbb{R}^3} |\hat{\mathbf{h}}[\mathbf{m}]|^2 + \int_\Omega \{\mathcal{W}(\mathbf{m}(\mathbf{x})) - \mathbf{m}(\mathbf{x})\cdot\mathbf{h}_0(\mathbf{x})\}\,d\mathbf{x} + \chi(\mathbf{m}) \tag{2.1}$$

is minimized subject to the constraint that

$$|\mathbf{m}(\mathbf{x})| = 1 \quad \text{at almost every } \mathbf{x}\in\Omega.$$

Here $\hat{\mathbf{h}}(\mathbf{m})$ is the *resultant magnetic field* generated by $\mathbf{m}$ (described further below), $\mathcal{W}$ is the *anisotropy energy density*, and $\chi(\mathbf{m})$ is the *exchange energy functional.* In this paper we consider a nonlocal version of the exchange energy of the form

$$\chi(\mathbf{m}) := -c\int_\Omega\int_\Omega \mathbf{m}(\mathbf{x})\cdot\mathbf{m}(\mathbf{y})k(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y}. \tag{2.2}$$

Here k is a positive, symmetric kernel concentrated at the origin and decaying at infinity and $c>0$ is a material constant. We consider the particular kernel

$$k(\mathbf{x},\mathbf{y}) := \frac{\gamma^2}{4\pi}\frac{e^{-\gamma|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}, \tag{2.3}$$

where γ is a positive constant and k has been normalized so that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} k(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = \frac{\gamma^2}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\gamma|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{x}\,d\mathbf{y} = 1. \tag{2.4}$$

This kernel arises naturally in nonlocal problems derived from statistical mechanics under the assumption that the probability of a nonlocal interaction between two particles is a function of the distance between the particles. This is a typical assumption in transport theory.

The vector field $\hat{\mathbf{h}}(\mathbf{m})$ is the resultant magnetic field produced by the magnetization. For a sufficiently smooth classical magnetization $\mathbf{m}$, the magnetic field is defined to be the solution of the equations

$$\operatorname{Curl}\hat{\mathbf{h}} = \mathbf{0}, \tag{2.5}$$

$$\operatorname{Div}\hat{\mathbf{h}} = -\operatorname{Div}\mathbf{m} \tag{2.6}$$

and the jump conditions

$$[[\hat{\mathbf{h}}]]\times\mathbf{n} = 0, \tag{2.7}$$

$$[[\hat{\mathbf{h}}]]\cdot\mathbf{n} = -[[\mathbf{m}]]\cdot\mathbf{n}, \tag{2.8}$$

on any surface of discontinuity of $\mathbf{m}$. Here $\mathbf{n}$ denotes a unit normal to the surface of discontinuity and $[[\cdot]]$ indicates the jump of a piecewise continuous function in the direction $\mathbf{n}$. More generally, for any $\mathbf{m} \in \mathbb{M}(\Omega)$, where

$$\mathbb{M}(\Omega) := \{\mathbf{m} \in L^\infty(\Omega) | \, |\mathbf{m}(\mathbf{x})| = 1 \text{ a.e.}\}, \tag{2.9}$$

the *resultant magnetic field* due to $\mathbf{m}$ is defined to be the unique function $\hat{\mathbf{h}}[\mathbf{m}]$ in the set

$$\mathcal{A} = \{\mathbf{h} \in L^2(\mathbb{R}^3) | \, \mathrm{Curl}\, \mathbf{h} = \mathbf{0} \text{ in } H^{-1}(\mathbb{R}^3)\}, \tag{2.10}$$

satisfying

$$\int_{\mathbb{R}^3} \hat{\mathbf{h}}[\mathbf{m}] \cdot \mathbf{h}^\sharp = -\int_\Omega \mathbf{m} \cdot \mathbf{h}^\sharp \quad \forall \mathbf{h}^\sharp \in \mathcal{A}. \tag{2.11}$$

Here $H^{-1}(\mathbb{R}^3)$ is the usual Sobolev space (cf., e.g., Adams [1]). The existence and uniqueness of solutions of (2.11) and their continuous dependence on $\mathbf{m}$ is guaranteed by the Lax-Milgram lemma. For more information on the magnetic field and the properties of the field energy see, e.g., [3, 5, 11, 12].

As we indicated in the introduction, the nonlocal model for ferromagnetic materials was introduced in order to study *measure-valued magnetizations*, which can be used to model wildly oscillating, weakly convergent sequences of classical magnetizations.

DEFINITION 2.1. We define an element of the set $\mathcal{M}(\Omega)$ of **measure-valued magnetizations** on the body Ω to be a nonnegative probability measure $\nu_\mathbf{x}$ on $\mathbb{R}^3$ parametrized by $\mathbf{x} \in \Omega$, with support on the unit sphere.

Note that a classical magnetization $\mathbf{m} \in \mathbb{M}(\Omega)$ can be viewed as a measure-valued magnetization with

$$\nu_\mathbf{x} := \delta(\mathbf{m}(\mathbf{x})). \tag{2.12}$$

Here $\delta(\mathbf{y})$ is the Dirac delta function centered at $\mathbf{y} \in \mathbb{R}^3$.

It can be shown that with any weak-star convergent sequence of classical magnetizations $\mathbf{m}^n \in \mathbb{M}(\Omega)$,

$$\mathbf{m}^n \overset{*}{\rightharpoonup} \mathbf{m} \quad \text{in } L^\infty(\Omega), \tag{2.13}$$

(i.e., $\int_\Omega \mathbf{m}^n \phi \to \int_\Omega \mathbf{m}\phi$ for every $\phi \in L^1(\Omega)$) one can associate a measure-valued magnetization $\nu \in \mathcal{M}(\Omega)$ satisfying

$$\lim_{n\to\infty} \int_\Omega F(x, \mathbf{m}^n(\mathbf{x}))d\mathbf{x} = \int_\Omega \int_{\mathbb{R}^3} F(\mathbf{x}, \lambda) d\nu_\mathbf{x}(\lambda) d\mathbf{x} \tag{2.14}$$

(at least for a subsequence) for any continuous function $F : \Omega \times \mathbb{R}^3 \to \mathbb{R}$. Conversely, for any measure-valued magnetization $\nu \in \mathcal{M}(\Omega)$ there exists a sequence $\mathbf{m}^n \in \mathbb{M}(\Omega)$ such that (2.13) and (2.14) hold. See [14] for a sketch of the proof of these results.

