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A UNIQUENESS THEOREM FOR A ROBIN BOUNDARY VALUE PROBLEM OF PHYSICAL GEODESY

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Abstract. We get a uniqueness theorem for a Robin type boundary value problem for the Laplace equation arising in Physical Geodesy in the context of the gravimetric determination of the geoid. The boundary is an oblate ellipsoid of revolution and we have uniqueness of solutions provided that its eccentricity is (approximately) less than 0.526428.

1. Introduction. The gravimetric determination of the *geoid* (a particular equipotential surface of the earth gravity field taken in Geodesy as a reference surface for heights in such a way as to be close to the sea surface) gives rise to a boundary value problem of the form: *to find u such that*

$$\begin{cases} \Delta u = 0 & \text{outside } \Sigma, \\ \dfrac{\partial u}{\partial n} - 2Hu = f & \text{on } \Sigma, \\ u(x) = c/|x| + O(|x|^{-3}) & \text{as } x \to \infty, \end{cases} \tag{1.1}$$

where

$$\Sigma = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{x_1^2 + x_2^2}{a^2} + \frac{x_3^2}{b^2} = 1 \right\}$$

is an oblate ellipsoid of revolution ($a \geq b$); H *is the mean curvature of* Σ; *and* n *is the unit normal to* Σ *pointing to the exterior* Σ^e *of* Σ (see, for example, [5, Section 2.13], [10, §2], [13]). Hereafter, by E, e and ε we shall denote the linear, the first and the second eccentricity of Σ, respectively; i.e., $E = (a^2 - b^2)^{1/2}$, $e = E/a \in [0, 1)$ and $\varepsilon = E/b \in [0, \infty)$. The condition at infinity in (1.1) means that the harmonic function u tends to zero and has no component of degree 1 in its spherical harmonic expansion outside a sphere of large enough radius. Since the dimension of the space spanned by spherical harmonics of degree 1 is 3, then this condition at infinity restricts u to a space of codimension 3.

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In the particular case $a = b = R$, we have $H = -1/R$, and (1.1) becomes *Stokes' problem* (see [5, Sections 2–14 and 2–16]):

$$\begin{cases} \Delta u = 0 & \text{outside } \Sigma_R, \\ \dfrac{\partial u}{\partial r} + \dfrac{2}{R} u = f & \text{on } \Sigma_R, \\ u(x) = c/|x| + O(|x|^{-3}) & \text{as } x \to \infty, \end{cases} \tag{1.2}$$

where Σ_R is a sphere of radius R and $\partial u/\partial r$ is the radial derivative of u. For this boundary problem the following result is well known: *The problem* (1.2) *can be uniquely solved if and only if the projection, in the* L^2 *norm, of* f *on the space spanned by spherical harmonics of degree* 1 *is zero* (see, for example, [6]); in this case there is an explicit representation for the solution known as (generalized) Stokes' formula (see [5, Eq. (2-163a)]).

For the problem (1.1), some formal solutions in terms of power series with respect to a parameter characterizing the deviation of the ellipsoid from a sphere ($e, \varepsilon, \dots$) have been proposed (see, for example, [9, Chapter III, §3] and [1]). Based as well on a series representation for the solutions of (1.1), in [12, Theorem 4.1] an existence theorem for (1.1) valid for sufficiently small values of ε has been stated in a nonquantitative way.

The purpose of the present paper is to give a first answer to the following still open problem: *for which values of* e *does the problem* (1.1) *admit at most one solution*? The main result of this paper is the following.

THEOREM A. Let $\Sigma \subset \mathbb{R}^3$ be an oblate ellipsoid of revolution such that $e < e_0$, where (approximately) $e_0 = 0.526428$. Suppose $u \in C^2(\Sigma^e) \cap C^1(\overline{\Sigma}^e)$ satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \Sigma^e = \text{ext}(\Sigma) \\ \dfrac{\partial u}{\partial n} - 2Hu = 0 & \text{on } \Sigma \\ u(x) = c/|x| + O(|x|^{-3}) & \text{as } x \to \infty. \end{cases} \tag{1.3}$$

Then $u \equiv 0$.

REMARK 1.1. The value of e_0 is given by Eqs. (3.9) and (3.10) in Sec. 3. The calculations that have led to this numerical approximation of e_0 have been done using MATHEMATICA (see [17]). Theorem A applies to the *earth reference ellipsoid*, whose first eccentricity e is ≈ 0.08 (see [11]).

REMARK 1.2. Since $H < 0$, a uniqueness theorem for (1.1) cannot be stated as a consequence of the *Hopf boundary point lemma and the strong maximum principle* (see [4, Lemma 3.4 and Theorem 3.5]). This kind of difficulty was surmounted by L. Hörmander [6, Chapter 1] in connection with what in Geodesy is called the *linearized Molodensky problem*.

