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# CIRCULARLY SYMMETRIC DEFORMATION OF SHALLOW ELASTIC MEMBRANE CAPS

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**Abstract.** We consider shallow elastic membrane caps that are rotationally symmetric in their undeformed state, and investigate their deformation under small uniform vertical pressure and a given boundary stress or boundary displacement. To do this we use the small-strain theory developed by Bromberg and Stoker, Reissner, and Dickey. We deal with the two-parameter family of membranes whose undeformed configuration is given in cylindrical coordinates as

$$z(x) = C(1 - x^{\gamma}), \tag{1}$$

which includes the spherical cap as a special case ($\gamma = 2$ and $C$ small). We show that if $\gamma > 4/3$ then a circularly symmetric deformation is possible for any positive boundary stress (or any boundary displacement) and any positive pressure, but if $1 < \gamma < 4/3$ then no circularly symmetric deformation is possible if the stress and pressure are positive and small (or for non-positive boundary displacement and small positive pressure).

**1. Introduction.** This study addresses how an elastic membrane cap deforms under the influence of particular stress and body force. We show, among other things, that under certain conditions a cap that is initially rotationally symmetric will fail to have a symmetric configuration after deformation.

To be specific, we consider an elastic membrane cap that is shallow (that is, nearly flat) and rotationally symmetric in its undeformed state (or reference configuration), and then investigate the shape that the cap takes on when radial stress is applied on the boundary and a small uniform vertical pressure $P$ is applied to the membrane. We use the small-strain, small-pressure theory developed by Bromberg and Stoker [2], Reissner [15], and Dickey [5, 6] (a generalization to curved membranes of the Föppl theory for plane membranes), which allows for large displacements of the membrane. Dickey [5, 6, 7] showed that under the assumptions of small strain and small pressure, the radial stress $\sigma_r$ on a membrane whose undeformed profile is given in cylindrical coordinates by

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$z = z(r)$ is determined by the following equation:

$$r^2 \frac{d^2}{dr^2}\sigma_r + 3r\frac{d}{dr}\sigma_r = \frac{E}{2}\left(\frac{d}{dr}z\right)^2 + \nu E^2 \frac{d}{dr}(rG)\frac{G}{\sigma_r} - \frac{E^3}{2}\left(\frac{G}{\sigma_r}\right), \tag{2}$$

where

$r$ is the radial variable;

$E$ is Young's modulus;

$\nu$ is the Poisson ratio, $0 < \nu < 1/2$;

and $G$ is given by

$$G(r) = \frac{1}{Ehr}\int_0^r tP(t)\,dt, \tag{3}$$

with $h$ the thickness of the membrane and $P(r)$ the applied pressure.

For simplicity we will assume that $P(r) \equiv P$ a constant and that the undeformed surface has the form

$$z(r) = C\left[1 - \left(\frac{r}{a}\right)^\gamma\right] \tag{4}$$

where $a$ is the radius of the cap, $C > 0$ is the height at the center of the cap, and $\gamma > 1$. We work with this family of caps because it is of a sufficiently simple and specific form to allow us to analyze the deformation, but it is general enough that we will be able to use it to demonstrate that even convex caps that can be described in a fairly simple way may, under the influence of rotationally symmetric pressure and boundary stress, fail to deform symmetrically. In addition, if $\gamma = 2$ and $C$ is small, (4) approximates the shape of a section of a sphere, as Dickey [7] showed. This spherical-cap case with $\nu = 0$ is a well-known problem, which has been studied by Reissner [15], Baxley [1], and Goldberg [10].

It is convenient to introduce the notation

$$x = r/a; \tag{5}$$

$$K = \left(\frac{EP^2a^2}{h^2}\right)^{1/3}; \tag{6}$$

$$\lambda = \frac{C\gamma h}{Pa^2}K; \tag{7}$$

$$S_r(x) = \frac{\sigma_r(r)}{K}. \tag{8}$$

In this new notation and with the added assumption that $\nu \ll 1/2$, (2) becomes (see [7])

$$x^2 S_r'' + 3xS_r' - \frac{\lambda^2}{2}x^{2\gamma-2} + \frac{x^2}{8S_r^2} = 0 \tag{9}$$

(where $'$ means $\frac{d}{dx}$). We will write (9) as

$$F(S_r) = 0. \tag{10}$$

Regularity of this equation at the origin requires

$$\lim_{x \to 0} x^3 S_r'(x) = 0, \tag{11}$$

and we also assume that $S_r(0)$ is finite. Given a solution $S_r$ to (10), one can then calculate the actual shape of the membrane using

$$U_{S_r}(x) = x(xS_r' + S_r), \tag{12}$$

$$W_{S_r}(x) = \int_x^1 \frac{t}{S_r}\, dt, \tag{13}$$

where $U$ is the radial displacement of the membrane and $W$ is the vertical displacement [7].

We can now specify two problems of interest. In the first the *stress* at the boundary is specified:

$$S_r(1) = S; \tag{14}$$

and in the other the *radial displacement* at the boundary is specified:

$$U_{S_r}(1) = \Gamma. \tag{15}$$

We will call the former the *stress problem* and the latter the *displacement problem*. We will deal with the stress problem first, and then use that solution to address the displacement problem.

Several special cases have already been investigated. Callegari and Reiss [3]; Callegari, Keller, and Reiss [4]; Dickey [8]; Weinitschke [16]; and Hencky [11] considered the plane membrane ($C = 0$); in [3] Callegari and Reiss showed that for the plane membrane, the stress problem always has a solution for $S > 0$, as does the displacement problem for $\Gamma = 0$ (the fixed-boundary case). Baxley [1] proved that for the spherical cap also, the displacement problem with $\Gamma = 0$ has a solution.