We define the total magnetostatic energy of a measure-valued magnetization μ to be

$$\begin{aligned}\widetilde{\mathcal{E}}(\mu) =& \frac{1}{2}\int_{\mathbb{R}^3} |\hat{\mathbf{h}}[\overline{\mathbf{m}}]|^2 + \int_\Omega \left[\int_{\mathbb{R}^3} \mathcal{W}(\lambda) d\mu_\mathbf{x}(\lambda) - \overline{\mathbf{m}}(\mathbf{x}) \cdot \mathbf{h}_0(\mathbf{x})\right] d\mathbf{x} \\ & - c \int_\Omega \int_\Omega \overline{\mathbf{m}}(\mathbf{x}) \cdot \overline{\mathbf{m}}(\mathbf{y}) k(\mathbf{x}, \mathbf{y})\, d\mathbf{y}\, d\mathbf{x}.\end{aligned} \tag{2.15}$$

Here $\overline{\mathbf{m}}$, the center of mass of μ, is given by

$$\overline{\mathbf{m}}(\mathbf{x}) := \int_{\mathbb{R}^3} \lambda \, d\mu_{\mathbf{x}}(\lambda). \tag{2.16}$$

The use of measure-valued magnetizations allows one to obtain a rigorous general existence theory of energy minimizers. For a further discussion of this energy see [12] and the work of DeSimone [5] and Pedregal [11].

3. A one-dimensional model problem. In this section we formulate a mathematical problem that models the plate experiment described in the introduction. The physical situation is identical to that studied by Miranker and Willner [10] using the micromagnetic model.

3.1. *Formulation.* Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be an orthonormal basis of vectors in $\mathbb{R}^3$ parallel to the $x, y,$ and z coordinate axes respectively. We consider the problem of a body of finite thickness $2L$ in the x direction, and infinite extent in the y and z directions:

$$\Omega := \{\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \in \mathbb{R}^3 \mid -L < x < L, -\infty < y < \infty, -\infty < z < \infty\}.$$

We seek to minimize the magnetostatic energy over magnetization fields that depend only on the x direction. Thus, with a slight abuse of notation, we consider classical magnetizations

$$\mathbf{m} \in \mathbb{M}(-L, L) := \{\mathbf{m} \in L^\infty(-L, L) \mid |\mathbf{m}(x)| = 1 \text{ a.e.}\}$$

and measure-valued magnetizations $\nu \in \mathcal{M}(-L, L)$ parametrized by $x \in (-L, L)$.

REMARK. Since measure-valued magnetizations depending only on the x coordinate could represent the limit of a sequence of functions that vary in the y and z directions, one might object to the restricting of the problem to one-dimensional magnetization. However, we shall see in Sec. 6 that minimizers obtained over the class of functions that depend only on x are in fact minimizers over the larger set of functions that are periodic in the y and z directions. This is a large enough class of functions to yield measure-valued magnetizations with prescribed first and second moments (cf. Theorem 6.2 of [12]). Thus, in order to ease our calculations, we concentrate on the one-dimensional problem for now and justify its relevance by doing the comparison with periodic functions in Sec. 6.

Because of the infinite extent of the body, the magnetostatic energy of any admissible magnetization would be infinite. Thus, we will seek to minimize the energy per unit area in the (y, z) plane.

We first calculate the field energy. We compute the resultant magnetic field due to a magnetization field

$$\mathbf{m}(x) = m_1(x)\mathbf{i} + m_2(x)\mathbf{j} + m_3(x)\mathbf{k}$$

(which represents either a classical magnetization or the center of mass of a measure-valued magnetization) using the classical equations (2.5), (2.6) and the jump conditions (2.7) and (2.8). We see that these are satisfied by

$$\hat{\mathbf{h}}[\mathbf{m}](\mathbf{x}) = \mathbf{h}(x, y, z) = \begin{cases} -m_1(x)\mathbf{i}, & |x| < L, \\ \mathbf{0}, & |x| \geq L. \end{cases} \tag{3.1}$$

Thus, the field energy per unit area is given by

$$\frac{1}{2}\int_{-L}^{L} m_1(x)^2 dx. \tag{3.2}$$

The anisotropy energy per unit area of a measure-valued magnetization ν is given by

$$\int_{-L}^{L} \alpha_1(g_1(x) + \alpha_2(g_2(x) + g_3(x)))\, dx, \tag{3.3}$$

where

$$g_i(x) := \int_{\mathbb{R}^3} \lambda_i^2 d\nu_x(\lambda), \quad i = 1, 2, 3, \tag{3.4}$$

are the symmetric second moments of the measure ν. This comes from the following form of the anisotropy energy for a uniaxial material:

$$\mathcal{W}_a(\mathbf{m}) := \alpha_1(\mathbf{m}_1^2 + \alpha_2(\mathbf{m}_2^2 + \mathbf{m}_3^2)).$$

(See [3].)

We consider a uniform applied field in the easy direction of magnetization $\mathbf{h}_0 = h_0\mathbf{i}$. The interaction energy is then given by

$$-h_0\int_{-L}^{L} m_1(x)\, dx. \tag{3.5}$$

As we indicated above, the exchange energy will also depend only on the center of mass $\mathbf{m}$ of a measure-valued magnetization. Its density per unit area is

$$\begin{aligned}
&\int_{-L}^{L}\int_{\Omega} k(\mathbf{x},\mathbf{y})\mathbf{m}(\mathbf{x})\cdot\mathbf{m}(\mathbf{y})\, d\mathbf{y}\, dx \\
&= -c\frac{\gamma^2}{4\pi}\int_{-L}^{L}\mathbf{m}(x)\cdot\left[\int_{-L}^{L}\mathbf{m}(y)\int_0^\infty\int_0^{2\pi}\frac{e^{-\gamma\sqrt{(x-y)^2+r^2}}}{\sqrt{(x-y)^2+r^2}} r\, dr\, d\theta\, dy\right] dx \\
&= -c\frac{\gamma}{2}\int_{-L}^{L}\int_{-L}^{L} e^{-\gamma|x-y|}\mathbf{m}(x)\cdot\mathbf{m}(y)\, dy\, dx.
\end{aligned} \tag{3.6}$$

