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SIMILARITY SOLUTIONS TO ROTATING COUETTE FLOW OF NON-NEWTONIAN FLUIDS IN THE PRESENCE OF A MAGNETIC FIELD

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Abstract. We study the rotating Couette flow of conducting power law fluids in the presence of a magnetic field. We analyze the coupled effects of the rheology and the magnetic field on the angular velocity and shear stress distributions for the case when an infinite long cylinder rotates in an unbounded fluid. For shear thickening fluids our exact similarity solutions exhibit traveling wave characteristics determining the existence of a moving shear front. We investigate the electrorheological effect on the propagation of the shear disturbances front.

1. Introduction. The unsteady shear flows with circular streamlines are at the present time of special interest in science and engineering. While rotating Couette flows with circular streamlines are extensively studied for Newtonian fluids, it appears that the case of non-Newtonian power law fluids in the presence of a constant magnetic field has not, to our knowledge, been presented in the literature. The interest in this particular class of unsteady flows has increased in the last few years due to possible applications in electrorheology where conducting non-Newtonian fluids are currently being used. The apparent viscosity of these fluids is shear rate dependent. For a shear thickening fluid (also known as a dilatant fluid) the apparent viscosity is a monotonically increasing function of shear rate, whereas for a shear thinning fluid (also known as a pseudoplastic fluid) the opposite occurs. The rheological properties of power law fluids lead to governing equations belonging to a class of nonlinear degenerate parabolic equations. As we will show in this paper, the exact similarity solutions of these equations have, for certain values of the rheological parameters, solutions with compact support. Consequently, due to the nonlinear rheological effects, the velocity and shear stress distributions exhibit traveling wave characteristics. Note that the flow domain will be well defined due to the existence of a moving shear front and that the shear disturbances will propagate with a finite velocity. In contrast, the case of Newtonian fluids, where the viscosity is constant, leads to linear governing equations. As a result, the propagation velocity of the shear

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disturbances is infinite, since a moving front does not exist, in which case the flow domain will be ill defined. It should be pointed out that establishing a relationship between the shear stress τ and the shear rate $\dot{\gamma}$, for non-Newtonian fluids is a difficult problem from a theoretical standpoint. Several constitutive equations are currently being used. It turns out that a large class of non-Newtonian fluids obeys the so-called "power-law" model, which is now well established and is expressed by the rheological equation

$$\tau = H \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y}$$

where, in one-dimensional flow, the shear rate $\dot{\gamma}$ is the velocity gradient $\dfrac{\partial u}{\partial y}$. For a shear thickening fluid we have $n > 1$, while for a shear thinning fluid we have $n < 1$, and $n = 1$ for a Newtonian fluid.

As shown in the literature on MHD of Newtonian fluids, the presence of a magnetic field has a great influence on the flow behavior. Obviously, mathematical models capable of addressing MHD flow problems for power-law fluids are particularly desirable. Unfortunately, these problems are difficult to solve analytically since one must deal with both the Navier-Stokes equations with a nonlinear rheological equation, which characterize the flow, and Maxwell's equations which govern the electromagnetic field. This is especially true when the two fields are strongly coupled. Actually, in general, even for Newtonian flows exact analytic solutions do not exist. However, there are situations of practical and theoretical interest where engineering MHD models can be simplified to the extent that one deals with decoupled governing equations. Such is the case when the imposed magnetic field is constant. As an example we mention the problem of a magnetic boundary layer where a constant magnetic field is applied in a transverse direction to the laminar boundary layer flow of a Newtonian fluid past a flat plate.

Other illustrative examples include magnetic Couette flows and channel flows (known as Hartman flows) in which the magnetic field is applied in a transverse direction to the channel walls. There is an extensive literature on this subject; thus, we will not discuss here these MHD engineering problems since the interested reader may refer to several excellent review papers (see, for example, [11]–[17]).

In order to illustrate the nonlinear rheological effects on MHD flows, the problems described above have been extended to the case of non-Newtonian power-law fluids (see, for example, references [5]–[10]). It appears, however, that little attention has been given to the unsteady rotating Couette flows of conducting power-law fluids in the presence of a constant magnetic field. In this case the flow has circular streamlines in a cylindrical geometry and the velocity depends only on the radial coordinate and time. The applied constant magnetic field is in the axial direction, i.e., parallel to the axis of symmetry of the cylinder. The problem that is investigated in this paper involves the impulsive spin-down of a cylinder immersed in a power-law fluid of infinite extent. The diffusion of a line vortex will also be discussed.

The study of rotating Couette flows in the presence of a constant magnetic field is stimulated by the possibility of determining the rheological properties of conducting power-law fluids by using measurements obtained from classical rotating viscosymmetric flows.

