

EFFECTS OF AN EXTERNAL PERIODIC BODY FORCE ON THE INTERFACIAL STABILITY OF A NEMATIC LAYER

BY

GALAL M. MOATIMID AND YUSRY O. EL-DIB

Department of Mathematics, Faculty of Education, Ain Shams University, Heliopolis, Cairo, Egypt

On using the Ericksen-Leslie continuum theory, the interfacial hydrodynamic stability of a nematic layer has been investigated. The layer is assumed to be influenced by an external vertical periodic body force. A general form of the deformations of the interface is considered. The normal modes technique has been utilized. The coupling between hydrodynamic motion and internal degrees of freedom (molecular orientation) includes some changes in the structure and dispersion of the surface modes. The problem contains nontraditional boundary-value equations that lead to a transcendental equation in the zero-order perturbation. The method of multiple time scales is used to analyze first-order perturbation equations. The solvability condition is obtained. It is found that a resonance mode may appear due to the periodicity of the external periodic body force. The transition curves are obtained. The analytical results are numerically confirmed taking into account the natural physical parameters of the MBBA and PAA nematics. It is found that the amplitude of the external periodic body force plays a destabilizing influence on the system. It is also found that the increase of the periodicity of the external force leads to a contraction in the instability of the system. It is also observed that the mechanism of the instability of MBBA differs from that of PAA.

1. Introduction. Liquid crystals are a state of matter intermediate between that of a crystalline solid and an isotropic liquid. They possess many of the mechanical properties of a liquid, e.g., high fluidity, inability to support shear, formation, and coalescence of droplets. At the same time they are similar to crystals in that they exhibit anisotropy in their optical, electrical, and magnetic properties. They are considered subjects of great interest for both fundamental and practical reasons [1]–[3]. Fundamentally, many mesophases in which liquid crystals exist and their corresponding phase transition characteristics offer a good testing ground for statistical mechanics.

Nematic liquid crystals [1] are anisotropic liquids in which the constituting molecules are, on average, aligned with their unique axis parallel to a preferred direction in space. This direction is labeled by the director \mathbf{n} . Though a nematic liquid flows as easily as an isotropic liquid consisting of similar molecules, an analysis of the viscosities turns out

Received November 10, 1994.

1991 *Mathematics Subject Classification.* Primary 82B24, 82D25.

to be rather complicated when the state of alignment, as given by \mathbf{n} , is considered. The direction of the alignment of liquid crystal molecules in the bulk nematic is arbitrary because the free energy of the system is rotationally invariant, i.e., it does not depend on the nematic director \mathbf{n} . This rotational symmetry of the system is broken by the presence of a limiting surface. As expected the flow depends on the angle between \mathbf{n} , the flow direction and the velocity gradient. This leads to the three Miesowicz viscosities η_1, η_2 , and η_3 [4]. In addition, the translational motions couple with the inner orientational motions and will cause the director to rotate. This behavior can be described by the hydrodynamic theory of Ericksen [5] and Leslie [6], which involves five independent viscosity coefficients.

The free surface of a liquid in equilibrium in a gravitational field is a plane. If the surface is moved from its equilibrium position at some point, the motion will occur in the liquid under the action of some external perturbation. This motion will be propagated over the whole surface in the form of waves, which are called gravity waves. The deformability of the free surface leads to capillary-gravity waves, which depend on forces tending to return the deformed surface to its equilibrium plane shape, unless we supply enough energy to overcome the viscosity inertia enabling overshooting of the Laplace over pressure and thus damped oscillations to occur. These transverse waves will only slightly depend on the exchange of solute between the bulk and the interface [7].

The surface (or capillary) waves on nematics provide a further application of the hydrodynamic equations. These waves have been observed by Langevin [8] in PAA and MBBA by light scattering from the free surface. The spectral distribution of the scattered light gives information on the surface tension and viscosity coefficients. The dispersion relation for surface waves in nematics has been determined from the hydrodynamic equations by Langevin and Bouchiat [9]. The form of this dispersion relation depends on the boundary conditions at the surface as well as the orientation of the molecules at the surface. In the case of surface waves in nematics, Langevin and Bouchiat [9] assumed that the forces that determine the orientation of the molecules at the surface are strong. When the surface is distorted by the wave, the director makes a fixed angle with the normal to the surface. In this case it has been shown by Langevin and Bouchiat [9] that the dispersion relation for surface waves on nematics is very similar to that for surface waves on normal liquids.

Surface effects of liquid crystals are important for both device applications and basic understanding of physical phenomena. The study of free-surface effects is even more interesting for understanding the anisotropy of molecular interactions. In this attempt, we have analyzed the instability of the interface of a nematic layer in a gravitational field. The layer is influenced by an external periodic body force. The method of multiple time scales is used to achieve the transition curves. On using the natural physical parameters of the nematics MBBA and PAA, the analytical results are numerically confirmed.