Dickey [7] showed that for general $\gamma > 1$ and $C > 0$, the stress problem has a solution if $S$ is large, and the displacement problem with $\Gamma = 0$ has a solution if the pressure $P$ is large. He also investigated in detail the case $\gamma = 4/3$, and showed that if $P$ is large then the stress problem has a solution for all $S > 0$ and the displacement problem has a solution for all $\Gamma$; but that if $P$ is small, the stress problem has a solution only for $S$ large and the displacement problem has a solution only for $\Gamma$ large.

In this study we find that $\gamma = 4/3$ is in fact a boundary case. Specifically, if $\gamma > 4/3$, then the stress problem and displacement problem have solutions regardless of the value of $P$ (Theorems 2.5 and 3.2); but if $1 < \gamma < 4/3$, then the stress [or displacement] problem has no solution if $P$ is small and $S$ is small (Theorem 2.4) [or $\Gamma$ is nonpositive (Theorem 3.3)]. The nonexistence result is perhaps most important, since it says that there is an entire family of membranes that will fail to deform symmetrically if they are subjected to small pressure and small stretching at the boundary. A nonsymmetric deformation presumably means that the membrane develops wrinkles; if this is so, we can interpret the nonexistence of a solution qualitatively to mean that the membrane will wrinkle if it is not stretched out enough by pressure or boundary stress to pull out the wrinkles.

Section 2 of this paper presents results for the stress problem, and Section 3 applies these to derive corresponding results for the displacement problem.

**2. The stress problem.** In this section we show existence of solutions to the stress problem for all values of $S > 0$ if $\gamma > 4/3$, and we show that for $1 < \gamma < 4/3$ no solution exists if $\lambda$ is sufficiently large (the pressure $P$ is sufficiently small) and $S$ is sufficiently small. To accomplish this we will use the method of shooting (cf. [13]). Thus we consider the initial-value problem

$$F(w) = 0, \tag{16a}$$

$$\lim_{x \to 0} x^3 w'(x) = 0, \tag{16b}$$

$$w(0) = A, \tag{16c}$$

where $A > 0$. (The case $A = 0$ presents additional difficulties, which this paper does not address. It is also disputable whether $A = 0$ is physically reasonable.) We will often make the initial value explicit by writing $w = w(x; A)$. We will say that a solution $w$ of (16) exists at $x > 0$ if it is positive on $[0, x)$. Thus the stress problem has a solution for a given $S > 0$ if there exists an $A$ such that the solution $w(x; A)$ to (16) exists at $x = 1$ and satisfies $w(1; A) = S$.

We can write (16a) as

$$w(x) = A + \int_0^x \frac{1}{t^3} \int_0^t \left( \frac{\lambda^2}{2} s^{2\gamma-1} - \frac{s^3}{8w^2} \right) ds\, dt, \tag{17}$$

and use this to obtain the simple bounds

$$w(x) \leq A + \frac{\lambda^2}{8\gamma(\gamma-1)} x^{2\gamma-2} \tag{18}$$

and

$$w'(x) \leq \frac{\lambda^2}{4\gamma} x^{2\gamma-3}. \tag{19}$$

Differentiating (17), we find that

$$w'(x) = \begin{cases} \dfrac{\lambda^2}{4\gamma} x^{2\gamma-3} + o(x^{2\gamma-3}) & \text{if } \gamma < 2; \\ \left( \dfrac{\lambda^2}{2} - \dfrac{1}{8A^2} \right) \dfrac{x}{4} + o(x) & \text{if } \gamma = 2; \\ \dfrac{-1}{32A^2} x + o(x) & \text{if } \gamma > 2, \end{cases} \tag{20}$$

which gives

$$\lim_{x \to 0} w'(x) = \begin{cases} \infty & \text{if } \gamma < 3/2; \\ \dfrac{\lambda^2}{4\gamma} & \text{if } \gamma = 3/2; \\ 0 & \text{if } \gamma > 3/2. \end{cases} \tag{21}$$

Similarly we can obtain

$$w(x) = \begin{cases} A + \dfrac{\lambda^2}{8\gamma(\gamma-1)} x^{2\gamma-2} + o(x^{2\gamma-2}) & \text{if } \gamma < 2; \\ A + \left( \dfrac{\lambda^2}{2} - \dfrac{1}{8A^2} \right) \dfrac{x^2}{8} + o(x^2) & \text{if } \gamma = 2; \\ A - \dfrac{1}{64A^2} x^2 + o(x^2) & \text{if } \gamma > 2. \end{cases} \tag{22}$$

We can prove in a manner similar to that of Callegari and Reiss [3] (see Johnson [12]) that despite the singularity at 0, solutions to (16) exist near 0 and are unique, and that $w(x;A)$ and $w'(x;A)$ are continuous functions of $A$. We can also show that solutions of (16) are monotonically increasing in $A$, as follows. Suppose $w$ and $\tilde{w}$ are solutions of the initial-value problem (16) with $w(0) = A$ and $\tilde{w}(0) = \widetilde{A}$, $\widetilde{A} > A$. Then

$$\tilde{w}(x) - w(x) = \widetilde{A} - A + \int_0^x \frac{1}{t^3} \int_0^t \left( \frac{s^3}{8w^2} - \frac{s^3}{8\tilde{w}^2} \right) ds\, dt. \tag{23}$$

By continuity of the solutions, $\tilde{w}(x) > w(x)$ for $x$ sufficiently small. Suppose there exists $x^*$ such that $\tilde{w}(x^*) = w(x^*)$ and $\tilde{w}(x) > w(x)$ for $x < x^*$. Then

$$\begin{aligned} 0 = \tilde{w}(x^*) - w(x^*) &= \widetilde{A} - A + \int_0^{x^*} \frac{1}{t^3} \int_0^t \left( \frac{s^3}{8w^2} - \frac{s^3}{8\tilde{w}^2} \right) ds\, dt \\ &> 0, \end{aligned} \tag{24}$$

which is a contradiction. So $\tilde{w}(x) > w(x)$ for all $x$.