REMARK 1.3. The index of the boundary value problem (1.1) is -3. In fact, the index of the Robin boundary value problem for the Laplace operator is 0 (see [16, Theorem 16.1]), and since u is restricted to a space of codimension 3, by the property that the index of the composition of two Fredholm operators is the sum of the indices (see, for example, [16, Theorem 12.6]), the assertion follows. Therefore, in the conditions of Theorem A, the boundary value problem (1.1) can be uniquely solved if and only if f satisfies three compatibility conditions.

Our approach for proving this main theorem follows the method of Hörmander. This paper is therefore essentially organized according to the two basic steps in this method:

(i) Energy integral estimates for the solutions of (1.3). These are obtained in Sec. 2 (Corollary 2.4) following a slightly different approach than in [6], but also based on a suggestion of L. Hörmander [6, p. 7, second paragraph]; see also [7];

(ii) Estimates for some spherical harmonic components. These estimates are easily deduced from Proposition 3.5. Here we have also followed ideas from [6, §1.3].

Throughout this paper we shall use ρ and ν to respectively denote the radius of curvature of the meridian ellipse and of the normal sections along the parallels of Σ. Since these are the principal directions at each point of Σ, then

$$H = -\frac{1}{2}\left(\frac{1}{\nu} + \frac{1}{\rho}\right).$$

As a parametrization for Σ we shall use the more regular in Geodesy (see, for example, [5])

$$\begin{cases} x_1 = \nu \cos\varphi \cos\lambda, \\ x_2 = \nu \cos\varphi \sin\lambda, \\ x_3 = (1+\varepsilon^2)^{-1} \nu \sin\varphi, \end{cases} \tag{1.4}$$

where $\lambda \in (0, 2\pi)$ is the longitude and $\varphi \in (-\pi/2, \pi/2)$ is the geodetic latitude (the angle that at each point makes the normal to Σ with the equatorial plane $x_3 = 0$). In these coordinates the radii of curvature ν and ρ are given by

$$\nu = \frac{a}{(1 - e^2 \sin^2\varphi)^{1/2}} = \frac{a\gamma}{(1+\varepsilon^2\cos^2\varphi)^{1/2}} \tag{1.5}$$

and

$$\rho = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}} = \frac{a\gamma}{(1+\varepsilon^2\cos^2\varphi)^{3/2}}, \tag{1.6}$$

where $\gamma = (1+\varepsilon^2)^{1/2}$. Observe that $\alpha = \nu/\rho = 1 + \varepsilon^2\cos^2\varphi \geq 1$.

We shall employ the notation $\mathcal{H}(\Omega)$ to denote the space of harmonic functions in an open set $\Omega \subset \mathbb{R}^3$. For unbounded Ω, $\mathcal{H}_\infty(\Omega)$ will denote the subset of $\mathcal{H}(\Omega)$ of functions regular at infinity.

2. A Rellich type inequality.

Let $u \in C^1(\overline{\Sigma}^e)$. On Σ we write $\nabla u = u'_n n + u'_s$ where $u'_n = \partial u/\partial n$ and u'_s is the tangential component of ∇u.

The main result of this section is the following inequality.

PROPOSITION 2.1. Let $u \in \mathcal{H}_\infty(\Sigma^e) \cap C^1(\overline{\Sigma}^e)$. Then

$$\int_\Sigma \nu |u'_s|^2\, dS \leq \int_\Sigma \nu |u'_n|^2\, dS - (1-\varepsilon^2)\int_{\Sigma^e} |\nabla u|^2\, dx.$$

REMARK 2.2. For harmonic functions in the exterior of oblate ellipsoids of revolution with $\varepsilon \leq 1$ this proposition gives the inequality

$$\int_\Sigma \nu |u'_s|^2\, dS \leq \int_\Sigma \nu |u'_n|^2\, dS.$$

For harmonic functions in the interior of a sphere, the inequality $\int_\sigma |u'_n|^2\,d\sigma \le \int_\sigma |u'_s|^2\,d\sigma$ was first proved by M. I. Vishik [14]. For harmonic functions in the exterior of a sphere and regular at infinity, the inequality $\int_\sigma |u'_s|^2\,d\sigma \le \int_\sigma |u'_n|^2\,d\sigma$ was given by L. Hörmander [6, Eq. (1.1.4)]. (Here $\int_\sigma f\,d\sigma \equiv \int_{\sigma\in S^2} f(\sigma)\,d\sigma$, where

$$S^2 = \{\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3 : \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 1\}$$

is the unit sphere.)