2. Formulation and equations of motion. An infinite nematic medium is considered to occupy the half-space $z < 0$. The unperturbed free surface of the nematic is the plane $z = 0$. Gravitational effects are taken into account. The medium is acted upon by

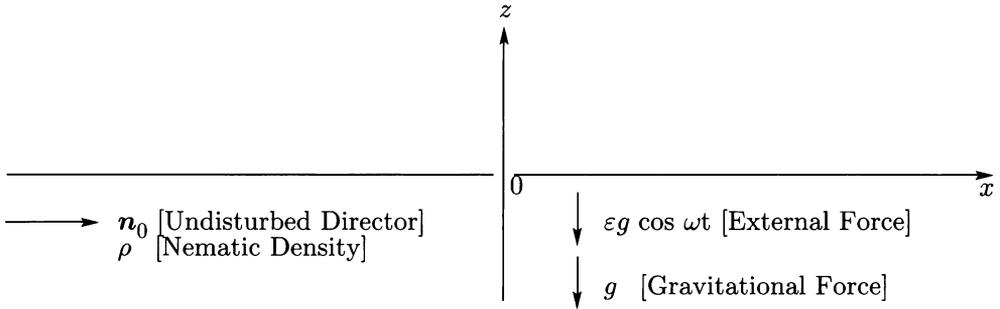


FIG. 0. Schematic diagram of configuration in the steady state

a vertical external periodic body force

$$\mathbf{F} = -\epsilon g \cos \omega t \mathbf{e}_z, \tag{2.1}$$

where g is the gravitational acceleration, ϵ is a small dimensionless parameter, ω is the frequency of the applied external body force and \mathbf{e}_z is the unit vector along the z -direction.

This may happen by varying the pressure in the air in some subtle way. The nematic director is assumed to be strongly anchored along the x -direction and is also assumed to be oriented uniformly along x throughout the sample in the steady state. Thus, in the steady state, the director is described by

$$\mathbf{n}_0 = (1, 0, 0). \tag{2.2}$$

We assume that everything does not depend on y : $\partial/\partial y = 0$; namely, a two-dimensional system in the (x, z) -plane is only considered. We also assume that the director \mathbf{n} always lies in this plane. Figure 0 is a schematic diagram of the configuration in the steady state.

The hydrodynamic equations applied to liquid crystals can be derived in a standard way from conservation laws [10]. The conservation of mass yields the equation of continuity

$$\frac{D}{Dt}(\rho) + \rho(\nabla \cdot \mathbf{V}) = 0, \tag{2.3}$$

where ρ is the mass density, \mathbf{V} is the fluid velocity, and D/Dt stands for the convective time derivative ($\partial/\partial t + \mathbf{V} \cdot \nabla$).

We shall restrict ourselves further to incompressible fluids, for which

$$\rho(x, z, t) = \text{constant}. \tag{2.4}$$

Consequently, certain phenomena such as the propagation of sound waves will not be considered. With this approximation, one gets

$$\text{div } \mathbf{V} = 0. \tag{2.5}$$

The conservation of linear momentum yields the force equation

$$\rho \frac{D}{Dt}(V_i) = F_i + \frac{\partial}{\partial x_j}(\sigma_{ij}), \quad (2.6)$$

where F_i is the i th component of any applied body force per unit volume and σ_{ij} is the stress tensor associated with the fluid. This tensor may be broken into three parts,

$$\sigma_{ij} = -P\delta_{ij} + \sigma_{0ij} + \sigma'_{ij}, \quad (2.7)$$

where P is a hydrostatic pressure, σ_{0ij} is a stress due to elastic deformations of the liquid, and σ'_{ij} is a viscous component. From [11], we have

$$\sigma_{0ij} = -\frac{\partial G}{\partial n_{k,j}} n_{k,i}, \quad (2.8)$$

where G is an elastic free energy and $n_{i,j} = \partial n_i / \partial x_j$.

Normally σ_{0ij} is quadratic in $n_{i,j}$ and can be ignored in linear treatments.

The viscous stress tensor σ'_{ij} has been derived by Leslie [11]:

$$\sigma'_{ij} = \alpha_1(n_k n_p A_{kp}) n_i n_j + \alpha_2 N_i n_j + \alpha_3 n_i N_j + \alpha_4 A_{ij} + \alpha_5 n_j n_k A_{ki} + \alpha_6 n_i n_k A_{kj}, \quad (2.9a)$$

where $\overset{\leftrightarrow}{A}$ is the strain rate tensor

$$A_{ij} = \frac{1}{2} \left(\frac{\partial}{\partial x_j}(V_i) + \frac{\partial}{\partial x_i}(V_j) \right), \quad (2.9b)$$

and

$$\mathbf{N} = \frac{D}{Dt}(\mathbf{n}) - \frac{1}{2}(\text{curl } \mathbf{V}) \times \mathbf{n}. \quad (2.9c)$$

The α 's are constants with the dimension of viscosity and are not all independent. It has been shown that [12]:

$$\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5.$$

The equation of the director may be represented as [4]:

$$I \frac{d}{dt} \left(\mathbf{n} \times \frac{d\mathbf{n}}{dt} \right) = \Gamma_{\text{el}} - \Gamma_{\text{visc}}, \quad (2.10)$$

where I is the moment of inertia per unit volume, and Γ_{el} and Γ_{visc} are the elastic and viscous torques respectively.