We can also show that if $\gamma \geq 2$ and $w(x;A)$ is a solution of (16) with $A < \frac{1}{2\lambda}$, then $w$ is a decreasing function of $x$ on any subset of $(0,1]$ on which it exists. From (17),

$$w'(x) = \frac{1}{x^3} \int_0^x \left( \frac{\lambda^2}{2} s^{2\gamma-1} - \frac{s^3}{8w(s)^2} \right) ds. \tag{25}$$

Since $2\gamma - 1 \geq 3$, for $x$ positive but small the second term in the integrand is dominant and so $w'(x) < 0$. Suppose there exists $0 < x^* \leq 1$ such that $w'(x^*) = 0$ and $w'(x) < 0$ for every $0 < x < x^*$. Then for every such $x$, $w(x) < \frac{1}{2\lambda}$; so

$$\frac{\lambda^2}{2} x^{2\gamma-1} - \frac{x^3}{8w^2} < \frac{\lambda^2}{2}(x^{2\gamma-1} - x^3) < 0. \tag{26}$$

Hence $w'(x^*) < 0$, which is a contradiction.

LEMMA 2.1. $w(x;A) > A/2$ for $0 \leq x < (2A)^{3/2}$.

The proof is in Appendix A.

At this point we have shown continuity of solutions of (16) in $A$ and have also established the existence of arbitrarily large solutions. Together these imply that given a solution $w(\cdot;A)$ that exists at $x^*$, for every $y > w(x^*)$ there exists $\widetilde{A} > A$ such that $w(x^*;\widetilde{A}) = y$. Put simply, we can always find solutions above any given solution. We will call this the "rising property" of problem (16). It means that, as long as there is some $A$ such that $w(x;A)$ exists at $x = 1$, the stress problem will have a solution for all $S \geq w(1;A)$.

LEMMA 2.2. Suppose that for some $\widetilde{A} > 0$ there exists a value $\tilde{y}$ such that $w(x;\widetilde{A}) \to 0$ as $x \uparrow \tilde{y}$. Then for every $y > \tilde{y}$ there exists $A$ such that $w(x;A) \to 0$ as $x \uparrow y$.

Again, the proof is in Appendix A.

With the addition of this lemma we can say that if there exists a solution with a zero at $x \leq 1$, then we can choose $A$ so as to hit any desired nonnegative value at $x = 1$. In other words, in this case the stress problem has a solution for all $S \geq 0$. On the other hand, if solutions of (16) have a positive lower bound at $x = 1$, independent of $A$, then the stress problem has no solution for $S$ small.

We will need the following theorem, found in [14].

THEOREM 2.3 (Max-min Principle). Suppose that $w(x)$ satisfies

$$w'' + H(x, w, w') = 0 \tag{27}$$

and the initial conditions $w(b) = \gamma_1$, $w'(b) = \gamma_2$; that $H(x,y,z), \partial H/\partial y, \partial H/\partial z$ are continuous; and that $\partial H/\partial y \leq 0$. If $z_1(x)$ satisfies

$$\begin{cases} z_1'' + H(x, z_1, z_1') \geq 0, \\ z_1(b) \geq \gamma_1, \quad z_1'(b) \geq \gamma_2, \end{cases} \tag{28}$$

and if $z_2(x)$ satisfies

$$\begin{cases} z_2'' + H(x, z_2, z_2') \leq 0, \\ z_2(b) \leq \gamma_1, \quad z_2'(b) \leq \gamma_2, \end{cases} \tag{29}$$

then we have the upper and lower bounds

$$z_2(x) + \gamma_1 - z_2(b) \leq w(x) \leq z_1(x) + \gamma_1 - z_1(b), \tag{30}$$

$$z_2'(x) \leq w'(x) \leq z_1'(x). \tag{31}$$

If we define

$$H(x,y,z) = \frac{3}{x}z - \frac{\lambda^2}{2}x^{2\gamma-4} + \frac{1}{8y^2}, \tag{32}$$

then (16) is equivalent to (27) and satisfies the hypotheses of the Max-min Principle.

THEOREM 2.4. Let $1 < \gamma < 4/3$. There exist $\overline{\lambda}(\gamma) > 0$ and $k(\gamma) > 0$ such that if $\lambda > \overline{\lambda}$ and $S < k$ then the stress problem has no solution.

*Proof.* Let $\alpha$ be such that $2\gamma - 2 < \alpha < 2 - \gamma$. Let

$$\overline{\lambda} = \sqrt{2}[(2-\gamma)(4-\gamma)]^{1/3}, \tag{33}$$

$\lambda \geq \overline{\lambda}$, and

$$k = \frac{\lambda^2}{4(2-\gamma)(4-\gamma)}. \tag{34}$$

Define $z(x) = kx^\alpha$. Then for $0 < x \leq 1$,

$$\begin{aligned} z'' + H(x,z,z') &= \frac{1}{x^2}F(z) && (35)\\ &\leq \frac{\lambda^2}{4}x^{\alpha-2} - \frac{\lambda^2}{2}x^{2\gamma-4} + \frac{2(2-\gamma)^2(4-\gamma)^2}{\lambda^4}x^{2\gamma-4} && (36)\\ &\leq \left(\frac{-\lambda^2}{4} + \frac{2(2-\gamma)^2(4-\gamma)^2}{\lambda^4}\right)x^{2\gamma-4} && (37)\\ &\leq 0. && (38) \end{aligned}$$

Choose $A > 0$. Since $z(x) = kx^\alpha$ and $z'(x) = \alpha k x^{\alpha-1}$ and since $\alpha > 2\gamma - 2$, we see from (22) and (20) that the boundary conditions in (29) are satisfied for $b$ positive and sufficiently small. So now we can apply the Max-min Principle to conclude that

$w(x) \geq z(x)$ for all $x \in [b, 1]$. In particular, $w(1) \geq z(1) = k$. So there is no $A > 0$ such that the solution to (16) satisfies $w(1; A) < k$. $\square$

Notice that this does not rule out the possibility of solutions to the stress problem if zero stress at the center of the membrane ($A = 0$) is allowed.