REMARK 2.3. The inequality of Proposition 2.1 is close to those known as *Rellich inequalities.* These kind of inequalities play an important role in the layer potential methods for boundary value problems (see [3, Lemma 1.1], for example).

Since $-\int_{\Sigma^e} |\nabla u|^2\,dx = \int_\Sigma u u'_n\,dS$ if $u \in \mathcal{H}_\infty(\Sigma^e) \cap C^1(\overline{\Sigma}^e)$, then we have the following immediate

COROLLARY 2.4. Let $u \in \mathcal{H}_\infty(\Sigma^e) \cap C^1(\overline{\Sigma}^e)$ be such that $u'_n = 2Hu$ on Σ. Then

$$\int_\Sigma \nu |u'_s|^2\,dS \le \int_\Sigma \nu^{-1}(1+\alpha)(\varepsilon^2+\alpha)u^2\,dS. \tag{2.1}$$

Hereafter by (t, β, λ) we shall denote the system of elliptic coordinates associated to Σ given by the equations

$$\begin{cases} x_1 = t(1+\varepsilon_t^2)^{1/2}\cos\beta\cos\lambda, \\ x_2 = t(1+\varepsilon_t^2)^{1/2}\cos\beta\sin\lambda, \\ x_3 = t\sin\beta, \end{cases}$$

where $t \in (0,\infty)$, $\beta \in (-\pi/2, \pi/2)$, $\lambda \in (0, 2\pi)$, and $\varepsilon_t = E/t$ (see, for example, [8, Ch. XI, §5]). Eliminating β and λ from these equations we get

$$\frac{x_1^2 + x_2^2}{t^2 + E^2} + \frac{x_3^2}{t^2} = 1, \tag{2.2}$$

which represents an oblate ellipsoid of revolution; so the t-system of surfaces consists of ellipsoids of this kind, all of which have the same foci $(\pm E, 0, 0)$. We denote by Σ_t the ellipsoid (2.2). Observe that ε_t is the second eccentricity of Σ_t. If $t = b$, then $\Sigma_b \equiv \Sigma$ and $\varepsilon_b \equiv \varepsilon$. Note in addition that if $t \ge b$ then $\varepsilon_t \le \varepsilon$.

The proof of Proposition 2.1 is based on the following identity.

LEMMA 2.5 ([6]). Let $\Omega \subset \mathbb{R}^3$ be an open set. Let $u \in C^2(\Omega)$ and let $f = (f_1, f_2, f_3)$ be a C^1 vector field defined in Ω. Then,

$$2\langle f, \nabla u\rangle \Delta u = \sum_k \partial\left(\sum_j f_j T_{jk}\right) \Big/ \partial x_k - Q[\nabla u; f] \tag{2.3}$$

where

$$Q[\nabla u; f] = \sum_{j,k} T_{jk}\frac{\partial f_j}{\partial x_k}$$

and

$$T_{jk} = 2\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k} - \delta_{jk}|\nabla u|^2.$$

(Here $\langle \cdot, \cdot \rangle$ denotes the inner product of vectors in $\mathbb{R}^3$.)

We wish to apply (2.3) to functions $u \in \mathcal{H}_\infty(\Sigma^e) \cap C^1(\overline{\Sigma}^e)$ where f is the vector field given by

$$f = (x_1, x_2, (1+\varepsilon_t^2)x_3). \tag{2.4}$$

The basic property of f we want to exploit is that its restriction to Σ_t coincides with the normal vector field $\nu_t n_t$ where ν_t denotes the radius of curvature of the normal sections along the parallels of Σ_t and n_t is a unit outer vector field on Σ_t. In particular, the restriction of f to Σ is equal to νn. Since $t^{-1} = r^{-1} + O(r^{-3})$ as $r \to \infty$, then $\varepsilon_t^2 x_3 = O(r^{-1})$ at infinity. This shows that $f - x \to 0$ as $r \to \infty$. (Here $r^2 = \sum_i x_i^2$.) Since $u \in \mathcal{H}(\Sigma^e)$, integration of (2.3) over the domain Σ^e gives

$$\int_{\Sigma^e} \sum_k \partial \left(\sum_j f_j T_{jk} \right) \Big/ \partial x_k \, dx = \int_{\Sigma^e} Q[\nabla u; f] \, dx. \tag{2.5}$$

In the next Lemma we evaluate the integral on the left-hand side of (2.5).

LEMMA 2.6. Let $u \in \mathcal{H}_\infty(\Sigma^e) \cap C^1(\overline{\Sigma}^e)$. Then,

$$\int_{\Sigma^e} \sum_k \partial \left(\sum_j f_j T_{jk} \right) \Big/ \partial x_k \, dx = \int_\Sigma \nu(|u'_s|^2 - |u'_n|^2) \, dS$$

where $f = x + \varepsilon_t^2 x_3 e_3$. (Here $e_3 = (0,0,1)$.)