This paper is also motivated by a desire to have a better understanding of the effects of the coupling of the nonlinear rheology and the magnetic field on the rotating Couette flow. The results obtained from our investigation allow us to emphasize the qualitative differences between Newtonian and non-Newtonian flow behaviors in the presence of a constant magnetic field.

2. Governing equations. In this section we are concerned with the derivation of the equations governing the unsteady flow with circular streamlines generated by a solid circular cylinder of infinite length rotating in a conducting power-law fluid. We consider the case of an unbounded fluid. The presence of a constant magnetic field applied parallel to the axis of symmetry of the cylinder introduces an additional term in the momentum equation. For illustrative purposes we will assume that this term depends linearly on the tangential velocity. In this case the Navier-Stokes equations for an unsteady creeping flow with circular streamlines become [18]

$$\frac{\rho v^2}{R} = \frac{\partial p}{\partial R} \tag{1}$$

and

$$\frac{\partial v}{\partial t} + \frac{\sigma B_0^2 v}{\rho} = \frac{1}{\rho R^2}\frac{\partial}{\partial R}(R^2 \tau_{R\theta}) \ , \tag{2}$$

where v is the tangential velocity, p is the pressure, ρ is the mass density, σ is the electrical conductivity, B_0 is the induction of the external transverse magnetic field and $\tau_{R\theta}$ is the shear stress, which for a power-law fluid is expressed in terms of shear rate $\dot{\gamma}$ in the form

$$\tau_{R\theta} = H\dot{\gamma}\left|\dot{\gamma}\right|^{n-1} \quad . \tag{3}$$

In the rheological equation (3), H is the consistency coefficient, n is the power-law exponent, and $\dot{\gamma}$ is the shear rate for which we have the relation

$$\dot{\gamma} = \frac{\partial v}{\partial R} - \frac{v}{R} \quad . \tag{4}$$

Taking into account the relation between the tangential velocity v and the angular velocity Ω,

$$v = R\Omega, \tag{5}$$

then (4) becomes

$$\dot{\gamma} = R\frac{\partial \Omega}{\partial R}, \tag{6}$$

and therefore from (3) we have

$$\tau_{R\theta} = HR\frac{\partial \Omega}{\partial R}\left|R\frac{\partial \Omega}{\partial R}\right|^{n-1} \quad . \tag{7}$$

The previous equations (2) and (7) lead to a nonlinear diffusion equation for determining the angular velocity distribution expressed in the form

$$\frac{1}{R^3}\frac{\partial}{\partial R}\left[-R^2\left(-R\frac{\partial \Omega}{\partial R}\right)^n\right] = a^2\left(\frac{\partial \Omega}{\partial t} + \delta\Omega\right) \tag{8}$$

where

$$\delta = \sigma B_0^2/\rho \quad \text{and} \quad a^2 = \rho/H \quad . \tag{9}$$

Equation (8) belongs to a class of nonlinear degenerate parabolic equations having, for certain values of n, solutions with compact support. Note that $\Omega(R,t)$ is a decreasing function of radial distance R, in which case $\left|R\frac{\partial\Omega}{\partial R}\right| = -R\frac{\partial\Omega}{\partial R}$.

We mention that the case $n > 1$ corresponds to a shear thickening fluid, while $n < 1$ corresponds to a shear thinning fluid. For $n = 1$, the case of a Newtonian fluid, we obtain from (8) the following linear equation:

$$\frac{1}{R^3}\frac{\partial}{\partial R}\left(R^3\frac{\partial\Omega}{\partial R}\right) = a^2\left(\frac{\partial\Omega}{\partial t} + \delta\Omega\right) \tag{10}$$

with $a^2 = \rho/\mu$, μ being the Newtonian viscosity.

3. Similarity solutions in closed form. To illustrate the nonlinear effects of electrorheological fluids on the behavior of unsteady rotating Couette flow in the presence of a constant magnetic field we focus in this section on finding some exact similarity solutions to Eq. (8). We are particularly interested in a class of unsteady shear flows generated by a line impulse of angular momentum. This class is associated with the Cauchy problem with a Dirac delta function as the initial condition, whose solution is known as the instantaneous point source solution to Eq. (8). It should be pointed out that this solution describes the rotating flow with circular streamlines generated by the instantaneous application of a given angular momentum to impulsively rotate the cylinder about its axis in an unbounded conducting power-law fluid. In this case, the angular velocity distribution $\Omega(R,t)$ must satisfy the integral

$$2\pi\int_0^\infty \Omega R^3 dR = M, \tag{11}$$

where M is the line impulse of angular momentum.