Dickey [7] showed using a phase-plane analysis that for $\gamma = 4/3$, no solution to the stress problem exists if $\lambda^2 > (\frac{64}{3})^{1/3} \approx 2.773$ and $S < (\frac{9}{64})^{1/3} \approx 0.520$. On the other hand, taking the limit as $\gamma \uparrow 4/3$ in (33) gives

$$\overline{\lambda}^2 \to \left[\sqrt{2}\left(\frac{16}{9}\right)^{1/3}\right]^2 = \left(\frac{2048}{81}\right)^{1/3} \approx 2.935; \tag{39}$$

so if we let $\lambda = \overline{\lambda}$, then using (34),

$$k \to \frac{\lambda^2}{64/9} = \left(\frac{9}{128}\right)^{1/3} \approx 0.413. \tag{40}$$

In other words, assuming some continuity in $\gamma$, the proof of Theorem 2.4 indicates that no solution to the stress problem should exist for the case $\gamma = 4/3$ if $\lambda^2 > 2.935$ and $S < 0.413$, values that are close to and consistent with the exact bounds that Dickey found.

THEOREM 2.5. Let $\gamma > 4/3$. Then the stress problem has a solution for any $\lambda > 0$ and $S > 0$.

*Proof.* First consider the case $\gamma \geq 2$. We showed earlier that solutions to (16) with $A < 1/(2\lambda)$ are decreasing on $(0, 1]$. Either (a) $w(x; A)$ exists on all of $[0, 1]$ for all $0 < A < 1/(2\lambda)$ or else (b) there exists $0 < A^* < 1/(2\lambda)$ such that $w(x; A^*)$ has a zero on $(0, 1]$. Suppose (a) held true. Let $\{A_n\}$ be a positive sequence with $A_n \to 0$. Then $w(x; A_n) \to 0$ uniformly on $(0, 1]$. But

$$w_n(x) \leq A_n + \frac{\lambda^2}{8\gamma(\gamma - 1)} x^{2\gamma-2} - \frac{x^2}{64A_n^2}, \tag{41}$$

which for fixed $x$ is negative for $n$ sufficiently large, yielding a contradiction to (a). So (b) must hold. By Lemma 2.2, for every $\tilde{x} > 1$ there exists $A(\tilde{x})$ such that

$$\lim_{x \uparrow \tilde{x}} w(x; A(\tilde{x})) = 0. \tag{42}$$

Let $\tilde{x} \downarrow 1$. Then there exists $A_0$ such that $A(\tilde{x}) \downarrow A_0$ and $w(1; A(\tilde{x})) \downarrow 0$. Thus we can apply the rising property to deduce that for every $S > 0$ there exists $A > A_0$ such that $w(1; A) = S$. Hence the stress problem has a solution for any $S > 0$.

Now consider the case $4/3 < \gamma < 2$. Choose $\varepsilon$ such that $2 - \gamma < \varepsilon < 2\gamma - 2$, which implies that $\varepsilon > 0$. Let $\alpha = 2\gamma - 2 - \varepsilon$, and choose a positive sequence $\{A_n\}$ with $A_n \to 0$. Define $z_n(x) = A_n(x^\alpha + 1)$ and let $w_n(x) = w(x; A_n)$.

Using (18) and (19) and the fact that $\alpha > 2\gamma - 2$ we can say that for every $n$ there exists $\hat{x}_n > 0$ such that if $0 < x \leq \hat{x}_n$ then $z_n(x) \geq w_n(x)$ and $z_n'(x) \geq w_n'(x)$. In particular, since $z_n'(x) = \alpha A_n x^{\alpha-1}$, we see that $z_n'(x) \geq w_n'(x)$ if

$$\alpha A_n x^{\alpha-1} \geq \frac{\lambda^2}{4\gamma} x^{2\gamma-3}, \tag{43}$$

which holds if $x$ is positive and

$$x \leq \left(\frac{4A_n\alpha\gamma}{\lambda^2}\right)^{1/\varepsilon}. \tag{44}$$

Similarly, $z_n \geq w_n$ if $x$ is positive and

$$A_n(x^\alpha + 1) \geq A_n + \frac{\lambda^2}{8\gamma(\gamma-1)}x^{2\gamma-2}; \tag{45}$$

that is, if

$$x \leq \left(\frac{8\gamma(\gamma-1)A_n}{\lambda^2}\right)^{1/\varepsilon}. \tag{46}$$

So we can define

$$\hat{x}_n = \min\left[\left(\frac{4A_n\alpha\gamma}{\lambda^2}\right)^{1/\varepsilon}, \left(\frac{8\gamma(\gamma-1)A_n}{\lambda^2}\right)^{1/\varepsilon}\right]. \tag{47}$$