Proof. Let $t > b$ and let Ω_t be the region bounded by Σ and Σ_t. Then

$$\int_{\Sigma^e} \sum_k \partial \left(\sum_j f_j T_{jk} \right) \Big/ \partial x_k \, dx = \lim_{t\to\infty} \int_{\Omega_t} \sum_k \partial \left(\sum_j f_j T_{jk} \right) \Big/ \partial x_k \, dx.$$

By the divergence theorem

$$\int_{\Omega_t} \sum_k \partial \left(\sum_j f_j T_{jk} \right) \Big/ \partial x_k \, dx = \int_{\Sigma_t} \sum_{j,k} f_j n_{t,k} T_{jk} \, dS_t - \int_\Sigma \sum_{j,k} f_j n_k T_{jk} \, dS$$

where $n_t = (n_{t,k})$. For the integral on Σ we have

$$\int_\Sigma \sum_{j,k} f_j n_k T_{jk} \, dS = \int_\Sigma (2\langle \nabla u, f \rangle u'_n - \nu |\nabla u|^2) \, dS = \int_\Sigma \nu(|u'_n|^2 - |u'_s|^2) \, dS.$$

On the other hand,

$$\lim_{t\to\infty} \int_{\Sigma_t} \sum_{j,k} f_j n_{t,k} T_{jk} \, dS_t = 0. \tag{2.6}$$

In fact,

$$I_t := \int_{\Sigma_t} \sum_{j,k} f_j n_{t,k} T_{jk} \, dS_t = \int_{\Sigma_t} \nu_t(|u'_n|^2 - |u'_t|^2) \, dS_t;$$

so

$$|I_t| \leq 2 \int_{\Sigma_t} \nu_t |\nabla u|^2 \, dS_t. \tag{2.7}$$

In the system of coordinates (t, β, λ) the element of surface area on Σ_t is given by (see, for example, [5, Section 1.19])

$$dS_t = t^2 [(1 + \varepsilon_t^2)(1 + \varepsilon_t^2 \sin^2 \beta)]^{1/2} \cos \beta \, d\beta \, d\lambda. \tag{2.8}$$

In addition, since $u \in \mathcal{H}_\infty(\Sigma^e)$, then for sufficiently large r there is a constant $k > 0$ such that (see, for example, [15, Theorem §23.2])

$$|\nabla u| \leq k r^{-2}.$$

Since $r \geq t$ we then also have for large t

$$|\nabla u| \leq k t^{-2}. \tag{2.9}$$

Finally, substituting (2.8) in (2.7), taking into account (2.9), and since $\nu_t \leq t(1 + \varepsilon_t^2)$, we have for large values of t

$$|I_t| \leq 8\pi k^2 t^{-1} (1 + \varepsilon^2)^2,$$

which proves (2.6). This completes the proof of this Lemma. □

We now want to estimate $Q[\nabla u; f]$ in terms of $|\nabla u|^2$ for the vector field (2.4). Firstly, observe that we can write

$$Q[\nabla u; f] = 2 \nabla u^{\mathrm{T}} A_f \nabla u - |\nabla u|^2 \operatorname{Tr}(A_f)$$

where the (j, k) entry of the matrix A_f is

$$\frac{1}{2} \left(\frac{\partial f_j}{\partial x_k} + \frac{\partial f_k}{\partial x_j} \right)$$

and $\operatorname{Tr}(A_f)$ denotes the trace of A_f.

To calculate A_f for the vector field (2.4) we need the first-order partial derivatives of the function t. Since

$$t^4 - t^2 (r^2 - E^2) - E^2 x_3^2 = 0,$$

differentiating both sides with respect to each of the x_i we get

$$\begin{cases} \partial t / \partial x_1 = \gamma_t (1 - \varepsilon_t^2 \sin^2 \beta)^{-1} \cos \beta \cos \lambda, \\ \partial t / \partial x_2 = \gamma_t (1 + \varepsilon_t^2 \sin^2 \beta)^{-1} \cos \beta \sin \lambda, \\ \partial t / \partial x_3 = \gamma_t^2 (1 + \varepsilon_t^2 \sin^2 \beta)^{-1} \sin \beta \end{cases}$$

where $\gamma_t = (1 + \varepsilon_t^2)^{1/2}$. After some computations we obtain

$$A_f = I + \delta C$$

where

$$C = \begin{bmatrix} 0 & 0 & -a \\ 0 & 0 & -b \\ -a & -b & c \end{bmatrix},$$

I is the unit matrix, and

$$\begin{aligned}\delta &= \varepsilon_t^2(1+\varepsilon_t^2\sin^2\beta)^{-1},\\ a &= \gamma_t \sin\beta\cos\beta\cos\lambda,\\ b &= \gamma_t \sin\beta\cos\beta\sin\lambda,\\ c &= 1-(2+\varepsilon_t^2)\sin^2\beta.\end{aligned}$$