By using the function $\phi(R,t)$, which is related to $\Omega(R,t)$ by

$$\Omega(R,t) = e^{-\delta t}\phi(R,t), \quad \delta > 0 \quad , \tag{12}$$

Eq. (8) is reduced to

$$\frac{1}{R^3}\frac{\partial}{\partial R}\left[-R^2\left(-R\frac{\partial\phi}{\partial R}\right)^n\right] = a^2\frac{\partial\phi}{\partial\tau} \tag{13}$$

where

$$\tau = \frac{1}{(n-1)\delta}\left(1 - e^{-(n-1)\delta t}\right) \quad ; \tag{14}$$

In the case when the magnetic field is absent, i.e., $\delta = 0$, (8) reduces to

$$\frac{1}{R^3}\frac{\partial}{\partial R}\left[-R^2\left(-R\frac{\partial\Omega}{\partial R}\right)^n\right] = a^2\left(\frac{\partial\Omega}{\partial t}\right), \tag{15}$$

which is similar to (13).

To find an exact similarity solution to Eq. (13), we employ the similarity transformation

$$\phi = \tau^{\alpha} f(\eta) \tag{16}$$

where the similarity variable is defined as

$$\eta = R\tau^{\beta} \quad . \tag{17}$$

The transformation (16) reduces Eq. (13) to the following ordinary differential equation in $f(\eta)$:

$$\frac{d}{d\eta}\left[-\eta^2\left(-\eta\frac{df}{d\eta}\right)^n\right] = a^2\eta^3\left(\alpha f + \beta\eta\frac{df}{d\eta}\right) \tag{18}$$

provided that α and β satisfy the relation

$$(1-n)\alpha = 1 + 2\beta \quad . \tag{19}$$

We now consider an instantaneous point source solution obtained from (18) for $\alpha = 4\beta$. In this case (18) becomes

$$\frac{d}{d\eta}\left[-\eta^2\left(-\eta\frac{df}{d\eta}\right)^n\right] = a^2\beta\frac{d}{d\eta}(\eta^4 f) \tag{20}$$

where from (19) for $\alpha = 4\beta$ one has

$$\alpha = -\frac{2}{2n-1} \quad \text{and} \quad \beta = -\frac{1}{2(2n-1)} \quad . \tag{21}$$

Equation (20) may be integrated exactly to yield

$$\eta^2\left(-\eta\frac{df}{d\eta}\right)^n = \frac{a^2}{2(2n-1)}\eta^4 f + C \tag{22}$$

where C is an integration constant.

An instantaneous point source solution requires $\left.\frac{df}{d\eta}\right|_{\eta=0} = 0$, in which case $C = 0$ in (22). As a result Eq. (22) may be integrated to yield

$$f(\eta) = \left[C^* - \frac{n-1}{2}\left(\frac{a^2}{2(2n-1)}\right)^{1/n}\eta^{2/n}\right]^{n/(n-1)} \quad . \tag{23}$$

Consequently, from (12), (16), and (23) we obtain

$$\Omega = e^{-\delta t}\tau^{\alpha}\left[C^* - \frac{n-1}{2}\left(\frac{a^2}{2(2n-1)}\right)^{1/n}\eta^{2/n}\right]^{n/(n-1)} \quad , \tag{24}$$

where C^* is also an integration constant.

We observe that Ω defined by (24) is a function with compact support if and only if $n > 1$, in which case we can write it in terms of a new constant η_1 in the form

$$\Omega = Ae^{-\delta t}\tau^{\alpha}\left(\eta_1^{2/n} - \eta^{2/n}\right)^{n/(n-1)}, \quad n > 1, \tag{25}$$

for $\eta < \eta_1$, and $\Omega = 0$ for $\eta \geq \eta_1$, where

$$A = \left[\frac{n-1}{2}\left(\frac{a^2}{2(2n-1)}\right)^{1/n}\right]^{n/(n-1)}. \tag{26}$$

The exact similarity solution (25) shows the existence of traveling wave characteristics for the angular velocity distribution if and only if the power-law fluid is of shear thickening behavior. The solution for the case without a magnetic field is recovered when in (25) $\delta = 0$, $\eta = Rt^{-1/2(2n-1)}$, and $\tau = t$.

For a shear thinning fluid, i.e., $n < 1$, a traveling wave solution does not exist, since for this case

$$\Omega = \frac{e^{-\delta t}\tau^{\alpha}}{A^*\left(\eta_1^{2/n} + \eta^{2/n}\right)^{n/(1-n)}}, \quad n < 1, \tag{27}$$

where

$$A^* = \left[\frac{1-n}{2}\left(\frac{a^2}{2(2n-1)}\right)^{1/n}\right]^{n/(1-n)}. \tag{28}$$

Obviously, the similarity solution (27) shows that $\Omega(\eta) = 0$ only for $\eta \to \infty$. As a result, the shear disturbances will propagate with an infinite velocity, resulting in an ill-defined domain of shear diffusion. It should be noted that this situation is also characteristic of a Newtonian fluid, where the governing equation (10) is linear. The case $n > 1$ leads to solution (25) which reveals that the flow domain is well defined, due to the existence of a moving shear front and, consequently, the shear disturbances will propagate with a finite velocity. The existence of solutions with compact support for $t > 0$ indicates a significant qualitative difference between the Newtonian and shear thickening power-law fluids in rotating Couette flow.