Putting all of this together, we get the following magnetostatic energy density for a measure-valued magnetization $\nu \in \mathcal{M}(-L, L)$:

$$\begin{aligned}
\mathcal{E}(\nu) = &\int_{-L}^{L}\left[\frac{1}{2}m_1^2(x) + \alpha_1(g_1(x) + \alpha_2(g_2(x) + g_3(x)) - h_0 m_1(x)\right] dx \\
&- c\frac{\gamma}{2}\int_{-L}^{L}\int_{-L}^{L} e^{-\gamma|\overline{x}-y|}\mathbf{m}(x)\cdot\mathbf{m}(y)\, dy\, dx.
\end{aligned} \tag{3.7}$$

We note that the energy of a measure-valued magnetization depends on the measure only through its center of mass and symmetric second moments. In fact, following the procedures outlined in [3, 12], it can be shown that, using the information obtained from minimizing the energy

$$\begin{aligned}\check{\mathcal{E}}(\mathbf{m},\mathbf{g}) := &\int_{-L}^{L}\left[\frac{1}{2}m_1^2 - h_0 m_1 + \alpha_1(g_1 + \alpha_2(g_2+g_3))\right]dx \\ &- c\frac{\gamma}{2}\int_{-L}^{L}\int_{-L}^{L} e^{-\gamma|x-y|}\mathbf{m}(x)\cdot\mathbf{m}(y)\,dy\,dx\end{aligned} \tag{3.8}$$

over the set of functions $(\mathbf{m},\mathbf{g}) \in L^\infty(-L,L)$ such that

$$m_i(x)^2 \le g_i(x), \quad i = 1,2,3, \tag{3.9}$$

and

$$g_1(x) + g_2(x) + g_3(x) = 1 \tag{3.10}$$

almost everywhere, one can construct measure-valued magnetizations of minimal energy.

While this relieves us of the difficulty of dealing directly with the Young-measures, we must still deal with a pointwise inequality constraint (3.9). To get around this difficulty we follow the procedure of [2] and introduce *slack variables* v_i, $i = 1,2,3$, such that

$$g_i(x) = m_i^2(x) + v_i^2(x), \quad i = 1,2,3. \tag{3.11}$$

We can now seek to minimize

$$\begin{aligned}\hat{\mathcal{E}}(\mathbf{m},\mathbf{v}) := &\int_{-L}^{L}\left[\frac{1}{2}m_1^2 - h_0 m_1 + \alpha_1(m_1^2 + v_1^2 + \alpha_2(m_2^2 + v_2^2 + m_3^2 + v_3^2))\right] \\ &- c\frac{\gamma}{2}\int_{-L}^{L}\int_{-L}^{L} e^{-\gamma|x-y|}\mathbf{m}(x)\cdot\mathbf{m}(y)\,dy\,dx\end{aligned} \tag{3.12}$$

over $(\mathbf{m},\mathbf{v}) \in L^\infty(-L,L)$ subject to the constraint

$$m_1^2(x) + v_1^2(x) + m_2^2(x) + v_2^2(x) + m_3^2(x) + v_3^2(x) = 1 \tag{3.13}$$

almost everywhere.

We can further simplify the problem by using the pointwise constraint (3.13) to reduce the number of unknowns by eliminating v_1; i.e., we seek to minimize

$$\begin{aligned}\overline{\mathcal{E}}(\mathbf{m},v_2,v_3) := &\int_{-L}^{L}\left\{\frac{m_1^2}{2} + \frac{\beta}{2}[m_2^2 + v_2^2 + m_3^2 + v_3^2] - h_0 m_1\right\}dx \\ &- c\frac{\gamma}{2}\int_{-L}^{L}\int_{-L}^{L} e^{-\gamma|x-y|}\mathbf{m}(y)\cdot\mathbf{m}(x)\,dx\,dy\end{aligned} \tag{3.14}$$

over

$$\overline{\mathcal{A}} := \{(\mathbf{m},v_2,v_3) \in L^\infty(-L,L) \mid m_1^2 + m_2^2 + m_3^2 + v_2^2 + v_3^2 \le 1\}. \tag{3.15}$$

Here

$$\beta := 2\alpha_1(\alpha_2 - 1) > 0. \tag{3.16}$$

As we indicated above, the classical variational problem of minimizing $\overline{\mathcal{E}}$ over $\overline{\mathcal{A}}$ is equivalent to the problem of minimizing $\mathcal{E}$ over the set of measure-valued magnetizations $\mathcal{M}(-L,L)$. (This is a direct consequence of Theorem 6.2 of [12].)

One relationship worth noting between the measures and their moments is that the measure-valued magnetizations are nontrivial (i.e., not a single delta function) exactly when we have strict inequality

$$m_i^2 < g_i \quad \text{or} \quad v_i^2 > 0 \tag{3.17}$$

for some i. In this case we refer to a solution as "truly measure-valued". When equality holds for each i we refer to the solutions as "classical".

3.2. *Necessary conditions.* In this section we derive necessary conditions for the existence of minimizers of $\overline{\mathcal{E}}$. Our necessary conditions will be for relative minimizers which we define as follows:

DEFINITION 3.1. We say that $(m_1, m_2, m_3, v_2, v_3) \in \overline{\mathcal{A}}$ is an L^p-relative minimizer of $\overline{\mathcal{E}}$ over $\overline{\mathcal{A}}$ ($p \in [1,\infty]$) if there exists an $\varepsilon > 0$ such that for any $(m_1^\sharp, m_2^\sharp, m_3^\sharp, v_2^\sharp, v_3^\sharp) \in \overline{\mathcal{A}}$ satisfying

$$\|(m_1, m_2, m_3, v_2, v_3) - (m_1^\sharp, m_2^\sharp, m_3^\sharp, v_2^\sharp, v_3^\sharp)\|_{L^p(-L,L)} < \varepsilon, \tag{3.18}$$

we have

$$\overline{\mathcal{E}}(m_1, m_2, m_3, v_2, v_3) \leq \overline{\mathcal{E}}(m_1^\sharp, m_2^\sharp, m_3^\sharp, v_2^\sharp, v_3^\sharp). \tag{3.19}$$

In addition, a point $(m_1, m_2, m_3, v_2, v_3) \in \overline{\mathcal{A}}$ is said to be stable when inequality (3.19) can be replaced by a strict inequality.

We begin with the following standard lemma.