The eigenvalues of the matrix $B_f = 2A_f - \mathrm{Tr}(A_f)I$ are

$$\begin{aligned}\lambda_1 &= -(1+\varepsilon_t^2),\\ \lambda_2 &= -(1+\delta c),\\ \lambda_3 &= -(1-\varepsilon_t^2).\end{aligned}$$

Since $|\delta c| \le \varepsilon_t^2$, then $\lambda_1 \le \lambda_2 \le \lambda_3$. Hence B_f is negative definite if and only if $\varepsilon_t < 1$. We have therefore proved the following.

LEMMA 2.7. Let Q be the quadratic form

$$Q[p;f] = 2p^{\mathrm{T}}A_f p - |p|^2\,\mathrm{Tr}(A_f)$$

where $p \in \mathbb{R}^3$ and $f = x + \varepsilon_t^2 x_3 e_3$. Then Q is negative definite if and only if $\varepsilon_t < 1$. In addition

$$-(1+\varepsilon_t^2)|p|^2 \le Q[p;f] \le -(1-\varepsilon_t^2)|p|^2.$$

Since outside of Σ we have $\varepsilon_t \le \varepsilon$, if $u \in C^1(\overline{\Sigma}^e)$ then from this Lemma we obtain

$$Q[\nabla u; f] \le -(1-\varepsilon^2)|\nabla u|^2;$$

so

$$\int_{\Sigma^e} Q[\nabla u; f]\,dx \le -(1-\varepsilon^2)\int_{\Sigma^e} |\nabla u|^2\,dx. \tag{2.10}$$

Finally, (2.5), Lemma 2.6, and (2.10) give Proposition 2.1.

3. Proof of the main Theorem. Let $g : \Sigma \to S^2$ be the Gauss map of Σ defined as $g(x) = n_x$, where n_x is the unit outer normal to Σ at $x \in \Sigma$. Using (1.4) we explicitly have

$$g(\nu\cos\varphi\cos\lambda, \nu\cos\varphi\sin\lambda, (1+\varepsilon^2)^{-1}\nu\sin\varphi) = (\cos\varphi\cos\lambda, \cos\varphi\sin\lambda, \sin\varphi).$$

This, of course, is a *diffeomorphism* and between $d\sigma$ on S^2 and dS on Σ we have $d\sigma = K\,dS$, where $K = (\nu\rho)^{-1}$ is the Gaussian curvature of Σ (for these results see, for example, [2]). Then, for any integrable function f on Σ we have

$$\int_\Sigma f\,dS = \int_\sigma FK^{-1}\,d\sigma$$

where F is the pull-back $f \circ g^{-1}$ of f to the unit sphere. In addition, we have the following.

LEMMA 3.1. Let $f \in C^1(\Sigma)$. Then

$$\alpha^{-1}|F'|^2\,d\sigma \le |f'|^2\,dS \le \alpha|F'|^2\,d\sigma$$

where $f' = \operatorname{grad} f$ and $F' = \operatorname{grad} F$.

(For the proof see Eq. (1.2.7) in [6] and simply observe that the function Q appearing in this Eq. (1.2.7) is in our case equal to α.)

Let $u \in C^1(\overline{\Sigma}^e)$ be a solution of (1.3). Since u'_s coincides with the gradient of the restriction of u to Σ, then from Corollary 2.4 and Lemma 3.1 we have

$$\int_\sigma \rho|U'_\sigma|^2\,d\sigma \le \int_\sigma \rho(1+\alpha)(\varepsilon^2+\alpha)U^2\,d\sigma \tag{3.1}$$

where $U = u \circ g^{-1}$ and U'_σ is the gradient of U. Parametrizing the unit sphere by λ and φ, the expressions for ρ and α in (3.1) are those given in Sec. 1. In addition, $d\sigma = \cos\varphi\,d\varphi\,d\lambda$.

The positive function $\rho(1+\alpha)(\varepsilon^2+\alpha)$ reaches its maximum value at $\varphi = \pm\pi/2$. Then

$$\rho(1+\alpha)(\varepsilon^2+\alpha) \le 2a(1+\varepsilon^2)^{3/2}.$$

On the other hand, $\rho \ge a(1+\varepsilon^2)^{-1}$; so we get

$$\int_\sigma |U'_\sigma|^2\,d\sigma \le 2\gamma^5 \int_\sigma U^2\,d\sigma$$

where $\gamma = (1+\varepsilon^2)^{1/2} = (1-e^2)^{-1/2}$.