In order to determine the shear stress distribution we have from (7)

$$\tau_{R\theta} = He^{-n\delta t}\tau^{n\alpha}\left(-\eta\frac{df}{d\eta}\right)^n, \tag{29}$$

and, consequently, from the previous relations we obtain

$$\tau_{R\theta} = HAe^{-n\delta t}\tau^{n\alpha}\left(\frac{a^2}{2(2n-1)}\right)\eta^2\left(\eta_1^{2/n} - \eta^{2/n}\right)^{n/(n-1)} \tag{30}$$

for $\eta < \eta_1$ and $\tau_{R\theta} = 0$ for $\eta \geq \eta_1$. As a result, the exact similarity solutions (25) and (30) lead to the shear front conditions $\Omega(R \geq l(t), t) = 0$ and $\left.\dfrac{\partial\Omega}{\partial R}\right|_{R\geq l(t)} = 0$, where $l(t)$ is the shear front location for which we have from (14) and (17)

$$l(t) = \eta_1\left[\frac{1 - e^{-(n-1)\delta t}}{(n-1)\delta}\right]^{\frac{1}{2(2n-1)}}, \quad n > 1, \tag{31}$$

where η_1 is a constant, as required by a similarity solution, which will be determined further on. From (31) the propagation velocity of shear disturbances, i.e., the front velocity $v_f = \frac{dl}{dt}$, will be

$$v_f = \frac{\eta_1(3-4n)}{2n-1}\left[(n-1)\delta\right]^{(4n-3)/(2(2n-1))} e^{-(n-1)\delta t}\left(1-e^{-(n-1)\delta t}\right)^{(3-4n)/(2(2n-1))} . \tag{32}$$

For $\delta = 0$, i.e., in the absence of a magnetic field, the previous relations (31) and (32) become

$$l(t) = \eta_1 t^{1/2(2n-1)}, \quad n > 1 \tag{33}$$

and

$$v_f = \frac{\eta_1 t^{(3-4n)/(2(2n-1))}}{2(2n-1)} . \tag{34}$$

In view of (31), it turns out that the presence of a magnetic field gives rise to a front extinction. When $t \to \infty$ we have from (31) the location

$$l^* = \lim_{t\to\infty} l(t) = \eta_1\left[(n-1)\delta\right]^{1/(2(1-2n))} \tag{35}$$

from which the shear front cannot advance. In contrast, in the situation when there is no magnetic field, the shear front continues to advance unboundedly in time, since from (33) we observe that $l(t) \to \infty$ for $t \to \infty$. This relevant qualitative difference between the cases with and without a magnetic field is in accordance with the physical expectations. In addition, we can see from (25) and (30) that for small time the presence of a magnetic field does not have any significant effect on the flow behavior. However, it is evident that its effect is a reduction in time of the angular velocity and shear stress as compared with the case $\delta = 0$; in other words, the magnetic field slows down the flow.

In order to determine the shear stress acting on the cylinder wall and the angular velocity of the cylinder, from (25) and (30) we obtain the following relations:

$$\tau_{R\theta}(R_0, t) = \frac{a^2}{2(2n-1)} HAR_0^2\eta_1^{2/(n-1)} e^{-n\delta t}\tau^{2\beta(2n+1)} \tag{36}$$

and

$$\Omega(R_0, t) = A\eta_1^{2/(n-1)} e^{-\delta t}\tau^{4\beta} \tag{37}$$

where R_0 is the cylinder radius and τ is expressed in terms of t by relation (14). Note that in (25) and (30) we have used the valid approximation $\eta_1^{2/n} - \eta_w^{2/n} \cong \eta_1^{2/n}$, since $\eta_1 \gg \eta_w$; $\eta_w = R_0\tau^\beta$.

From (37) it turns out that the line impulse of angular momentum generates a variable angular velocity which declines in time. The constant η_1 can be determined from the integral (11), which due to the presence of a magnetic field, i.e., $\delta \neq 0$, is a function of time expressed as

$$M(t) = 2\pi\int_0^{l(t)} \Omega R^3 dR . \tag{38}$$

Taking into account (25) this integral is written in terms of the similarity variable η in the form

$$M(t) = 2\pi A\eta_1^{(2n-1)/(n-1)} e^{-\delta t} \int_0^1 \xi^3 \left(1 - \xi^{2/n}\right)^{n/(n-1)} d\xi \tag{39}$$

where $\xi = \eta/\eta_1$.