LEMMA 3.2. Suppose $(m_1, m_2, m_3, v_2, v_3)$ is an L^∞-relative minimizer of $\overline{\mathcal{E}}$ over $\overline{\mathcal{A}}$. Then for any $(u_1, u_2, u_3, w_2, w_3) \in L^\infty(-L,L)$ for which there exists $\varepsilon > 0$ such that

$$(m_1, m_2, m_3, v_2, v_3) + \tau(u_1, u_2, u_3, w_2, w_3) \in \overline{\mathcal{A}} \tag{3.20}$$

for all $\tau \in (-\varepsilon, \varepsilon)$, we have

$$\begin{aligned} 0 = &\int_{-L}^{L} [(m_1 - h_0)u_1 + \beta(m_2 u_2 + m_3 u_3 + v_2 w_2 + v_3 w_3]\, dx \\ &- c\gamma \sum_{i=1}^{3} \int_{-L}^{L} \int_{-L}^{L} e^{-\gamma|x-y|} m_i(x) u_i(y)\, dy\, dx. \end{aligned} \tag{3.21}$$

Also, for any $u \in L^\infty(-L,L)$ for which there exists $\varepsilon > 0$ such that (3.20) holds for all $\tau \in [0,\varepsilon)$, we have

$$\begin{aligned} 0 \leq &\int_{-L}^{L} [(m_1 - h_0)u_1 + \beta(m_2 u_2 + m_3 u_3 + v_2 w_2 + v_3 w_3)]\, dx \\ &- c\gamma \sum_{i=1}^{3} \int_{-L}^{L} \int_{-L}^{L} e^{-\gamma|x-y|} m_i(x) u_i(y)\, dy\, dx. \end{aligned} \tag{3.22}$$

Proof. We consider the first case and define $f : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$f(\tau) := \overline{\mathcal{E}}((m_1, m_2, m_3, v_2, v_3) + \tau(u_1, u_2, u_3, w_2, w_3)). \tag{3.23}$$

The function f is quadratic and is minimized at zero. Setting the first derivative at $\tau = 0$ to be zero leads to (3.21).

In the second case we define $f : [0, \varepsilon) \to \mathbb{R}$ as above and note that it is minimized at the left-hand endpoint of its domain. Requiring the first derivative to be nonnegative at $\tau = 0$ leads to (3.22). □

We can use this lemma to prove the following.

THEOREM 3.3. If $(m_1, m_2, m_3, v_2, v_3)$ is an L^∞-relative minimizer of $\overline{\mathcal{E}}$ over $\overline{\mathcal{A}}$, then the following relations hold almost everywhere:

$$v_2(x) = v_3(x) = 0, \tag{3.24}$$

$$m_1(x)[c\gamma K(m_1)(x) - m_1(x) + h_0] \geq 0, \tag{3.25}$$

and

$$m_i(x)[c\gamma K(m_i)(x) - \beta m_i(x)] \geq 0, \quad i = 2, 3, \tag{3.26}$$

where

$$K(m)(x) := \int_{-L}^{L} e^{-\gamma|x-y|} m(y)\, dy. \tag{3.27}$$

Proof. Let U be any measurable set in $(-L, L)$ and let χ_U be its characteristic function. Letting $(u_1, u_2, u_3, w_2, w_3) = (0, 0, 0, -v_2\chi_U, 0)$ (which is an admissible "one-sided" variation) and using (3.22) we get

$$0 \leq -\int_U v_2^2(x)\, dx.$$

Since U is arbitrary, this and an analogous calculation for v_3 yield (3.24). Inequalities (3.25) and (3.26) follow from the same type of procedure. □

Note that when

$$m_2 = m_3 = v_2 = v_3 = 0, \tag{3.28}$$

(i.e., when the magnetization lies completely in the easy direction) conditions (3.24) and (3.26) are satisfied. Thus, it is natural to search for minimizers for which m_1 is the only nontrivial component. We speak of such functions as "completely transverse to the plate". For such magnetizations we define

$$\begin{aligned} E(m) &:= \overline{\mathcal{E}}(m, 0, 0, 0, 0) \\ &= \int_{-L}^{L} \left\{ \frac{m^2}{2} - h_0 m \right\} - \frac{c\gamma}{2} \int_{-L}^{L} \int_{-L}^{L} e^{-\gamma|x-y|} m(x) m(y)\, dx\, dy. \end{aligned} \tag{3.29}$$

We now derive necessary conditions for completely transverse magnetizations to be energy minimizers. Part of our result is a smoothness condition that depends crucially on our choice of a kernel through the following lemma. (The lemma is elementary and can be verified via direct computation.)

LEMMA 3.4. For any $m \in L^\infty(-L, L)$, it follows that $K(m) \in W^{2,\infty}(-L, L)$. Furthermore, if m is continuous on $[-L, L]$, then $K(m) \in C^2([-L, L])$ and satisfies the differential equation

$$K(m)''(x) - \gamma^2 K(m)(x) = -2\gamma m(x) \tag{3.30}$$

and the boundary conditions

$$K(m)'(L) = -\gamma K(m)(L), \tag{3.31}$$

$$K(m)'(-L) = \gamma K(m)(-L). \tag{3.32}$$

Conversely, given $m \in L^\infty(-L, L)$, $K(m)$ is the unique solution of (3.30), (3.31), and (3.32).

We now derive basic necessary conditions for transverse minimizers.

THEOREM 3.5. Suppose $(m, 0, 0, 0, 0) \in \overline{\mathcal{A}}$ is an $L^\infty(-L, L)$-relative minimizer of $\overline{\mathcal{E}}$ over $\overline{\mathcal{A}}$. Then the following hold.

1. The function m (has a representative that) is continuous and piecewise smooth.
2. Whenever

$$|m| < 1, \tag{3.33}$$

the Euler-Lagrange equation

$$m(x) = c\gamma K(m)(x) + h_0 \tag{3.34}$$

is satisfied.

3. The constraint equation

$$m(x) = 1 \tag{3.35}$$

holds if and only if the variational inequality

$$c\gamma K(m)(x) + h_0 \geq 1 \tag{3.36}$$

holds.

4. Similarly,

$$m(x) = -1 \tag{3.37}$$

if and only if

$$c\gamma K(m)(x) + h_0 \leq -1. \tag{3.38}$$

Proof. Assume that $(m, 0, 0, 0, 0) \in \overline{\mathcal{A}}$ is an $L^\infty(-L, L)$ minimizer of $\overline{\mathcal{E}}$ over $\overline{\mathcal{A}}$. We first claim that (3.34) holds almost everywhere on the measurable set $\overline{U} := \{x \in (-L, L) \mid |m(x)| < 1\}$. This follows from (3.21) since we can take a "two-sided" variation $(u, 0, 0, 0, 0)$ in (3.20) to be the characteristic function of any measurable subset of the set $\overline{U}_\varepsilon := \{x \in (-L, L) \mid |m(x)| \leq 1 - \varepsilon\}$ for any $\varepsilon > 0$.