Let $I = \int_\sigma |U'_\sigma|^2\,d\sigma - k\int_\sigma U^2\,d\sigma$ where $k = 2\gamma^5$. Denoting by U_n $(n \ge 0)$ the L^2 projection of U on the space spanned by spherical harmonics of degree n, we have

$$I = \sum_{n\ge 0}(n^2+n-k)\|U_n\|^2.$$

(Here $\|U_n\|^2 = \int_\sigma U_n^2\,d\sigma$.) It should be noted that since $k \ge 2$, then $n^2+n-k > 0$ only if $n \ge 2$. In addition, if $k < 6$ ($\Leftrightarrow \varepsilon < (3^{2/5}-1)^{1/2})$), then $k_0 = (6-k)/7$ is the greatest positive number for which the following inequality

$$0 < k_0(n^2+n+1) \le n^2+n-k$$

holds, for all $n \ge 2$. Then

$$\begin{aligned} I &= -k\|U_0\|^2 + (2-k)\|U_1\|^2 + \sum_{n\ge 2}(n^2+n-k)\|U_n\|^2 \\ &\ge -k\|U_0\|^2 + (2-k)\|U_1\|^2 + k_0\sum_{n\ge 2}(n^2+n+1)\|U_n\|^2; \end{aligned}$$

so

$$I + a_0\|U_0\|^2 + a_1\|U_1\|^2 \ge k_0\sum_{n\ge 0}(n^2+n+1)\|U_n\|^2 = k_0\int_\sigma (U^2+|U'_\sigma|^2)\,d\sigma$$

where $a_0 = k_0 + k = 6(1+k)/7$ and $a_1 = 3k_0 + k - 2 = 4(1+k)/7$.

Summing up, we have proved the following.

PROPOSITION 3.2. Let Σ be an oblate ellipsoid of revolution such that $\varepsilon < (3^{2/5}-1)^{1/2}$. If $u \in \mathcal{H}_\infty(\Sigma^e) \cap C^1(\overline{\Sigma}^e)$ satisfies $u'_n = 2Hu$ on Σ, then

$$\int_\sigma (U^2 + |U'_\sigma|^2)\, d\sigma \le \frac{1+2\gamma^5}{3-\gamma^5}(3\|U_0\|^2 + 2\|U_1\|^2) \tag{3.2}$$

where $\gamma = (1+\varepsilon^2)^{1/2}$.

REMARK 3.3. Observe that if the right-hand side of (3.2) is not larger than $k' \int_\sigma (U^2 + |U'_\sigma|^2)\, d\sigma$ for some $k' < 1$, then it follows that $U = 0$ and hence $u = 0$ in $\overline{\Sigma}^e$.

In order to estimate the L^2 norm of U_0 and U_1 we recall that

$$U_0 = \frac{1}{4\pi}\int_\sigma U\, d\sigma$$

and $U_1(\sigma) = \sum_{i=1}^3 U_{1i}Y_{1i}(\sigma)$ where $Y_{1i}(\sigma) = \sqrt{3/(4\pi)}\sigma_i$ and

$$U_{1i} = \int_\sigma UY_{1i}\, d\sigma.$$

The next Lemma characterizes the behaviour $u(x) = c/|x| + O(|x|^{-3})$ at infinity of harmonic functions (cf. [6, §1.3]).

LEMMA 3.4. Let ω be a C^1 closed surface in $\mathbb{R}^3$, and let Ω be the unbounded connected component of $\mathbb{R}^3 - \{\omega\}$. Let $u \in \mathcal{H}_\infty(\Sigma^e) \cap C^1(\overline{\Sigma}^e)$. Then, $u(x) = c/|x| + O(|x|^{-3})$ as $x \to \infty$ if and only if

$$\int_\omega \left(x_i\frac{\partial u}{\partial n} - n_i u\right) d\omega = 0 \quad \forall i \in \{1,2,3\},$$

where $n = (n_i)$ is the unit outer normal to ω.