The elastic torque is given by [13]:

$$\Gamma_{\text{el}} = \mathbf{n} \times \{ \mathbf{h}_S + \mathbf{h}_T + \mathbf{h}_B \} = \mathbf{n} \times \mathbf{h}, \quad (2.11)$$

where the various contributions ($_S, _T$ and $_B$ denote splay, twist and bend respectively) are:

$$\mathbf{h}_S = K_{11} \text{grad}(\text{div } \mathbf{n}), \quad (2.12a)$$

$$\mathbf{h}_T = -K_{22} [(\mathbf{n} \cdot \text{rot } \mathbf{n}) \text{rot } \mathbf{n} + \text{rot}((\mathbf{n} \cdot \text{rot } \mathbf{n})\mathbf{n})], \quad (2.12b)$$

$$\mathbf{h}_B = K_{33} [(\mathbf{n} \times \text{rot } \mathbf{n}) \text{rot } \mathbf{n} + \text{rot}(\mathbf{n} \times (\mathbf{n} \times \text{rot } \mathbf{n}))] \quad (2.12c)$$

and the viscous torque is given by [13]

$$\Gamma_{\text{visc}} = \mathbf{n} \times ((\alpha_3 - \alpha_2)\mathbf{N} + (\alpha_3 + \alpha_2)\overleftrightarrow{A} \cdot \mathbf{n}). \quad (2.13)$$

At low frequencies, the inertial term in Eq. (2.10) is much smaller than the elastic and viscous torques. Then Eq. (2.10) is referred to as the balance of torques, after inserting the results. From Eqs. (2.11) and (2.13), we get

$$\mathbf{n} \times \mathbf{h} = \mathbf{n} \times ((\alpha_3 - \alpha_2)\mathbf{N} + (\alpha_3 + \alpha_2)\overleftrightarrow{A} \cdot \mathbf{n}). \quad (2.14)$$

3. Perturbation equations. Consider the effect of a small wave disturbance on the interface $z = 0$, propagating in the positive x -direction. The surface deflection is assumed to be in the form

$$\zeta = \gamma(t) \exp iqx, \quad (3.1)$$

where $\gamma(t)$ is an arbitrary function of time that determines the behavior of the amplitude of the disturbance of the interface, and q is the wavenumber, which is assumed to be positive.

For a small departure from the equilibrium state, the quantities \mathbf{V} , \mathbf{n} , and P receive the increments \mathbf{V}_1 , \mathbf{n}_1 , and P_1 to yield

$$\mathbf{V} = \mathbf{V}_1, \quad \mathbf{n} = \mathbf{n}_0 + \mathbf{n}_1, \quad \text{and} \quad P = P_0 + P_1. \quad (3.2)$$

The subscript “0” refers to quantities in the equilibrium state. The small fluctuation \mathbf{n}_1 must be perpendicular to \mathbf{n}_0 since necessarily $n^2 = 1$. It follows that $\mathbf{n}_1 = (0, 0, n_z)$. The perturbation of the velocity \mathbf{V}_1 has the components u , v , and w .

Analyzing the disturbances into normal modes, there is no loss of generality in confining attention to perturbations that are independent of y . Thus, the perturbation quantities may be represented in the following form:

$$\psi(x, z, t) = \hat{\psi}(z, t) \exp iqx. \quad (3.3)$$

The zero-order solutions of the equations of motion are

$$P_0 = \lambda(t) - \rho g(1 + \varepsilon \cos \omega t)z, \quad (3.4)$$

where $\lambda(t)$ is the time-dependent integration constant.

The linearization of the director equation (2.14) becomes

$$\left(\gamma_1 \frac{\partial}{\partial t} - K_{11} \frac{\partial^2}{\partial z^2} + q^2 K_{33} \right) \hat{n}_z + \alpha_3 \frac{\partial \hat{u}}{\partial z} + iq\alpha_2 \hat{w} = 0. \quad (3.5)$$

Equation (3.5) shows that, in our theoretical model, the deformations that act on the nematic liquid crystals involve the splay and bend moduli K_{11} and K_{33} respectively and not the twist module K_{22} . This is the same role as in the Freedericksz effect [2]. This equation can be simplified since the elastic terms are small. These terms are to be compared with $\omega\gamma_1$ where $\gamma_1 \cong 0.1$ poise and ω is the frequency shift, which may be several MHz. The numerical values of these constants [8] are given in Table 1. The negligence of the elastic terms leads to

$$\frac{\partial \hat{n}_z}{\partial t} = -\frac{1}{\gamma_1} \left(\alpha_3 \frac{\partial \hat{u}}{\partial z} + iq\alpha_2 \hat{w} \right). \quad (3.6)$$

Properties	PAA near 125° C	MBBA near 25° C
<u>Frank constant :</u>		
k_{11} (dyne)	4.5×10^{-7}	6×10^{-7}
k_{22}	2.9×10^{-7}	4×10^{-7}
k_{33}	9.5×10^{-7}	7.5×10^{-7}
<u>Viscosity Coefficients :</u>		
γ_1 (centipoise)	6.7	77
γ_2	-7.0	-80
α_1	4.3	6.5
α_2	-6.9	-77.5
α_3	-.2	-1
α_4	6.8	83
α_5	4.7	46
α_6	-2.3	-35
<u>Density :</u>		
ρ (gm cm ⁻³)	1.159	1.088
<u>Surface Tension :</u>		
σ (dyn/cm)	38	40

TABLE 1. Properties of PAA near 125° C and MBBA near 25° C

It might be interesting to note that the director equation, Eq. (3.5), is identically satisfied in the case of normal liquids.