Let us also obtain a lower bound on $z_n'' + H(x, z_n, z_n')$. We compute

$$\begin{aligned} z_n'' + H(x, z_n, z_n') &= \frac{1}{x^2}F(z_n) \qquad (48)\\ &\geq -\frac{\lambda^2}{2}x^{2\gamma-4} + \frac{1}{32A_n^2} \quad \text{if } x \leq 1 \\ &\geq 0 \quad \text{if } x \geq (4A_n\lambda)^{\frac{1}{2-\gamma}}. \qquad (49) \end{aligned}$$

Let us define

$$\hat{\hat{x}}_n = (4A_n\lambda)^{1/(2-\gamma)}. \tag{50}$$

Now we would like to apply the Max-min Principle. In order that a $b$ exist such that (28) is satisfied, we need the interval $(0, \hat{x}_n]$ on which the boundary condition holds to overlap with the interval $[\hat{\hat{x}}_n, 1]$ on which the differential inequality holds. That is, we need $\hat{x}_n \geq \hat{\hat{x}}_n$. So we set

$$(4A_n\lambda)^{\frac{1}{2-\gamma}} \leq \min\left[\left(\frac{4\gamma\alpha A_n}{\lambda^2}\right)^{1/\varepsilon}, \left(\frac{8\gamma(\gamma-1)A_n}{\lambda^2}\right)^{1/\varepsilon}\right], \tag{51}$$

which gives

$$A_n^{\left(\frac{1}{2-\gamma}-\frac{1}{\varepsilon}\right)} \leq \left(\frac{1}{4\gamma}\right)^{\frac{1}{2-\gamma}} \min\left[\left(\frac{4\gamma\alpha}{\lambda^2}\right)^{1/\varepsilon}, \left(\frac{8\gamma(\gamma-1)}{\lambda^2}\right)^{1/\varepsilon}\right], \tag{52}$$

which is satisfied for $n$ sufficiently large if $1/(2-\gamma) > 1/\varepsilon$. In other words, if $2-\gamma < \varepsilon$ then for each $n$ sufficiently large there is a $b_n$ for which the hypotheses of the Max-min Principle are satisfied.

Note that we also need $\alpha > 0$ in our calculations above, which imposes the restriction $\varepsilon < 2\gamma - 2$.

So we conclude that for $n$ sufficiently large, there exists a $b_n$ such that $w_n(x) \leq z_n(x)$ on $[b_n, 1]$ by the Max-min Principle. In particular, we can say that for $n$ large, $w_n(1) \leq 2A_n$. But $A_n \to 0$, and so this combined with the rising property of (16) implies that the stress problem has a solution for any given boundary stress $S > 0$. □

**3. The displacement problem.** We now address the displacement problem, making use of the existence for the stress problem established by Theorems 2.4 and 2.5. Since the expression (13) for the displacement of the membrane involves $S_r'$, we first need the following lemma to establish the behavior of $S_r'$.

LEMMA 3.1. Let $\{S_n\}$ be a positive sequence with $S_n \to 0$ as $n \to \infty$, and suppose for each $n$ there exists a solution $w_n$ to the problem

$$F(w_n) = 0, \tag{53a}$$

$$\lim_{x \to 0} x^3 w_n'(x) = 0, \tag{53b}$$

$$w_n(1) = S_n. \tag{53c}$$

Then $w_n'(1) \to -\infty$ as $n \to \infty$.

The proof of this lemma is rather technical and is in the Appendix.

THEOREM 3.2. If $\gamma > 4/3$, for any boundary displacement value $\Gamma^*$ the displacement problem has a solution.

*Proof.* We again want to use shooting; specifically, we wish to find $A$ such that $U_{w(\cdot;A)}(1) = \Gamma^*$ (where $U$ is as given in (12)). If we consider a sequence $\{S^n\}$ with $S^n \in \mathbb{R}^+$ for all $n$ and $S^n \to 0$, then Theorem 2.5 implies that there exists a solution $S_r^n(x)$ to the boundary-value problem

$$F(S_r^n) = 0, \tag{54a}$$

$$\lim_{x \to 0} x^3 S_r^{n\prime}(x) = 0, \tag{54b}$$

$$S_r^n(1) = S^n. \tag{54c}$$

If we let $A_n = S_r^n(0)$, then $S_r^n$ also satisfies the initial-value problem

$$F(S_r^n) = 0, \tag{55a}$$

$$\lim_{x \to 0} x^3 S_r^{n\prime}(x) = 0, \tag{55b}$$

$$S_r^n(0) = A_n. \tag{55c}$$

By Lemma 3.1,

$$U_{S_r^n}(1) \to -\infty. \tag{56}$$

Now let $\{\widetilde{A}_n\}$ be a sequence with $\widetilde{A}_n \to \infty$, and let $\tilde{w}_n$ be the sequence of solutions to the problem

$$F(\tilde{w}_n) = 0, \tag{57a}$$

$$\lim_{x \to 0} x^3 \tilde{w}_n'(x) = 0, \tag{57b}$$

$$\tilde{w}_n(0) = \widetilde{A}_n. \tag{57c}$$

By Lemma 2.1, $\tilde{w}_n(1) \to \infty$. Moreover, by the same lemma there exists an $N$ such that if $n > N$ then

$$\tilde{w}_n(x) > \frac{1}{2\lambda}\sqrt{\frac{\gamma}{2}} \tag{58}$$

on $[0,1]$. For such an $n$,

$$\tilde{w}_n'(1) > \frac{\lambda^2}{4\gamma} - \int_0^1 \frac{s^3}{8(\frac{1}{2\lambda}\sqrt{\frac{\gamma}{2}})^2}\,ds > 0. \tag{59}$$

Therefore

$$U_{\tilde{w}_n}(1) \to \infty. \tag{60}$$

So there exists an $M$ such that $U_{S_r^M}(1) < \Gamma^*$ (by (56)) and $U_{\tilde{w}_M}(1) > \Gamma^*$ (by (60)). We know $w(1;A)$ and $w'(1;A)$ are continuous functions of $A$; so $U_w(1)$ is as well. Hence there exists an $A^*$ such that the solution $w(\cdot;A^*)$ to problem (16) has $U_{w(\cdot;A^*)}(1) = \Gamma^*$. □

THEOREM 3.3. Let $1 < \gamma < 4/3$. Then there exists $\overline{\lambda}(\gamma)$ such that if $\lambda > \overline{\lambda}$ and $\Gamma \le 0$, then the displacement problem has no solution.