We now note that (3.25) implies

$$m(x)[c\gamma K(m)(x) + h_0] \geq m(x)^2; \tag{3.39}$$

so

$$\operatorname{sgn}(m(x)) = \operatorname{sgn}(c\gamma K(m)(x) + h_0) \tag{3.40}$$

almost everywhere that $m \neq 0$. These imply

$$|m(x)| \leq |c\gamma K(m)(x) + h_0|. \tag{3.41}$$

Note that (3.41) implies that the inequality (3.36) (respectively (3.38)) holds at almost every point at which (3.35) (respectively (3.37)) holds.

Since $c\gamma K(m) + h_0$ is independent of the representative of the equivalence class $m \in L^\infty(-L, L)$, it follows that there is a representative of m such that conditions 1, 2, and 3 hold for each $x \in (-L, L)$. By Lemma 3.4, $c\gamma K(m) + h_0$ is in $W^{2,\infty}(-L, L)$ and thus is in $C^1([-L, L])$. The continuity of $c\gamma K(m) + h_0$ implies that the representative of m chosen above is continuous; a simple bootstrap argument and Lemma 3.4 imply that it is piecewise smooth. □

We introduce the following terminology for a function satisfying the necessary conditions for minimization.

DEFINITION 3.6. We refer to a continuous, piecewise smooth function $m : (-L, L) \to \mathbb{R}$ satisfying

$$|m(x)| \leq 1, \tag{3.42}$$

and (3.34), (3.36), and (3.38) as a *transverse stationary point* of the energy $\overline{\mathcal{E}}$.

Equations (3.30) and (3.34) can be combined with the first result of Theorem 3.5 to give us the following local description of a transverse stationary point.

COROLLARY 3.7. Let m be a transverse stationary point of the energy $\overline{\mathcal{E}}$. Then the interval $(-L, L)$ can be decomposed into a finite collection of subintervals such that on each subinterval exactly one of the following hold:

1. $m \equiv 1$,
2. $m \equiv -1$, or
3. $|m| < 1$, and the differential equation

$$m'' = \gamma^2(1 - 2c)m - \gamma^2 h_0 \tag{3.43}$$

is satisfied.

4. A continuum of stationary points. In this section we exhibit a family of stationary points for the one-dimensional model problem formulated in the previous section. We reiterate the following remarks about our solutions:

- Our results depend in a critical way on the specific choice of the anisotropy energy and the kernel of the nonlocal exchange energy.
- As one would expect, the character of our solutions depends on the strength of the exchange forces as embodied in the parameter c.

Our family of stationary points exhibits the following properties:

1. When the applied field h_0 is large in magnitude, the saturated state $m \equiv \operatorname{sgn}(h_0)$ is a stable stationary point. (In fact, it is a global minimizer for very large values of h_0.)

2. At a critical value of the applied field, the saturated state loses stability to an "edge effect" solution that is saturated in the middle of the plate and oscillatory (measure-valued) on the surfaces.

3. If the exchange parameter c is sufficiently small, the oscillatory edges encroach on the plate as the magnitude of the field is reduced. Below another critical magnitude of the applied field, there is a stationary point that is measure-valued in the entire plate.

4. For relatively small values of c we have a complete picture of the magnetization curve: as the applied field is reduced, solutions go from saturated to measure-valued with transitions exhibiting edge effects in between. For larger values of c the picture is less clear. We can show existence of stationary points exhibiting edge effects for applied fields close to the fields at which saturated states lose stability. However, we have not "connected" these solutions to a family of solutions defined for all h_0, and (as we shall see in the next section) we have no stability result for nonsaturated solutions in this case.

4.1. *Saturated solutions.* We refer to states with $m \equiv 1$ or $m \equiv -1$ as *saturated.* In [3] we examined (in a much more general setting) the question of when saturated states were relative minimizers of the magnetostatic energy. In this section we review our previous results for the specialized problem we are addressing in this paper. We first determine when a saturated state is a transverse stationary point. For definiteness we examine $m \equiv 1$. For this function we need only check (3.36), i.e., we require that for $x \in (-L, L)$ we have

$$\begin{aligned} 1 &< c\gamma K(1)(x) + h_0 \\ &= 2c[1 - e^{-\gamma L}\cosh \gamma x] + h_0. \end{aligned} \tag{4.1}$$

This holds if

$$h_0 > 1 - c(1 - e^{-2\gamma L}). \tag{4.2}$$

Similarly, $m \equiv -1$ is a transverse stationary point if

$$h_0 < -[1 - c(1 - e^{-2\gamma L})]. \tag{4.3}$$

In fact, we can say much more: the stationary points are actually relative minimizers. The following theorem is a special case of Theorem 4.5 of [3].

THEOREM 4.1. If $|h_0| \geq 1$ then the saturated state $m \equiv \operatorname{sgn} h_0$ is a global minimizer of $\overline{\mathcal{E}}$ over $\overline{\mathcal{A}}$.

Furthermore, if

$$\pm h_0 > 1 - c(1 - e^{-2\gamma L}), \tag{4.4}$$

then the saturated state $m \equiv \pm 1$ is a stable L^1-relative minimizer of $\overline{\mathcal{E}}$ over $\overline{\mathcal{A}}$.

Of course, this raises the natural question: what happens when the saturated state loses stability? In the next section we show that as the applied field is varied past the limit of stability of the saturated state, a stationary point exhibiting edge effects exists.

4.2. *Edge effect solutions.* We now show the existence of transverse stationary points such that

1. m is even,
2. there exists $\overline{x} \in (0, L)$ such that $m \equiv 1$ on $[-\overline{x}, \overline{x}]$,

and

3. $|m| < 1$ on $[-L, -\overline{x}) \cup (\overline{x}, L]$.

While it is possible to find such an m using the local conditions described in Corollary 3.7, we have found it easier to search for $K(m)$. More specifically, we seek $\overline{x} \in (0, L)$ and $K : [-L, L] \to \mathbb{R}$ such that

E-1. K is even.