Proof. Let $v_i(x) := x_i$. Let Σ_R be a sphere of sufficiently large radius R such that ω is strictly contained in the interior of Σ_R. Applying the Green formula to u and v_i in the region bounded by ω and Σ_R we get

$$\int_\omega \left(x_i\frac{\partial u}{\partial n} - n_i u\right) d\omega = \sqrt{\frac{4\pi}{3}}\int_{\Sigma_R} Y_{1i}\left(R\frac{\partial u}{\partial r} - u\right) d\sigma_R.$$

If $u = \sum_{n=0}^\infty (R/r)^{n+1}u_n$ $(r \ge R)$ where u_n is a spherical harmonic of degree n, then we have on Σ_R

$$R\frac{\partial u}{\partial r} - u = -\sum_{n=0}^\infty (n+2)u_n.$$

Hence

$$\int_{\Sigma_R} Y_{1i}\left(R\frac{\partial u}{\partial r} - u\right) d\sigma_R = -3R^2\int_{S^2} Y_{1i}u_1\, d\sigma$$

and this completes the proof. □

PROPOSITION 3.5. Let Σ be an oblate ellipsoid of revolution. Let $u \in \mathcal{H}_\infty(\Sigma^e) \cap C^1(\overline{\Sigma}^e)$ be such that $u'_n = 2Hu$ on Σ. Then

$$U_0 = \frac{1}{4\pi}\int_\sigma F_0 U\, d\sigma \tag{3.3}$$

where

$$F_0 = 2r^{-1}HK^{-1} + r^{-3}x_nK^{-1} + 1. \tag{3.4}$$

(Here $r(\sigma) = |g^{-1}(\sigma)|$ and $x_n = \langle x, \sigma\rangle$ where $x(\sigma) = g^{-1}(\sigma)$.)

If, in addition, $u(x) = c/|x| + O(|x|^{-3})$ as $x \to \infty$ we have

$$U_{1i} = \frac{1}{\sqrt{12\pi}} \int_\sigma F_i U \, d\sigma \tag{3.5}$$

where

$$F_i = 2a^{-2}x_iHK^{-1} - a^{-2}\sigma_iK^{-1} + 3\sigma_i. \tag{3.6}$$

Proof. The first part is quite clear since the Green formula applied to u and r^{-1} gives

$$2\int_\Sigma r^{-1}Hu\,dS = -\int_\Sigma r^{-3}x_nu\,dS,$$

and so

$$\begin{aligned}\int_\sigma U\,d\sigma &= \int_\Sigma Ku\,dS = \int_\Sigma (2r^{-1}H + r^{-3}x_n + K)u\,dS \\ &= \int_\sigma (2r^{-1}HK^{-1} + r^{-3}x_nK^{-1} + 1)U\,d\sigma.\end{aligned}$$

For the proof of the second part we first observe that from Lemma 3.4 we have

$$\int_\Sigma n_iu\,dS = 2\int_\Sigma x_iHu\,dS$$

and so

$$3\int_\Sigma n_iu\,dS = 2\int_\Sigma (x_iH + n_i)u\,dS.$$

On the other hand, writing

$$\int_\Sigma n_iKu\,dS = a^{-2}\int_\Sigma n_iu\,dS + \int_\Sigma n_i(K - a^{-2})u\,dS$$

we conclude that

$$\begin{aligned}\int_\sigma \sigma_iU\,d\sigma &= \int_\Sigma n_iKu\,dS = \frac{1}{3}\int_\Sigma [2a^{-2}(x_iH + n_i) + 3n_i(K - a^{-2})]u\,dS \\ &= \frac{1}{3}\int_\sigma (2a^{-2}x_iHK^{-1} - a^{-2}\sigma_iK^{-1} + 3\sigma_i)U\,d\sigma.\end{aligned}$$

The Lemma is proved. □

REMARK 3.6. Observe that if $e = 0$ then $F_0 \equiv 0$ and $F_1 \equiv 0$. In this case we then have $U_0 = 0$ and $U_1 = 0$ and so $U \equiv 0$. We also expect that for "small" values of e, F_0 and F_1 differ but little from zero.

By Schwartz's inequality and this proposition we have the following estimates for the L^2 norm of U_0 and U_1:

$$\|U_0\|^2 \leq \left[\int_\sigma |F_0|^2\, d'\sigma\right] \int_\sigma U^2\, d\sigma \leq \left[\int_\sigma |F_0|^2\, d'\sigma\right] D^2, \tag{3.7}$$

$$\|U_1\|^2 \leq \frac{1}{3}\left[\sum_{i=1}^{3}\int_\sigma |F_i|^2\, d'\sigma\right] \int_\sigma U^2\, d\sigma \leq \frac{1}{3}\left[\sum_{i=1}^{3}\int_\sigma |F_i|^2\, d'\sigma\right] D^2 \tag{3.8}$$

where $D^2 = \int_\sigma (U^2 + |U'_\sigma|^2)\, d\sigma$ and $d'\sigma = 1/(4\pi)\, d\sigma$.