Expressing the integral in (39) by means of the Euler beta function B^* we obtain

$$M(t) = 2\pi A\eta_1^{(2n-1)/(n-1)} e^{-\delta t} B^*(p,q) \quad , \tag{40}$$

where

$$B^*(p,q) = \frac{n\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad , \quad p > 0 \quad \text{and} \quad q > 0, \tag{41}$$

with

$$p = 2n \quad \text{and} \quad q = \frac{2n-1}{n-1} \quad , \quad n > 1, \tag{42}$$

and Γ being the gamma function.

Relation (40) shows that M is no longer a constant as in the case without a magnetic field but varies in time as

$$M(t) = M_0 e^{-\delta t} \quad , \tag{43}$$

where M_0 = constant is the line impulse of angular momentum initially applied at the cylinder. The previous relations (40) and (43) determine η_1 = constant in terms of the rheological parameter n,

$$\eta_1 = \left[\frac{M_0}{2\pi n A B^*(p,q)}\right]^{(n-1)/(2n-1)} \quad . \tag{44}$$

On the other hand, based on the results shown above, we can point out that the Cauchy problem, having an instantaneous point source solution, is equivalent to the first boundary-value problem with the condition (37) and the shear front conditions $\Omega(R \geq l(t), t) = 0$ and $\left.\dfrac{\partial \Omega}{\partial R}\right|_{R \geq l(t)} = 0$.

The Newtonian case, i.e., $n = 1$, can be easily recovered from the previous results by means of the relation

$$\lim_{n\to 1} \left[1 - \frac{n-1}{2}\left(\frac{a^2}{2(2n-1)}\right)^{1/n} \eta^{2/n}\right]^{n/(n-1)} = \exp\left(-\frac{a^2\eta^2}{4}\right) \quad . \tag{45}$$

Figures 1 and 2 show the effect of the nonlinear rheology on the dimensionless angular velocity and shear stress distributions respectively, obtained from (25) and (30), where

$$\Omega^* = \Omega\left[A\eta_1^{2/(n-1)} e^{-\delta t} \tau^{\alpha}\right]^{-1}$$

and

$$\tau^* = \tau_{R\theta}\left[HA\eta_1^{2/(n-1)}\left(\frac{a^2}{2(2n-1)}\right) e^{-n\delta t}\tau^{n\alpha}\right]^{-1} \quad . \tag{46}$$