*Proof.* Let $k, \alpha, \overline{\lambda}, \lambda$, and $z$ be as in the proof of Theorem 2.4. Then we can apply the Max-min Principle to show that for any $A > 0$,

$$w(x;A) \ge z(x) \quad \text{for } x \in [0,1] \tag{61}$$

(as in the proof of Theorem 2.4), and also

$$w'(x;A) \ge z'(x) = k\alpha x^{\alpha-1} \quad \text{for } x \in [0,1]. \tag{62}$$

Therefore

$$U_w(1) > k\alpha + k > 0, \tag{63}$$

which means that if $\Gamma \le 0$ then the displacement problem has no solution. □

Thus for $1 < \gamma < 4/3$, if $\lambda$ is large (pressure $P$ is small), the boundary of the membrane must be stretched out from its original position in order for a rotationally symmetric solution to exist.

**A. Appendix.** Here we present proofs of some of the more technical lemmas.

*Proof of Lemma* 2.1. Let $w(x) = w(x;A)$, and suppose the lemma is false. Then there exists $z$ with $0 < z < (2A)^{3/2}$ such that $w(z) = A/2$ and $w(x) > A/2$ for $x \in I = [0,z)$. On $I$,

$$\begin{aligned} w'(x) &\ge \frac{1}{x^3}\int_0^x \left(\frac{\lambda^2}{2}t^{2\gamma-1} - \frac{t^3}{8(A/2)^2}\right)dt \\ &\ge \frac{-x}{8A^2}. \end{aligned} \tag{64}$$

Let $b(x) = A - \frac{x^2}{16A^2}$. Then on $I$, $w'(x) \ge b'(x)$. So

$$\begin{aligned} w(z) &= A + \int_0^z w'(x)\,dx \\ &\ge A + \int_0^z b'(x)\,dx \\ &= b(z) \\ &> b((2A)^{3/2}) \\ &= A/2, \end{aligned}$$

which is a contradiction. □

In the following it will be helpful if we define for $x > 0$

$$\delta(x) = \frac{x^{2-\gamma}}{2\lambda}. \tag{65}$$

Then we can write the integral equation (17) as

$$w(x) = A + \int_0^x \frac{1}{t^3} \int_0^t \frac{s^3}{8} \left( \frac{1}{\delta^2(s)} - \frac{1}{w^2(s)} \right) ds\, dt. \tag{66}$$

In order to prove Lemma 2.2 we first need the following result.

LEMMA A.1. There is a continuous positive function $f$ defined on $\mathbb{R}^+$ such that if $w$ is a solution to (16) and $w(x^*) < f(x^*)$ then $w'(x^*) < 0$.

*Proof of Lemma* A.1. Let us consider the cases $\gamma = 2, \gamma > 2$, and $\gamma < 2$ separately. For $\gamma = 2$, $\delta(x) \equiv \frac{1}{2\lambda}$ and (17) becomes

$$w'(x; A) = \frac{1}{x^3} \int_0^x \frac{t^3}{8} \left( 4\lambda^2 - \frac{1}{w^2(t; A)} \right) dt. \tag{67}$$

If $A < \frac{1}{2\lambda}$, then (67) implies that $w'(x; A)$ is negative for $x$ positive but small. Suppose there were an $x_1$ such that $w'(x_1; A) = 0$ and $w'(x; A) < 0$ for every $0 < x < x_1$. Then (67) says $w'(x_1; A) < 0$; so no such $x_1$ exists, and $w'(x; A) < 0$ for all $x > 0$ such that the solution exists. Similarly, if $A > \frac{1}{2\lambda}$ then $w'(x; A) > 0$ for all $x > 0$; and if $A = \frac{1}{2\lambda}$ then $w(x; \frac{1}{2\lambda}) \equiv \frac{1}{2\lambda}$ is a solution. So $w'(x^*; A) < 0$ if and only if $w(x^*; A) < \frac{1}{2\lambda}$, and we can define $f(x) = \frac{1}{2\lambda} = \delta(x)$.

For $\gamma > 2$,

$$\lim_{x \to 0} \delta(x) = \infty; \tag{68}$$

so for $x$ sufficiently small, $w(x) < \delta(x)$. If $w(x^*) < \delta(x^*)$ and $w(x) < \delta(x)$ for all $x < x^*$ then

$$w'(x^*) = \frac{1}{x^{*3}} \int_0^{x^*} \frac{s^3}{8} \left( \frac{1}{\delta^2(s)} - \frac{1}{w^2(s)} \right) ds < 0. \tag{69}$$

If $w(x^*) < \delta(x^*)$ and there exists $0 < x < x^*$ such that $w(x) > \delta(x)$, then let

$$x_1 = \sup_{x < x^*} \{x : w(x) = \delta(x)\}. \tag{70}$$

By continuity of $w$ and $\delta$, $x_1 < x^*$. Notice also that since $w(x) < \delta(x)$ for $x > x_1$, $w'(x_1) \le \delta'(x_1) < 0$. So

$$w'(x^*) = \left( \frac{x_1}{x^*} \right)^3 w'(x_1) + \frac{1}{x^{*3}} \int_{x_1}^{x^*} \frac{t^3}{8} \left( \frac{1}{\delta^2(t)} - \frac{1}{w^2(t)} \right) dt < 0. \tag{71}$$

So again we can define $f(x) = \delta(x)$.