By making use of Eq. (3.6), the linearization of the equations of motion (2.5, 2.6) yields

$$\left(\rho \frac{\partial}{\partial t} + q^2(2\nu_1 - \nu_3)\right) \hat{u} - \nu_3 \frac{\partial^2 \hat{u}}{\partial z^2} + iq\hat{P} = 0, \quad (3.7)$$

$$\left(\rho \frac{\partial}{\partial t} + q^2\nu_3\right) \hat{w} + (\nu_3 - 2\nu_2) \frac{\partial^2 \hat{w}}{\partial z^2} + \frac{\partial \hat{P}}{\partial z} = 0, \quad (3.8)$$

$$iq\hat{u} + \frac{\partial \hat{w}}{\partial z} = 0. \quad (3.9)$$

Apart from the different viscosity coefficients, these equations are exactly the same ones for a normal fluid [14]. The viscosity coefficients entering Eqs. (3.7), (3.8), and (3.9) are

$$\nu_1 = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5)/2, \quad (3.10a)$$

$$\nu_2 = \alpha_4/2, \quad (3.10b)$$

$$\nu_3 = (\alpha_4 + \alpha_5 - \gamma_2\alpha_2/\gamma_1)/2. \quad (3.10c)$$

This is the notation of Forster et al. [8], but it differs from that of Miesowicz [4]. It is worthwhile to notice that in the case of normal liquids these notations become

$$\nu_1 = \nu_2 = \nu_3 = \alpha_4/2 = \eta, \quad (3.11)$$

the case that has been extensively studied earlier [14].

The associated boundary conditions [14] are

$$\hat{w}(0, t) = \frac{d\gamma}{dt}, \quad z = 0, \tag{3.12}$$

$$iq\hat{w} + \frac{\partial \hat{u}}{\partial z} = 0, \quad z = 0, \tag{3.13}$$

$$\{\sigma q^2 + \rho g(1 + \varepsilon \cos \omega t)\}\gamma + 2\nu_2 \frac{\partial \hat{w}}{\partial z} = \hat{P}, \quad z = 0, \tag{3.14}$$

where σ is the surface tension coefficient.

4. Method of solution. The system of Eqs. (3.7), (3.8), (3.9), (3.12), (3.13), and (3.14) is rather complicated. Moreover, this system cannot be solved by the method of separation of variables. Due to the complexity of this system, an asymptotic expansion technique is utilized. In order to carry out the stability analysis, we will use the method of multiple time scales [15], which depends on a smallness parameter. This method is applied in the assumption that the amplitude of the periodic external body force is small.

On applying the method of multiple time scales, we limit ourselves to $O(\varepsilon)$ and use only

$$T_0 = t \quad \text{and} \quad T_1 = \varepsilon t. \tag{4.1}$$

The derivatives may be transformed according to

$$\frac{\partial}{\partial t} = D_0 + \varepsilon D_1, \quad D_n = \frac{\partial}{\partial T_n}. \tag{4.2}$$

Moreover, one may assume that the solutions have the following expansions:

$$\gamma(t; \varepsilon) = \gamma_0(T_0, T_1) + \varepsilon \gamma_1(T_0, T_1) + \dots, \tag{4.3}$$

$$\hat{\psi}(z, t; \varepsilon) = \hat{\psi}_0(z, T_0, T_1) + \varepsilon \hat{\psi}_1(z, T_0, T_1) + \dots, \tag{4.4}$$

where $\hat{\psi}$ stands for \hat{u}, \hat{w} , and \hat{P} .

It is clear that the zero order of ε corresponds to the case of absence of the external periodic body force, while the first-order perturbation represents the general case.

Substituting the above expansions (4.3 and 4.4) into Eqs. (3.7), (3.8), (3.9) and the related boundary conditions (3.12), (3.13), (3.14) and equating the coefficients of equal powers of ε we obtain

$$(\rho D_0 + q^2(2\nu_1 - \nu_3))\hat{u}_0 - \nu_3 \frac{\partial^2 \hat{u}_0}{\partial z^2} + iq\hat{P}_0 = 0, \tag{4.5}$$

$$(\rho D_0 + q^2\nu_3)\hat{w}_0 + (\nu_3 - 2\nu_2) \frac{\partial^2 \hat{w}_0}{\partial z^2} + \frac{\partial \hat{P}_0}{\partial z} = 0, \tag{4.6}$$

$$iq\hat{u}_0 + \frac{\partial \hat{w}_0}{\partial z} = 0, \tag{4.7}$$

$$\hat{w}_0 = D_0\gamma_0 \quad \text{at} \quad z = 0, \tag{4.8}$$

$$iq\hat{w}_0 + \frac{\partial \hat{u}_0}{\partial z} = 0, \quad z = 0, \tag{4.9}$$

$$\{\sigma q^2 + \rho g\}\gamma_0 + 2\nu_2 \frac{\partial \hat{w}_0}{\partial z} - \hat{P}_0 = 0, \quad z = 0. \tag{4.10}$$

The normal modes wave solutions of the zero-problem may be represented as:

$$\gamma_0(T_0, T_1) = \gamma_{00}(T_1)e^{i\omega_0 T_0} + \text{c. c.}, \tag{4.11a}$$

$$\hat{\psi}_0(z, T_0, T_1) = \psi_0^*(z, T_1)e^{i\omega_0 T_0} + \text{c. c.}, \tag{4.11b}$$

where c. c. stands for the complex conjugate of the preceding terms and ω_0 is the wave frequency, which is assumed to be real.