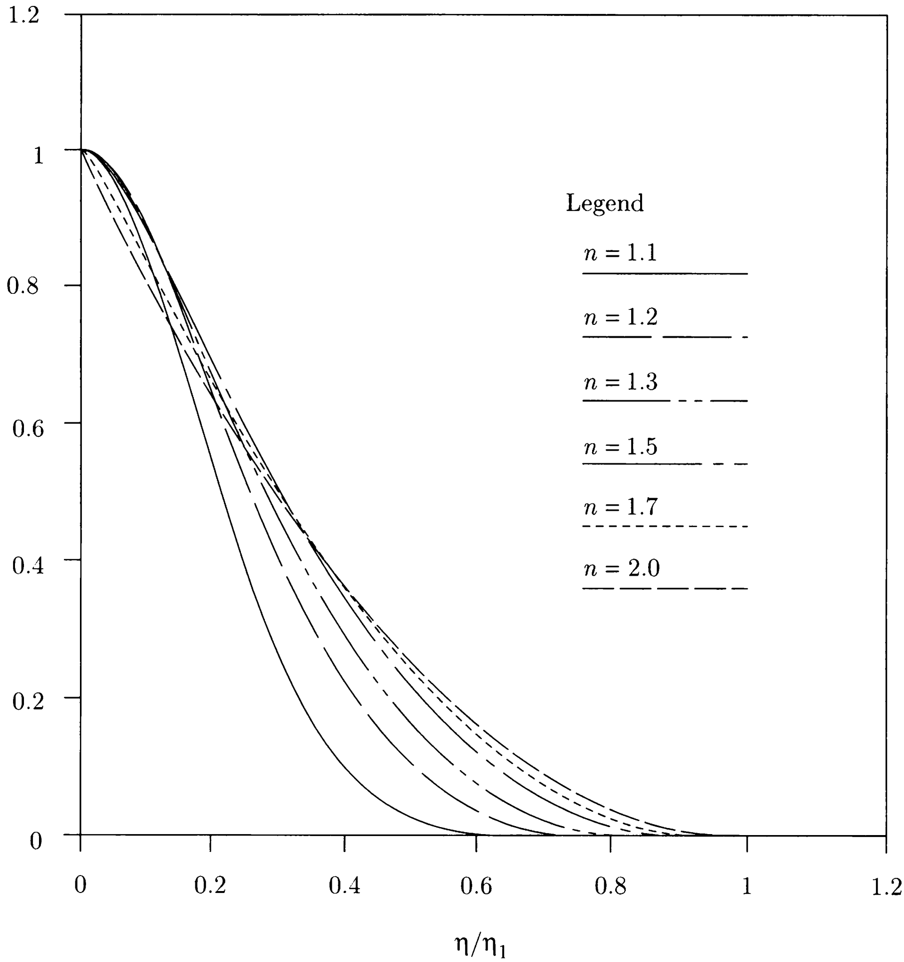

FIG. 1. Effect of rheology on the dimensionless angular velocity

We now focus our attention on the case of the first boundary-value problem when the angular velocity, $\Omega(\eta_0)$, of the cylinder is a certain function of time required by a similarity solution. We can determine an exact similarity solution when $\alpha = 0$ in (18) and (19), which leads to

$$\frac{d^2 f}{d\eta^2} + \left(\frac{2+n}{n}\right)\frac{1}{\eta}\frac{df}{d\eta} - \frac{a^2}{2n}\eta^{2-n}\left(-\frac{df}{d\eta}\right)^{2-n} = 0 \quad , \tag{47}$$

where

$$\beta = -1/2 \quad \text{and} \quad \eta = R\tau^{-1/2} \quad . \tag{48}$$

Equation (47) may be reduced to a Bernoulli's equation so that its solution may be

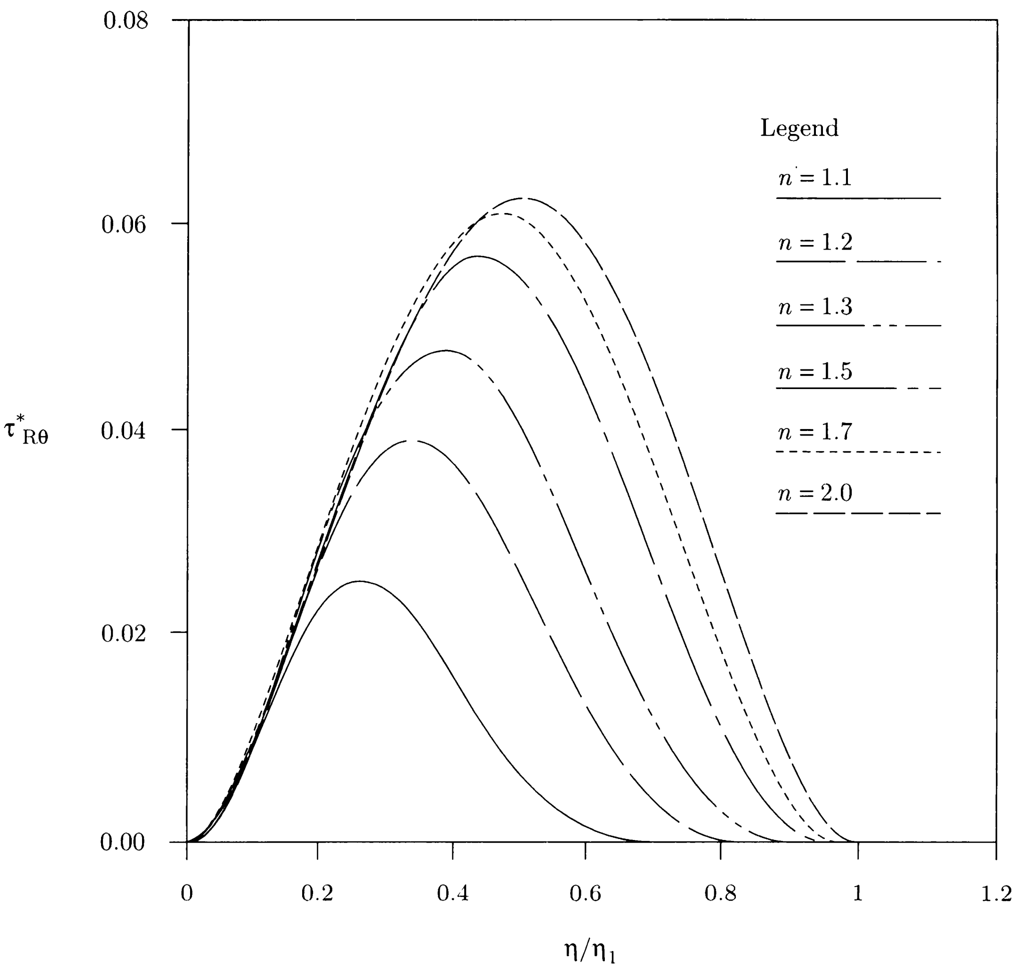

FIG. 2. Effect of rheology on the dimensionless shear stress

written as

$$\frac{df}{d\eta} = -\eta^{-(2+n)/n}\left[C - \frac{(n-1)a^2}{4(2n-1)}\eta^{2(2n-1)/n}\right]^{1/(n-1)} . \tag{49}$$