E-2. In $(-L, -\overline{x}) \cup (\overline{x}, L)$, K satisfies

$$K'' - \gamma^2(1 - 2c)K = -2\gamma h_0, \tag{4.5}$$

and

$$|c\gamma K + h_0| < 1. \tag{4.6}$$

E-3. In $(-\overline{x}, \overline{x})$, K satisfies

$$K'' - \gamma^2 K = -2\gamma \tag{4.7}$$

and, in addition, the variational inequality (3.36) holds, i.e.,

$$c\gamma K + h_0 > 1.$$

E-4. $K \in C^2([-L, L])$. In particular, K and K' are continuous at $\pm\overline{x}$, and

$$c\gamma K(\overline{x}) + h_0 = 1. \tag{4.8}$$

E-5. K satisfies the boundary conditions (3.31) and (3.32), i.e.,

$$K'(L) = -\gamma K(L),$$
$$K'(-L) = \gamma K(-L).$$

Constructing m from this K is straightforward.

THEOREM 4.2. If $\overline{x} \in (0, L)$ and $K : [-L, L] \to \mathbb{R}$ satisfy the conditions E-1–E-5, then

$$m(x) := \begin{cases} c\gamma K(x) + h_0, & x \in [-L, -\overline{x}) \cup (\overline{x}, L], \\ 1, & x \in [-\overline{x}, \overline{x}] \end{cases} \tag{4.9}$$

is a transverse stationary point.

Finding K satisfying the conditions above is a tedious but routine exercise in ordinary differential equations. Solutions can be found as follows.

4.2.1. *Case* I. $c = 1/2$. In this case, after implementing conditions E-1 through E-5 and using (4.9) we get

$$m(x) := \begin{cases} c\gamma(B + A|x| - \gamma h_0 x^2) + h_0, & x \in [-L, -\overline{x}) \cup (\overline{x}, L], \\ 1, & x \in [-\overline{x}, \overline{x}] \end{cases} \tag{4.10}$$

where

$$A := [1 + \gamma(L - \overline{x})]^{-1}\{h_0(2\gamma L + \gamma^2(L^2 - \overline{x}^2) + 2) - 2\} \tag{4.11}$$

$$B := \gamma^{-1}[h_0(2L\gamma + L^2\gamma^2) - A(1 + L\gamma)], \tag{4.12}$$

and where h_0 and $\overline{x}$ satisfy

$$h_0 = \frac{2}{1 + \gamma(L - \overline{x})}\left[2\tanh \gamma\overline{x} - 2\gamma\overline{x} + \frac{2\gamma L + \gamma^2(L^2 - \overline{x}^2) + 2}{1 + \gamma(L - \overline{x})}\right]^{-1} \tag{4.13}$$
$$=: T_1(\overline{x}).$$

It is easy to show that T_1 is strictly monotone, so that this equation has solutions $\overline{x} \in [0, L]$ for $h_0 \in [(1 + \gamma L + \gamma^2 L^2/2)^{-1},\ 1 - 1/(2(1 - e^{-2\gamma L}))]$.

4.2.2. *Case* II. $c < 1/2$. Similarly, in this case we get

$$m(x) := \begin{cases} \mu_1(x), & x \in [-L, -\overline{x}) \cup (\overline{x}, L], \\ 1, & x \in [-\overline{x}, \overline{x}] \end{cases} \tag{4.14}$$

where

$$\mu_1(x) := c\gamma \left(B \cosh \gamma\sqrt{1-2c}x + A \sinh \gamma\sqrt{1-2c}|x| + \frac{2h_0}{\gamma(1-2c)} \right) + h_0. \tag{4.15}$$

As above, A and B can be found in terms of $h_0, \overline{x}, L, \gamma$, and c by solving a system of linear equations obtained from E-1, E-4, and E-5. Finally, using the constants obtained above and Eq. (4.8) we get a transcendental relationship between h_0 and $\overline{x}$ of the form

$$h_0 = T_2(\overline{x}). \tag{4.16}$$

Again, it is easy to show that T_2 is monotone so that there is a unique solution $\overline{x} \in [0, L]$ for $h_0 \in [h_c, 1 - 1/(2(1 - e^{-2\gamma L}))]$ where

$$h_c := \frac{\gamma(1-2c)}{2}[1 - (\cosh \gamma\sqrt{1-2c}L + \sqrt{1-2c}\sinh \gamma\sqrt{1-2c}L)^{-1}]^{-1}. \tag{4.17}$$

4.2.3. *Case* III. $c > 1/2$. As we shall see below, when $c \leq 1/2$ we can show a complete continuum of stationary points parametrized by h_0. In these cases the stationary points go from saturated to edge-effect to measure-valued as the magnitude of the applied field is decreased. For $c > 1/2$ this is not possible. As we indicated above, the saturated states $m \equiv \pm 1$ are stable when $\pm h_0 > 1 - c(1 - e^{-2\gamma L})$. However, in this case we can only show the existence of edge-effects solutions *locally* in a neighborhood of $\overline{x} = L$, $h_0 = 1 - c(1 - e^{-2\gamma L})$.

In particular, we get

$$m(x) := \begin{cases} \mu_2(x), & x \in [-L, -\overline{x}) \cup (\overline{x}, L], \\ 1, & x \in [-\overline{x}, \overline{x}] \end{cases} \tag{4.18}$$

where

$$\mu_2(x) := c\gamma \left(B \cos \gamma\sqrt{2c-1}x + A \sin \gamma\sqrt{2c-1}|x| + \frac{2h_0}{\gamma(1-2c)} \right) + h_0, \tag{4.19}$$

and where A and B can be obtained in the same way as above. Once again, (4.8) can be manipulated to get a transcendental relationship of the form $h_0 = T_3(\overline{x})$, where $T_3(L) = 1 - 1/(2(1 - e^{-2\gamma L}))$. In this case, we cannot show that T_3 is monotone. However, $T_3'(L) < 0$ so that there is a solution with edge effects with $(h_0, \overline{x})$ close to $(L, 1 - 1/(2(1 - e^{-2\gamma L}))$.

4.3. *Purely measure-valued solutions.* As we remarked in the previous section, for $c \leq 1/2$, the edge-effect solutions found above lose stability to stationary points that are purely measure-valued. We construct these measure-valued solutions by seeking $K : [-L, L] \to \mathbb{R}$ satisfying the following conditions.

M-1. K is even and in $C^2([-L, L])$.