With the parametrization of S^2 by $\lambda \in (0, 2\pi)$ and $\varphi \in (-\pi/2, \pi/2)$, it is straightforward to see that $x_n = a^2\nu^{-1}$; so

$$F_0 = -\xi(1+\alpha) + \xi\xi_a^2 + 1$$

where $\xi = \rho r^{-1}$, $\xi_a = ar^{-1}$, and $r^2 = \nu^2[1 - (1-\gamma^{-4})\sin^2\varphi]$. As we see, F_0 does not depend on λ and it depends on φ only through the square of $t = \sin\varphi$. By computational convenience, we make in (1.5) and (1.6) the substitution $\varepsilon^2 = \gamma^2 - 1$; so we may consider F_0 as a function of t and γ. Hence

$$W(\gamma) := \int_\sigma |F_0|^2\, d'\sigma = \int_0^1 |F_0(t,\gamma)|^2\, dt.$$

The functions F_i may be written in the form

$$\begin{cases} F_1 = \widehat{F}_{12}\cos\varphi\cos\lambda, \\ F_2 = \widehat{F}_{12}\cos\varphi\sin\lambda, \\ F_3 = \widehat{F}_3 \sin\varphi, \end{cases}$$

where

$$\widehat{F}_{12} = 3 - a^{-2}\nu\rho(2+\alpha)$$

and

$$\widehat{F}_3 = 3 - a^{-2}\nu\rho[1 + \gamma^{-2}(1+\alpha)].$$

Again, we observe that $\widehat{F}_{12}$ and $\widehat{F}_3$ do not depend on λ and they depend on φ only through t^2, and as before, using $\varepsilon^2 = \gamma^2 - 1$, the functions involved depend on t and γ. Since $\int_0^{2\pi} |\cos\lambda|^2\, d\lambda = \int_0^{2\pi} |\sin\lambda|^2\, d\lambda = \pi$, then

$$Z(\gamma) := \sum_{i=1}^{3}\left(\int_\sigma |F_i|^2\, d'\sigma\right) = \int_0^1 (1-t^2)|\widehat{F}_{12}(t,\gamma)|^2\, dt + \int_0^1 t^2|\widehat{F}_3(t,\gamma)|^2\, dt.$$

With the notation we have just introduced, (3.7) and (3.8) read as follows:

$$\|U_0\|^2 \leq W(\gamma)D^2,$$

$$\|U_1\|^2 \leq \frac{1}{3}Z(\gamma)D^2.$$

Recalling Remark 3.3, if we apply these estimates in the right-hand side of (3.2), we get a uniqueness theorem for the boundary value problem (1.1) provided that the ellipsoid Σ is such that

$$\gamma \in [1, 3^{1/5}) \cap \mathcal{S}$$

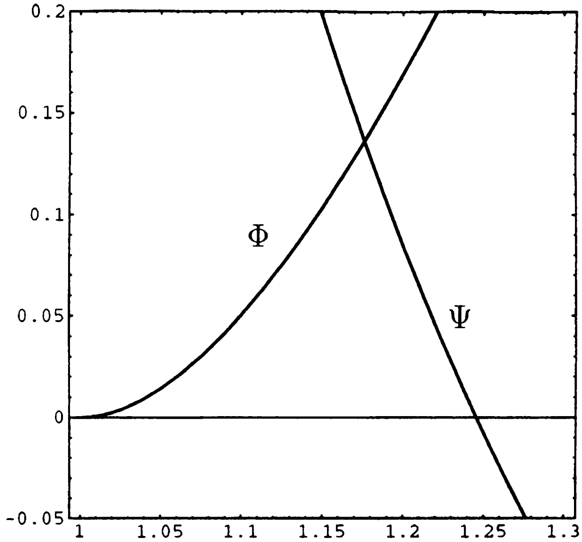

FIG. 1. Functions Φ and Ψ

where

$$\mathcal{S} = \{\gamma \geq 1 : \Phi(\gamma) < \Psi(\gamma)\},$$

and

$$\Phi(\gamma) = 3W(\gamma) + \tfrac{2}{3}Z(\gamma),$$
$$\Psi(\gamma) = \frac{3-\gamma^5}{1+2\gamma^5}.$$

Using MATHEMATICA we get (see Fig. 1)

$$[1, 3^{1/5}) \cap \mathcal{S} = [1, \gamma_0) \tag{3.9}$$

where (approximately) $\gamma_0 = 1.176166$. Since $e = \sqrt{1-\gamma^{-2}}$, and the function $f : [1,\infty) \to [0,1)$ defined as $f(x) = \sqrt{1-x^{-2}}$ is a homeomorphism, our main result (Theorem A) has just been proved with

$$e_0 = \sqrt{1-\gamma_0^{-2}}. \tag{3.10}$$

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