Now consider the case $\gamma < 2$. Here

$$\lim_{x \to 0} \delta(x) = 0; \tag{72}$$

so for $A > 0$ and $x$ small, $w(x; A) > \delta(x)$. Let us pick $\tilde{x}$ and $\tilde{y}$ such that $\tilde{y} < \delta(\tilde{x})$. Suppose for some $A$, $w = w(\cdot; A)$ is a solution to (16) passing through $(\tilde{x}, \tilde{y})$, and suppose $w'(\tilde{x}) \geq 0$.

Now we claim that if for some $\overline{x}$, $w(\overline{x}; A) < \delta(\overline{x})$ and $w'(\overline{x}; A) < 0$, then $w'(x; A) < 0$ for every $x > \overline{x}$ where the solution exists. To show this, suppose $x_1 > \overline{x}, w'(x_1; A) = 0$, and $w'(x; A) < 0$ for every $x \in [\overline{x}, x_1)$. Then we can bound $w'(x_1; A)$ as we bounded $w'(x^*)$ in (71) to show that $w'(x_1; A) < 0$, which is a contradiction.

By the claim above, if here we define $x_1$ such that $\delta(x_1) = \tilde{y}$, then $w'(x) \geq 0$ for every $x_1 < x < \tilde{x}$. Now

$$
\begin{aligned}
w'(\tilde{x}) &\leq \frac{1}{\tilde{x}^3}\left[\int_0^{\tilde{x}} \frac{s^3}{8\delta^2(s)} ds - \int_{x_1}^{\tilde{x}} \frac{s^3}{8\tilde{y}^2} ds\right] \\
&= \frac{\lambda^2}{4\gamma}\tilde{x}^{2\gamma-3} - \frac{\tilde{x}}{32\tilde{y}^2} + \frac{(2\lambda)^{\frac{4}{2-\gamma}}}{32}\left(\frac{\tilde{y}^{\frac{2\gamma}{2-\gamma}}}{\tilde{x}^3}\right),
\end{aligned}
\tag{73}
$$

and we notice that for fixed $\tilde{x}$ this is negative if $\tilde{y}$ is sufficiently small, which gives the desired contradiction. So we define $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\tilde{y} = f(\tilde{x})$ makes (73) negative. $\square$

*Proof of Lemma* 2.2. Suppose the claim is false; so there exists a point $(r_1, \varepsilon)$ such that $r_1 > \tilde{y}$ and there is no $A$ such that $w(r_1; A) = \varepsilon$. By the rising property of the initial-value problem, for any $p \leq \varepsilon$ no solution passes through $(r_1, p)$. Letting $w(\cdot) = w(\cdot, \widetilde{A})$, if $w(x) < f(x)$ (where $f$ is as in Lemma A.1) for all $x > 0$ then let $s = 0$; otherwise, let $s = \max\{x : f(x) = w(x)\}$. Let

$$
\zeta = \inf_{x \in (s, r_1)} f(x). \tag{74}
$$

By the definition of $s$, if $s = 0$ then $\lim_{x \to 0} f(x) > 0$. Therefore, since $f$ is a positive continuous function on $\mathbb{R}^+$, $\zeta$ is positive. Let

$$
r = \sup\{x < r_1 : \text{ some solution passes through } (x, \zeta)\}. \tag{75}
$$

Then $r \geq \tilde{y}$, again by the rising property. Hence $r > s$. Let us define $c(x)$ by $w(x; c(x)) = \zeta$ for $x$ on the interval $I = (s, r)$. Note that $w'(x; c(x)) < 0$ for every $x \in I$.

Choose $x_0 \in I$. Let

$$
Q = \min_{t \in [0, x_0]} w(t; c(x_0)). \tag{76}
$$

Then if $x \in [x_0, r)$,

$$
\min_{t \in [0, x_0]} w(t; c(x)) \geq Q \tag{77}
$$

by monotonicity and

$$
\min_{t \in [x_0, x]} w(t; c(x)) = \zeta \geq Q; \tag{78}
$$

so

$$
Q \leq \min_{t \in [0, x]} w(t; c(x)). \tag{79}
$$

Thus

$$w'(x, c(x)) > -\frac{1}{x^3}\int_0^x \frac{t^3}{8Q^2}\,dt = \frac{-x}{32Q^2}. \tag{80}$$

Take a sequence $\{x_n\} \subset [x_0, r)$ with $x_n \to r$. Let $w_n(\cdot) = w(\cdot; c(x_n))$.

(a) Suppose there does not exist an $n$ such that $w_n$ exists at $r$. Then for every $n$, $w_n(x) \to 0$ as $x \to z_n < r$. Define $\{y_n\}$ by $w_n(y_n) = \zeta/2$. Then $y_n \to r$ as $n \to \infty$. By the Mean Value Theorem, for every $n$ there exists $v_n \in (x_n, y_n)$ such that

$$w_n'(v_n) = \frac{-\zeta/2}{y_n - x_n}. \tag{81}$$

(b) Note that $w_n'(v_n) \to -\infty$ as $n \to \infty$. However, $w_n$ is bounded below on $[0, y_n]$ by $B = \min(Q, \zeta/2)$. So for $x \in [x_n, y_n]$,

$$\begin{aligned} w_n'(x) &> -\frac{1}{x^3}\int_0^x \frac{t^3}{8B^2}\,dt \\ &> -\frac{x_0}{32B^2}, \end{aligned} \tag{82}$$

which is a contradiction to (b). So (a) must be false; in other words, there exists $n$ such that $w_n(r)$ exists. Hence we can continue the solution to see that $w_n(x)$ exists for some $x > r$. □

*Proof of Lemma* 3.1. Let $\{S_n\}$ and $\{w_n\}$ be as in the statement of the lemma. Suppose that $w_n'(1) \nrightarrow -\infty$, and let us label this assumption (A1). Then there exists a number $B > 0$ and there exists a subsequence $\{w_{n_j}\}$ such that for all $j$, $w_{n_j}'(1) \geq -B$. For simplicity, let us denote the subsequence by $\{w_n\}$.