M-2. In all of $(-L, L)$, K satisfies

$$K'' - \gamma^2(1-2c)K = -2\gamma h_0, \tag{4.20}$$

and, in addition,

$$|c\gamma K + h_0| < 1. \tag{4.21}$$

M-3. K satisfies the boundary conditions (3.31) and (3.32), i.e.,

$$K'(L) = -\gamma K(L),$$
$$K'(-L) = \gamma K(-L).$$

Once again, constructing m from K is straightforward.

THEOREM 4.3. If $K : [-L, L] \to \mathbb{R}$ satisfies conditions M-1–M-3, then

$$m := c\gamma K + h_0 \tag{4.22}$$

is a transverse stationary point.

4.3.1. *Case* I. $c = 1/2$. In this case, after implementing conditions M-1 through M-3 and using (4.9) we get

$$m(x) := h_0[1 + c\gamma(2L + \gamma(L^2 - x^2))]. \tag{4.23}$$

Note this satisfies the inequality (4.21) if $|h_0| < (1 + \gamma L + \gamma^2 L^2/2)^{-1}$.

4.3.2. *Case* II. $c < 1/2$. Similarly, after implementing conditions M-1 through M-3 we get

$$m(x) := h_0\left[\frac{2c}{(1-2c)}\left(1 - \frac{\cosh\gamma\sqrt{1-2c}x}{(\cosh\gamma\sqrt{1-2c}L + \sqrt{1-2c}\sinh\gamma\sqrt{1-2c}L)}\right) + 1\right]. \tag{4.24}$$

Again, note that (4.21) is satisfied if $|h_0| < h_c$, where h_c is defined in (4.17).

4.4. *Summary.* Let us put the various types of solutions found above together to get a more complete picture. We display the solutions in a *magnetization diagram* which plots the applied fields h_0 versus the average magnetization $M := \frac{1}{2L}\int_{-L}^{L} m(x)\,dx$. The cases $c \le 1/2$ exhibit similar overall pictures (see Fig. 2 on p. 282).

1. For $|h_0| > h_s$, where

$$h_s := 1 - 1/(2(1 - e^{-2\gamma L})), \tag{4.25}$$

the saturated solution $m = \operatorname{sgn} h_0$ is a stationary point.

2. For $h_c < |h_0| < 1 - 1/(2(1 - e^{-2\gamma L}))$ there are stationary points exhibiting edge effects. The solutions are saturated in the center of the plate with measure-valued (oscillatory) edges. The measure-valued regions become larger as the magnitude of the applied field is increased. The critical field h_c is given by

$$h_c := \frac{\gamma(1-2c)}{2}[1 - (\cosh\gamma\sqrt{1-2c}L + \sqrt{1-2c}\sinh\gamma\sqrt{1-2c}L)^{-1}]^{-1}, \tag{4.26}$$

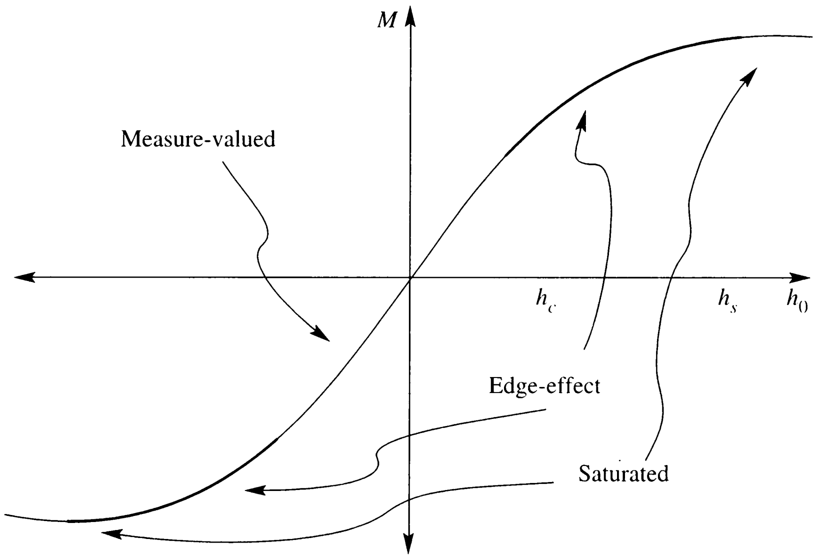

FIG. 2. Magnetization curve for $c \leq 1/2$

for $c < 1/2$ and

$$(1 + \gamma L + \gamma^2 L^2/2)^{-1} \tag{4.27}$$

when $c = 1/2$.

3. For $|h_0| < h_c$ the stationary points are purely measure-valued. The function m is linear in h_0.

For large values of c we can no longer show the existence of a continuum of stationary points. In particular, as was noted in [2], if $h_s < 0$ then there is a range of values of h_0 in which both saturated solutions are relative minimizers of the energy (i.e., there is hysteresis). Since we have multiple equilibria, we expect some sort of hysteresis loop of stationary points. We cannot at this time show the existence of such a structure of solutions analytically. However, we have shown that close to the critical value of the applied field at which the saturated states lose stability, there exists a continuum of stationary points exhibiting edge effects. (See Fig. 3.)

5. Stability of edge-effect stationary points. In the previous section, we noted our result from [3] concerning the stability of saturated solutions. In this section we discuss stability of the nonsaturated solutions found above. We show stability (actually global minimization) only for values of $c \leq 1/2$ for which there is no hysteresis.

THEOREM 5.1. Let h_0, γ, and L be given and K be as defined above. Let $c \leq 1/2$ and suppose that $m : [-L, L] \to [-1, 1]$ is a stationary point of the form constructed above, i.e.,

M-1. m is even and continuous.

M-2. There exists $\overline{x} \in [0, L)$ such that $m(x) \equiv 1$ and $h_0 \geq 1 - c\gamma K(m)(x)$ for $x \in (-\overline{x}, \overline{x})$.