From (29), with $\alpha = 0$, and (49) the shear stress distribution will be

$$\tau_{R\theta} = He^{-n\delta t}\eta^{-2}\left[C - \frac{(n-1)a^2}{4(2n-1)}\eta^{2(2n-1)/n}\right]^{n/(n-1)} , \tag{50}$$

where C is an integration constant. This similarity solution also reveals the existence of traveling wave characteristics if and only if $n > 1$, in which case we can write (50) in the form

$$\tau_{R\theta} = DHe^{-n\delta t}\eta^{-2}\left[\eta_1^{2(2n-1)/n} - \eta^{2(2n-1)/n}\right]^{n/(n-1)} \tag{51}$$

for $\eta < \eta_1$ and $\tau_{R\theta} = 0$ for $\eta \geq \eta_1$, where

$$D = \left[\frac{(n-1)a^2}{4(2n-1)}\right]^{n/(n-1)} . \tag{52}$$

Since $\alpha = 0$, we have from (12) and (16), $\dfrac{d\Omega}{d\eta} = e^{-\delta t}\dfrac{df}{d\eta}$ and, consequently, the integration of (49) yields

$$\Omega(\eta) = \Omega(\eta_0) - D^{1/n}\eta_1^{2/(n-1)}e^{-\delta t}J_n\left(\frac{\eta}{\eta_1}\right) , \tag{53}$$

where $\eta_0 = R_0\tau^{-1/2}$ and

$$J_n\left(\frac{\eta}{\eta_1}\right) = \int_{\eta_0/\eta_1}^{\eta/\eta_1} \xi^{-(2+n)/n}\left(1 - \xi^{2(2n-1)/n}\right)^{1/(n-1)} d\xi . \tag{54}$$

To determine $\eta_1 =$ constant in (53), as required by a similarity solution, we observe that the angular velocity of the cylinder must be of the form $\Omega(\eta_0) = \Omega_0 e^{-\delta t}$, where Ω_0 is the initial angular velocity. On the other hand, taking into account the shear front condition $\Omega(\eta_1) = 0$, from (53) it turns out that

$$\eta_1 = \left[\frac{\Omega_0}{D^{1/n}J_n(1)}\right]^{(n-1)/2} \tag{55}$$

so that (53) may now be rewritten in the form

$$\Omega = \Omega_0 e^{-\delta t}\left[1 - \frac{J_n(\eta/\eta_1)}{J_n(1)}\right] \tag{56}$$

for $\eta < \eta_1$ and $\Omega = 0$ for $\eta \geq \eta_1$, where $J_n(1)$ is determined from (54) for $\eta = \eta_1$.

The expression for the shear stress acting on the cylinder wall is obtained from (51) and expressed as

$$\tau_{R\theta}(R_0, t) = DHR_0^{-2}\eta_1^{2(2n-1)/(n-1)}e^{-n\delta t}\left(1 - e^{-(n-1)\delta t}\right)/((n-1)\delta) . \tag{57}$$

Note that (57) was obtained from (51) by taking into account the fact that $\eta_1 \gg \eta_0$, where $\eta_0 = R_0\tau^{-1/2}$, τ being given by (14).

For $\alpha = 0$ the shear front location is

$$l(t) = \eta_1\left[\left(1 - e^{-(n-1)\delta t}\right)/((n-1)\delta)\right]^{1/2} , \quad n > 1, \tag{58}$$

whereas for the front velocity we obtain

$$v_f = \frac{dl}{dt} = \frac{1}{2}\eta_1 e^{-(n-1)\delta t}\left[\left(1 - e^{-(n-1)\delta t}\right)/((n-1)\delta)\right]^{-1/2} . \tag{59}$$

For small time, i.e., $t \to 0$, we can use the approximation $1 - e^{-(n-1)\delta t} \cong (n-1)\delta t$. As a result, instead of (57), (58), and (59) we have

$$\tau_{R\theta}(R_0, t) = DHR_0^{-2}\eta_1^{2(2n-1)/(n-1)}te^{-n\delta t} , \tag{60}$$

$$l(t) = \eta_1 t^{1/2}, \tag{61}$$

and

$$v_f = \frac{dl}{dt} = \frac{1}{2}\eta_1 t^{-1/2}, \tag{62}$$

which reveal that in the case of small time the effect of the magnetic field on the flow behavior is insignificant.