Combining Eqs. (4.5), (4.6), and (4.7), in view of Eq. (4.11), one gets

$$\left\{ \frac{\partial^4}{\partial(qz)^4} - \left[\frac{\rho D_0}{q^2 \nu_3} + 2 \left(\frac{\nu_1 - \nu_3 + \nu_2}{\nu_3} \right) \right] \frac{\partial^2}{\partial(qz)^2} + \left(\frac{\rho D_0}{q^2 \nu_3} + 1 \right) \right\} \hat{w}_0^*(z, T_1) = 0. \tag{4.12}$$

The appropriate solution of \hat{w}_0^* becomes

$$\hat{w}_0^*(z, T_1) = e^{qz m_1} A(T_1) + e^{qz m_2} B(T_1), \tag{4.13}$$

where $A(T_1)$ and $B(T_1)$ are arbitrary constants to be determined from the related boundary conditions (4.8), (4.9), (4.10); m_1 and m_2 are satisfied by

$$m_1^2 + m_2^2 = \frac{i\rho\omega_0}{q^2\nu_3} + 2 \left[\frac{\nu_1 - \nu_3 + \nu_2}{\nu_3} \right], \tag{4.14a}$$

$$m_1^2 m_2^2 = \left(\frac{i\rho\omega_0}{q^2\nu_3} + 1 \right). \tag{4.14b}$$

Moreover, as shown in [10], we have

$$m_1 m_2 = \Delta \left(\frac{i\rho\omega_0}{q^2\nu_3} + 1 \right)^{1/2}, \tag{4.14c}$$

$$m_1 + m_2 = \left\{ \frac{i\rho\omega_0}{q^2\nu_3} + 2 \left[\frac{\nu_1 - \nu_3 + \nu_2}{\nu_3} \right] + 2\Delta \left(\frac{i\rho\omega_0}{q^2\nu_3} + 1 \right)^{1/2} \right\}^{1/2}. \tag{4.14d}$$

Taking into account the positive root, Δ is a sign that is determined from

$$\Delta = \text{sign Re} \left\{ \left\{ \frac{i\rho\omega_0}{q^2\nu_3} + 2 \left[\frac{\nu_1 - \nu_3 + \nu_2}{\nu_3} \right] + 2 \left(\frac{i\rho\omega_0}{q^2\nu_3} + 1 \right)^{1/2} \right\}^{1/2} - \left\{ \frac{i\rho\omega_0}{q^2\nu_3} + 2 \left[\frac{\nu_1 - \nu_3 + \nu_2}{\nu_3} \right] - 2 \left(\frac{i\rho\omega_0}{q^2\nu_3} + 1 \right)^{1/2} \right\}^{1/2} \right\}. \tag{4.14e}$$

From Eqs. (4.13) and (4.5), (4.7), it is easy to find the solutions of \hat{u}_0 and \hat{P}_0 . Utilizing these solutions with the aid of the boundary conditions (4.8), (4.9), (4.10) and relations (4.14), the following dispersion relation is obtained:

$$\rho\omega_0^2 - m_1 m_2 (\rho\omega_0^2 - 2i\omega_0(\nu_1 + \nu_2)q^2) + q(m_1 + m_2)(\sigma q^2 + \rho g) = 0. \tag{4.16}$$

Squaring both sides of Eq. (4.16), after some rearrangements, one gets

$$a_3\omega_0^6 + a_2\omega_0^4 + a_1\omega_0^2 + a_0 = 0, \tag{4.17}$$

where the constants a_j ($j = 0, 1, 2, 3$) are given in the Appendix.

Equation (4.17) is a cubic equation in ω_0^2 . In the absence of the periodic external body force, the stability analysis—in the zero-order problem—is based on the algebraic

properties of this equation. Since ω_0^2 is a pure real number, it follows that ω_0^2 must be a real positive number. It follows that

$$L^3 + M^2 \leq 0, \tag{4.18}$$

where

$$L = \frac{1}{9a_3^2}(3a_1a_3 - a_2^2),$$

$$M = \frac{1}{54a_3^2}(9a_3(a_1a_2 - 3a_0a_3) - 2a_2^3),$$

together with the conditions

$$\frac{a_2}{a_3} < 0, \quad \frac{a_1}{a_3} > 0, \quad \text{and} \quad \frac{a_0}{a_3} < 0. \tag{4.19}$$

5. Stability analysis. The first-order problem includes the effect of the periodic external body force, which appears for the first time in this study. Thus the first-order in ε of Eqs. (3.7), (3.8), and (3.9) is given by

$$(\rho D_0 + q^2(2\nu_1 - \nu_3))\hat{u}_1 - \nu_3 \frac{\partial^2 \hat{u}_1}{\partial z^2} + iq\hat{P}_1 = -\rho D_1 \hat{u}_0, \tag{5.1}$$

$$(\rho D_0 + q^2\nu_3)\hat{w}_1 + (\nu_3 - 2\nu_2) \frac{\partial^2 \hat{w}_1}{\partial z^2} + \frac{\partial \hat{P}_1}{\partial z} = -\rho D_1 \hat{w}_0, \tag{5.2}$$

$$iq\hat{u}_1 + \frac{\partial \hat{w}_1}{\partial z} = 0. \tag{5.3}$$

The first-order forms of the boundary conditions (3.12), (3.13), and (3.14) are

$$\hat{w}_1 = D_1\gamma_0 + D_0\gamma_1, \quad z = 0, \tag{5.4}$$

$$iq\hat{w}_1 + \frac{\partial \hat{u}_1}{\partial z} = 0, \quad z = 0, \tag{5.5}$$

$$\{\sigma q^2 + \rho g\}\gamma_1 + \rho g \cos \omega T_0 \gamma_0 + 2\nu_2 \frac{\partial \hat{w}_1}{\partial z} - \hat{P}_1 = 0, \quad z = 0. \tag{5.6}$$