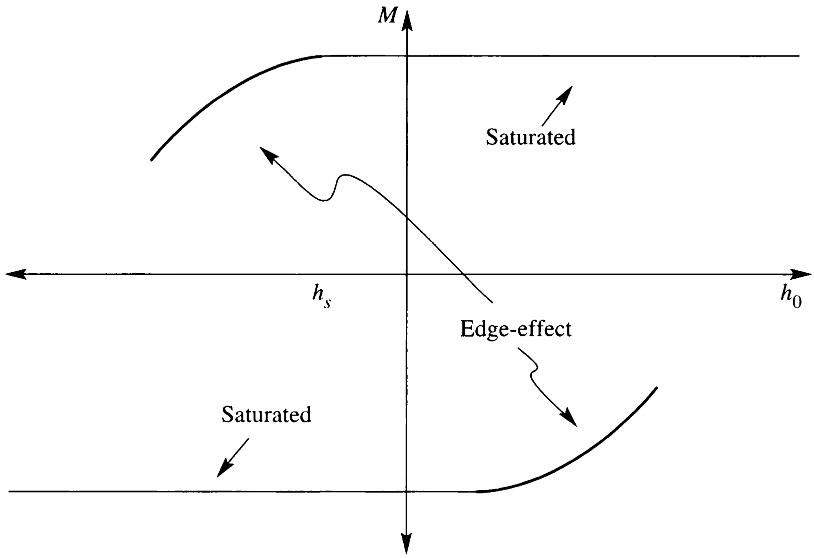

FIG. 3. Magnetization curve for $c >> 1/2$

M-3. For $x \in [-L,-\overline{x}) \cup (\overline{x}, L]$ we have $|m(x)| < 1$ and

$$m(x) = c\gamma K(m)(x) + h_0. \tag{5.1}$$

Then m is a stable global minimizer of E (defined in (3.29)).

Proof. Let $v \not\equiv 0$ be an admissible perturbation of m, i.e.,

$$|m+v| \leq 1. \tag{5.2}$$

Note that this implies

$$-2 \leq v(x) \leq 0 \quad \text{for } x \in [\overline{x}, \overline{x}]. \tag{5.3}$$

Some routine calculations using (5.1) and the Cauchy-Schwarz inequality yield

$$\begin{aligned} E(m+v) - E(m) =& \frac{1}{2}\int_{-L}^{L} v^2 - \frac{c\gamma}{2}\int_{-L}^{L}\int_{-L}^{L} e^{-\gamma|x-y|}v(x)v(y)\,dx\,dy \\ &+ \int_{-\overline{x}}^{\overline{x}} v(x)[1 - h_0 - c\gamma K(m)(x)]\,dx \\ \geq& \frac{1}{2}\int_{-L}^{L} v^2 - \frac{c\gamma}{2}\int_{-L}^{L}\int_{-L}^{L} e^{-\gamma|x-y|}v(x)v(y)\,dx\,dy \\ \geq& \left[\frac{1}{2} - c(1-e^{-\gamma L})\right]\int_{-L}^{L} v^2 \\ >& 0. \end{aligned} \tag{5.4}$$

This completes the proof. □

REMARK. Note that the case where $m \equiv -1$ on $(-\overline{x}, \overline{x})$ can be treated in the same way.

6. Conclusions. We conclude with a few remarks.

We first note that we have not ruled out the existence of stationary points that would oscillate between saturated regions of opposite sign with the saturated regions separated by measure-valued regions. Existence and metastability of such solutions (particularly for large values of c) remains an open problem.

Second, we note that one might object to the setting of the one-dimensional problem in the first place. After all, Young-measures that depend only on the variable x arise as the limit of sequences that are periodic in the y and/or z directions. Fortunately, there is no way to lower the energy per unit area of a plate whose (measure-valued) magnetization depends on x by a perturbation that is periodic in the y and z directions.

To see this, consider a periodic classical magnetization $\mathbf{m}(x, y, z) \in \mathbb{M}(\Omega)$ such that

$$\mathbf{m}(x, y, z) = \mathbf{m}(x, y + L_y, z + L_z) \quad \text{for every } (x, y, z) \in \Omega. \tag{6.1}$$

For such a magnetization, the energy per unit area of the plate is given by

$$\mathcal{E}_p(\mathbf{m}) := \frac{1}{L_y L_z} \int_{-L}^{L} \int_0^{L_y} \int_0^{L_z} D(x, y, z)\, dz\, dy\, dx, \tag{6.2}$$

where the energy density D is defined by

$$\begin{aligned} D(x, y, z) := &- \frac{1}{2}\mathbf{m}(x, y, z) \cdot \hat{\mathbf{h}}[\mathbf{m}](x, y, z) + \mathcal{W}(\mathbf{m}(x, y, z)) - h_0 \cdot \mathbf{m}(x, y, z) \\ &- \int_{-L}^{L} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x, y, z; x', y', z')\mathbf{m}(x, y, z)\mathbf{m}(x', y', z')\, dx'\, dy'\, dz'. \end{aligned} \tag{6.3}$$

Note that if we assume that the kernel takes the form

$$k(x, y, z; x', y', z') = g(|(x, y, z) - (x', y', z')|) \tag{6.4}$$

then the energy density is periodic in y and z if $\mathbf{m}$ is periodic as well.

Now suppose that the one-dimensional measure-valued magnetization μ minimizes the energy $\mathcal{E}$ over $\mathcal{M}(-L, L)$. Suppose further that there is a periodic magnetization $\mathbf{m} \in \mathbb{M}(\Omega)$ such that

$$\mathcal{E}_p(\mathbf{m}) < \mathcal{E}(\mu). \tag{6.5}$$

We now define

$$\mathbf{m}_n(x, y, z) := \mathbf{m}(x, ny, nz) \tag{6.6}$$

and note that

$$\mathbf{m}_n \overset{*}{\rightharpoonup} \mathbf{m} \quad \text{in } L^{\infty}(\Omega). \tag{6.7}$$

Furthermore, we have

$$\mathcal{E}_p(\mathbf{m}) = \mathcal{E}_p(\mathbf{m}_n), \qquad n = 1, 2, 3, \ldots . \tag{6.8}$$

Thus, if we let $\nu \in \mathcal{M}(\Omega)$ be the measure-valued magnetization corresponding to the sequence $\mathbf{m}_n$ we have

$$\mathcal{E}(\nu) \leq \lim_{n \to \infty} \mathcal{E}_p(\mathbf{m}_n) = \mathcal{E}_p(\mathbf{m}) < \mathcal{E}(\mu), \tag{6.9}$$

which contradicts the assumption that μ minimizes the energy.

In light of this observation, we could just as well have posed the problem of minimizing the energy per unit area of any magnetization that is periodic parallel to the plate. We can always lower the energy of such a function by rescaling, taking a weak limit, and using the resulting measure-valued magnetization which depends only on the x-direction.

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