For each $n$, either (a) there exists $a_n$ such that $w_n'(x) \geq -B$ on $[a_n, 1]$ and $w_n'(x) < -B$ on some interval $(b_n, a_n)$, or else (b) $w_n'(x) \geq -B$ on $[0, 1]$. In this latter case let us define $a_n = 1/2$.

Now suppose

$$\limsup_{n\to\infty} a_n = 1 \tag{83}$$

so that there is a subsequence $\{a_{n_j}\}$ with $a_{n_j} \to 1$. Of course, if this is true, then for $j$ large case (a) above must apply. So writing

$$w_{n_j}''(x) = \frac{-3}{x}w_{n_j}'(x) + \frac{\lambda^2}{2}x^{2\gamma-4} - \frac{1}{8w_{n_j}^2(x)}, \tag{84}$$

we see that for $j$ large, $w_{n_j}''(x) < 0$ on $[a_{n_j}, 1]$. Therefore $w_{n_j}'(a_{n_j}) > w_{n_j}'(1)$, which contradicts the definition of $a_n$ under (a).

Thus

$$\limsup_{n\to\infty} a_n < 1. \tag{85}$$

That is, there exists a uniform interval $[a, 1]$ with $a < 1$ on which $w_n'(x) \geq -B$. Since $w_n(1) = S_n$, on $[a, 1]$ $w_n$ must satisfy $w_n(x) \leq S_n + (1 - x)B$.

Let

$$r_1 = \min_{x\in[\frac{1}{2},1]} f(x) \tag{86}$$

where $f$ is as in Lemma A.1. Choose $N$ such that $S_N < r_1$, and let

$$b_1 = \max\left(\frac{1}{2}, 1 - \frac{r_1 - S_N}{B}\right). \tag{87}$$

Let $n > N$. Then on $[b_1, 1], w_n(x) < f(x)$ (and hence $w_n'(x) < 0$, by Lemma A.1). Let $b = \max(b_1, a)$. On $[b, 1]$,

$$-B \leq w_n'(x) < 0 \tag{88}$$

and

$$w_n(x) \leq S_n + (1 - x)B. \tag{89}$$

But

$$w_n'(1) = b^3 w'(b) + \int_b^1 \left(\frac{\lambda^2}{2} t^{2\gamma-1} - \frac{t^3}{8w_n(t)^2}\right) dt \tag{90}$$

$$\leq \frac{\lambda^2}{4\gamma}(1 - b^{2\gamma}) - \int_b^1 \frac{t^3}{8[S_n + (1-t)B]^2}\, dt. \tag{91}$$

This expression goes to $-\infty$ as $S_n \to 0$ (that is, as $n \to \infty$). This is a contradiction to the original assumption (A1). □

## References


[1] J. V. Baxley, *A singular nonlinear boundary value problem: Membrane response of a spherical cap*, SIAM J. Appl. Math. **48**, 497–585 (1988)

[2] E. Bromberg and J. J. Stoker, *Non-linear theory of curved elastic sheets*, Quart. Appl. Math. **3**, 246–265 (1945/46)

[3] A. J. Callegari and E. L. Reiss, *Non-linear boundary value problems for the circular membrane*, Arch. Rat. Mech. Anal. **31**, 390–400 (1968)

[4] A. J. Callegari, H. B. Keller, and E. L. Reiss, *Membrane buckling: a study of solution multiplicity*, Comm. Pure and Appl. Math. **24**, 499–527 (1971)

[5] R. W. Dickey, *Membrane caps*, Quart. Appl. Math. **45**, 697–712 (1987); *Erratum*, Quart. Appl. Math. **46**, 192 (1988)

[6] R. W. Dickey, *Membrane caps under hydrostatic pressure*, Quart. Appl. Math. **46**, 95–104 (1988)

[7] R. W. Dickey, *Rotationally symmetric solutions for shallow membrane caps*, Quart. Appl. Math. **47**, 571–581 (1989)

[8] R. W. Dickey, *The plane circular elastic surface under normal pressure*, Arch. Rat. Mech. Anal. **26**, 219–236 (1967)

[9] A. Föppl, *Vorlesungen Über Technische Mechanik*, Teubner, Leipzig, 1907

[10] M. A. Goldberg, *An iterative solution for rotationally symmetric non-linear membrane problems*, Internat. J. Non-linear Mech. **1**, 169–178 (1966)

[11] H. Hencky, *Über den Spannungszustand in kreisrunden Platten*, Z. Math. Phys. **63**, 311–317 (1915)

[12] K. N. Johnson, *Circularly Symmetric Deformations of Shallow Elastic Membrane Caps*, Ph.D. Thesis, University of Wisconsin-Madison, 1994

[13] H. B. Keller, *Numerical Solution of Two Point Boundary Value Problems*, SIAM, Philadelphia, 1976

[14] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1967

[15] E. Reissner, *Rotationally symmetric problems in the theory of thin elastic shells*, 3rd U.S. Natl. Congress of Applied Mechanics, 1958

[16] H. J. Weinitschke, *On finite displacements of circular elastic membranes*, Math. Meth. in the Appl. Sci. **9**, 76–98 (1987)