We may solve the above set of equations in the following manner:

$$\gamma_1(T_0, T_1) = \gamma_{12}(T_1) \exp 2i\omega_0 T_0 + \text{c. c.}, \tag{5.7}$$

$$\hat{\psi}_1(z, T_0, T_1) = \hat{\psi}_{11}(z, T_1) \exp i\omega_0 T_0 + \hat{\psi}_{12}(z, T_1) \exp 2i\omega_0 T_0 + \text{c. c.}, \tag{5.8}$$

where $\hat{\psi}_1$ stands for \hat{u}_1, \hat{w}_1 , and \hat{P}_1 .

The above set of the first-order perturbations contains the nonhomogeneous equations (5.1), (5.2), and (5.3). The uniform solution is required to eliminate secular terms. This elimination reduces the solvability condition corresponding to the terms containing the factor $(\exp i\omega_0 T_0)$. Here the solvability condition is divided into two cases. The first is the case when the frequency ω is away from ω_0 . This is called the non-resonant case. The second one is the resonant case arising when the frequency ω approaches the frequency ω_0 . Away from the details, the solvability condition yields

$$\{(R_1 + iR_2)D_1 + (E_1 + iE_2) \cos \omega T_0\}\gamma_{00}(T_1) \exp i\omega_0 T_0 + \text{c. c.} = 0, \tag{5.9}$$

where the constants $(R_1 + iR_2)$ and $(E_1 + iE_2)$ are complex, and they depend on the nature of the roots of the dispersion equation (4.17). These coefficients may be written in the following forms:

$$\begin{aligned} R_1 + iR_2 &= 4[i\omega_0\rho + q^2(\nu_1 + \nu_2)] \left[\frac{i\omega_0\rho}{q^2\nu_3} + 1 \right] + \frac{i\omega_0\rho}{q^2\nu_3} [i\omega_0\rho + q^2(2\nu_1 - \nu_3 + 2\nu_2)] \\ &\quad \times \left[-\frac{i\omega_0\rho}{q^2\nu_3} (m_1 + m_2)^{-2} + 1 \right] + i\omega_0\rho m_1 m_2 \left[-\frac{i\omega_0\rho}{q^2\nu_3} (m_1 + m_2)^{-2} + 3 \right], \\ E_1 + iE_2 &= 2\rho q g (m_1 + m_2) m_1 m_2. \end{aligned}$$

In the non-resonant case, the solvability condition becomes

$$(R_1 + iR_2)D_1\gamma_{00} = 0. \quad (5.10)$$

Inverting to the original variable t , it follows that the non-resonant region is definitely stable.

Secondly, for the resonant case, which arises when the frequency ω approaches the frequency ω_0 , we introduce a detuning parameter δ defined by

$$\omega = 2\omega_0 + 2\varepsilon\delta; \quad (5.11)$$

hence, we can write

$$-i(\omega_0 - \omega)T_0 = i\omega_0 T_0 + 2i\delta T_1. \quad (5.12)$$

The solvability condition in this base becomes

$$(R_1 + iR_2)D_1\gamma_{00} + \frac{1}{2}(E_1 + iE_2)\bar{\gamma}_{00} \exp 2i\delta T_1 = 0, \quad (5.13)$$

where $\bar{\gamma}_{00}$ is the complex conjugate of γ_{00} .

Inverting to the original variable t , we obtain

$$(R_1 + iR_2)\frac{d\gamma}{dt} + \frac{\varepsilon}{2}(E_1 + iE_2)\bar{\gamma} \exp 2i\varepsilon\delta t = 0. \quad (5.14)$$

We seek a solution for Eq. (5.14) in the form

$$\gamma(t) = (\phi_1(t) + i\phi_2(t)) \exp i\varepsilon\delta t, \quad (5.15)$$

with real $\phi_{1,2}$, separate real and imaginary parts, and obtain

$$R_1\dot{\phi}_1 + \varepsilon \left(\frac{E_1}{2} - R_2\delta \right) \phi_1 - R_2\dot{\phi}_2 + \varepsilon \left(\frac{E_2}{2} - R_1\delta \right) \phi_2 = 0, \quad (5.16a)$$

$$R_2\dot{\phi}_1 + \varepsilon \left(\frac{E_2}{2} + R_1\delta \right) \phi_1 + R_1\dot{\phi}_2 - \varepsilon \left(\frac{E_1}{2} + R_2\delta \right) \phi_2 = 0; \quad (5.16b)$$

here the dot denotes the time derivative.

Equations (5.16) admit a solution in the form

$$\phi_j(t) = \Lambda_j \exp \theta t, \quad j = 1, 2 \quad (5.17)$$

with constants $\Lambda_{1,2}$.

When the real part of θ is negative, the response decays; when the real part of θ is zero, the response is finite and bounded; and when the real part of θ is positive, the response grows. Thus the values of the parameters for which the real part of θ is zero divide the

parameter space into regions of stability and instability (stability charts). From Eqs. (5.16a,b) and (5.17), we must have

$$\theta^2 = \varepsilon^2 \left(\frac{E_1^2 + E_2^2}{4(R_1^2 + R_2^2)} - \delta^2 \right). \tag{5.18}$$

The motion is unstable when θ^2 is positive definite. Therefore the motion is unstable when

$$-\delta_1 < \delta < \delta_1, \tag{5.19}$$

where

$$\delta_1 = \frac{1}{2} \left(\frac{E_1^2 + E_2^2}{R_1^2 + R_2^2} \right)^{1/2}; \tag{5.20}$$

otherwise it is stable.