It should be pointed out that the exact similarity solution corresponding to $\alpha = 0$ indicates similar conclusions to the case $\alpha = 4\beta$ (i.e., the Cauchy problem) concerning the effect of the magnetic field on the unsteady flow behavior, except for the fact that η_1 is determined differently in the two cases.

4. Diffusion of a vortex in a power-law fluid in the presence of a magnetic field. Vortices in unsteady shear flows are of special interest in engineering. As is well known, these shear flows describe the process of diffusion of vorticity, since we are dealing with creeping flows. In general, the solutions of the equations governing the flows of power-law fluids can be found only by numerical methods. However, as we have shown in the previous sections, for a certain class of unsteady flows, exact similarity solutions can be found.

Thus far, little attention has been given to the viscous diffusion of a vortex in a non-Newtonian fluid of power-law behavior. For illustrative purposes, here we will investigate the diffusion of a rectilinear vortex in an unbounded power-law fluid, for which the vorticity $\omega_z(R,t)$ satisfies the relation

$$\omega_z = \frac{\partial v}{\partial R} + \frac{v}{R} \quad , \quad \omega_R = \omega_\theta = 0, \tag{63}$$

or, in terms of angular velocity Ω, we have

$$\omega_z = R\frac{\partial \Omega}{\partial R} + 2\Omega \quad . \tag{64}$$

From relations (12), (16), and (17) we can write (64) in the form

$$\omega_z = e^{-\delta t}\tau^{\alpha}\left(\eta\frac{df}{d\eta} + 2f\right) \quad . \tag{65}$$

Based on the instantaneous point source solution (25), it turns out that for $\alpha = 4\beta$ a decaying vortex generated by a line impulse of angular momentum, expressed as a Dirac delta function, leads to the following expression for the vorticity:

$$\omega_z = 2Ae^{-\delta t}\tau^{\alpha}\left[\left(\eta_1^{2/n} - \eta^{2/n}\right)^{n/(n-1)} - \frac{1}{n-1}\eta^{2/n}\left(\eta_1^{2/n} - \eta^{2/n}\right)^{1/(n-1)}\right] \tag{66}$$

for $\eta < \eta_1$ and $\omega_z = 0$ for $\eta \geq \eta_1$.

The solution (56), corresponding to the first boundary-value problem, yields for the vorticity

$$\omega_z = e^{-\delta t}\left[2\Omega_0\left(1 - \frac{J_n(\eta/\eta_1)}{J_n(1)}\right) - D\eta^{-2/n}\left(\eta_1^{2(2n-1)/n} - \eta^{2(2n-1)/n}\right)^{1/(n-1)}\right] \tag{67}$$

for $\eta < \eta_1$ and $\omega_z = 0$ for $\eta \geq \eta_1$. Note that A in (66) is given by (26), while D in (67) is given by (52).

The exact similarity solutions (66) and (67) allow us to assess the effects associated with the presence of a constant magnetic field on the diffusion of a rectilinear vortex. These effects are similar to those previously shown for the angular velocity and shear stress distributions. In addition, we observe that near the axis one can assume that $1 - (\eta/\eta_1)^{2/n} \sim 1$, in which case from (66) we obtain

$$\omega_z \cong 2Ae^{-\delta t}\tau^{\alpha}\eta_1^{2/(n-1)} \tag{68}$$

which compared with (37) shows that near the axis the vorticity is twice the angular velocity at the cylinder wall.

5. Concluding remarks. We have developed a mathematical model for the unsteady rotating Couette flow of power-law fluids in the presence of a constant magnetic field. The governing equations belong to a class of nonlinear degenerate parabolic equations. The exact similarity solutions of these equations for the Cauchy problem with a Dirac delta function as the initial condition, known as an instantenous point source solution, and for the first boundary value problem have been found. These closed-form solutions indicate that for shear thickening fluids, the angular velocity and shear stress distributions are functions with compact support and, consequently, have traveling wave characteristics. Thus, for shear thickening fluids there exists a moving shear disturbance front that determines a well-defined flow domain. We have determined the shear front location as a function of time and the velocity of propagation of the front. In addition, we found that in the presence of a constant magnetic field the location of the moving shear front tends to a finite value in time; this means a front extinction. In contrast, for Newtonian and shear thinning fluids, the propagation velocity of the shear disturbances is infinite, resulting in an ill-defined flow domain. We have analyzed the effects of the magnetic field on the flow behavior and have emphasized the fact that the magnetic field reduces the magnitude of the angular velocity and shear stress with increasing time. In other words, the magnetic field slows down the flow. We have also shown that for small time the magnetic field effect on the flow is insignificant.

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