The value of δ_1 is given in the Appendix.

The values for which $\delta = \delta_1$ and $\delta = -\delta_1$ are called the transition values. The loci of transition values are called transition curves, which separate the stable from the unstable regions.

Substituting from Eqs. (5.19), (5.20) into Eq. (5.11), we obtain the following transition curves:

$$\omega = 2\omega_0 + 2\varepsilon\delta_1, \tag{5.21}$$

$$\omega = 2\omega_0 - 2\varepsilon\delta_1. \tag{5.22}$$

6. Numerical discussions and conclusions. In what follows, we shall give numerical discussions for the stability of the system under consideration by drawing the transition curves. The transition curves are represented by Eqs. (5.21) and (5.22) in the parametric space. In the following figures, the letter *S* denotes stable regions while the letter *U* stands for unstable ones.

The values of ω , as described by Eqs. (5.21) and (5.22), are the critical values of the disturbances. These critical values, which are known as the transition curves, separate the stable from the unstable regions. According to the Floquet theory [15], the region bounded by the two branches of the transition curves is unstable but the area outside these curves is a stable region. In our numerical calculations, we considered the natural physical parameters of the nematics MBBA and PAA.

In Fig. 1 (see p. 424), ε is plotted versus q for a single value of the frequency of the external periodic body force ($\omega = 70$ Hz). In this figure, we consider the physical parameters, see Table (1), of the nematic MBBA. The system is stable in the absence of the external periodic body force. The presence of the frequency ω produces an unstable region that approaches twice the disturbance frequency $2\omega_0$. The broken line represents the sharp resonance case ($\omega = 2\omega_0$) that occurs at $q = 1.15 \text{ cm}^{-1}$. The (ε, q) -plane has been divided into two regions by this broken line. The right region corresponds to the case of $\omega < 2\omega_0$ while the left one satisfies the case of $\omega > 2\omega_0$. At the exact resonance the system is unstable. The region bounded by the two transition curves (5.20), (5.21) is unstable; otherwise stability reveals. We observe that the instability region increases as ε is increased. This shows that the amplitude of the periodic external force plays

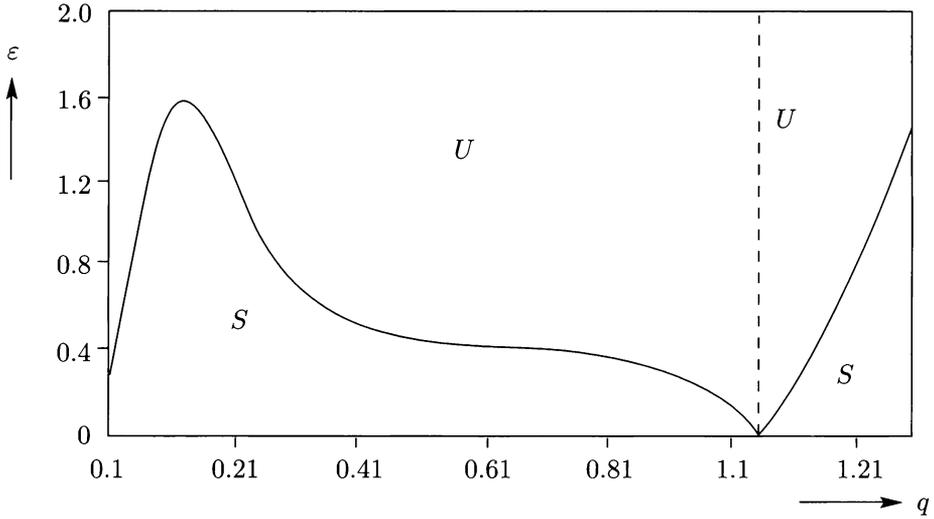


FIG. 1. For a system of a nematic MBBA taking into account $\delta = 70$ Hz

a destabilizing role in the stability criterion. It is noticed that the unstable area lying in the left region ($\omega > 2\omega_0$) is larger than that in the right one ($\omega < 2\omega_0$). Thus the maximum destabilizing effect of the amplitude (ε) takes place when $\omega > 2\omega_0$.

In Fig. 2, the same system considered in Fig. 1 is considered but when $\omega = 60$ Hz. The effect of the decrease of the frequency of the external periodic body force appears in this figure. It is observed that the value of the resonance point has been decreased and the broken line is shifted to the left; it occurs at $q = 1.098 \text{ cm}^{-1}$, in the (ε, q) -plane. The unstable region has been decreased as ω is increased. Thus the increase of ω leads to a contradiction in the unstable region. Therefore, the effect of increasing ω makes a damping in the nematic motion. This damping mechanism involves a transfer of the energy from the external periodic force to the perturbed interface.

In Fig. 3, ω is plotted versus ε for a single value of the wavenumber ($q = 0.8 \text{ cm}^{-1}$). The system considered here is the nematic wave MBBA. The transition curves are represented in the figure by straight lines, where the resonance point occurs at $\omega = 30$ Hz.

In Fig. 4 (see p. 426), ε is plotted versus q for a single value of the frequency of the external periodic body force ($\omega = 34$ Hz). In the figure, we consider the physical parameters, see Table (1), of the nematic PAA. The broken line, representing the sharp resonance case, occurs at $q = 0.635 \text{ cm}^{-1}$. It is observed that the unstable area lying in the right region (where $\omega < 2\omega_0$) is larger than that in the left one (where $\omega > 2\omega_0$). Thus the maximum destabilizing effect of the amplitude (ε) is present when $\omega < 2\omega_0$. A comparison between Figs. 1 and 4 shows that the mechanism of instability of MBBA differs from that of PAA.

In Fig. 5 (see p. 426), ω is plotted versus q for a single value of the amplitude of the external periodic body force ($\varepsilon = 1.0$). The system considered here is the nematic PAA. We see that the resonance point appears at $\omega = 1.7$ Hz.

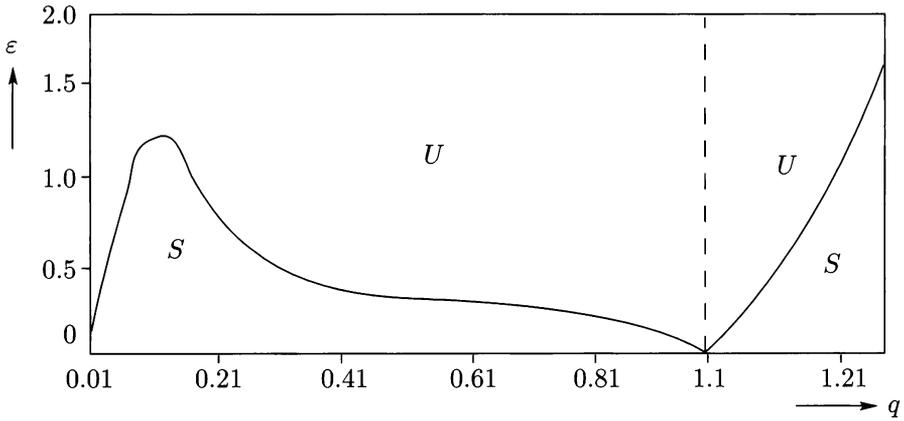


FIG. 2. For a system of a nematic MBBA taking into account $\omega = 60$ Hz

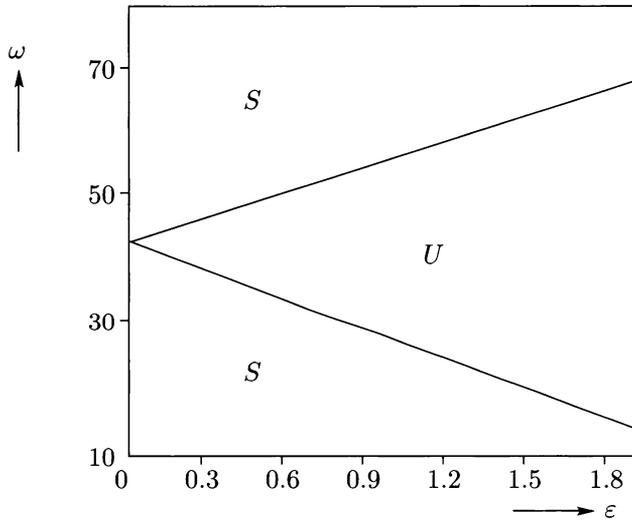


FIG. 3. For a system of a nematic MBBA taking into account $q = 0.8 \text{ cm}^{-1}$

In Fig. 6 (see p. 427), ω is plotted versus ε for a single value of the wavenumber ($q = 0.8 \text{ cm}^{-1}$). The system considered here is the nematic PAA. The transition curves are represented in the figure by straight lines, where the resonance point occurs at $\omega = 30$ Hz. A comparison between Figs. 3 and 6 shows that the disturbance of the surface waves in PAA is more stabilizing than that of MBBA.

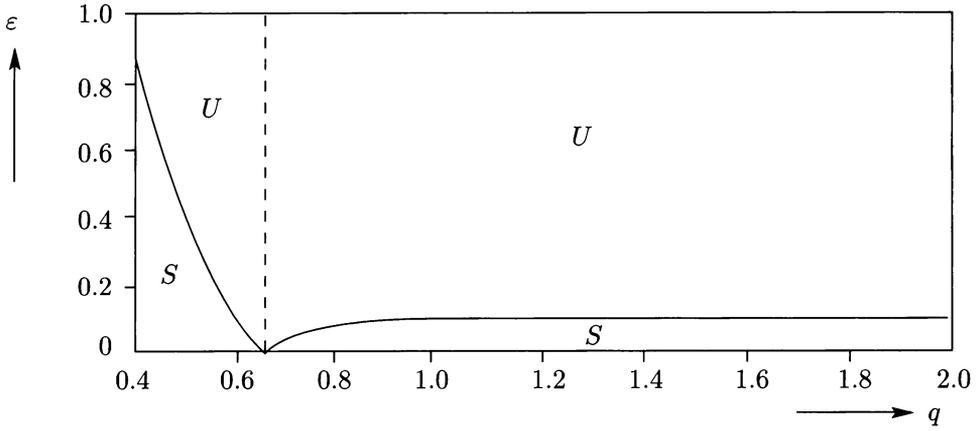


FIG. 4. For a system of a nematic PAA taking into account $\omega = 34$ Hz

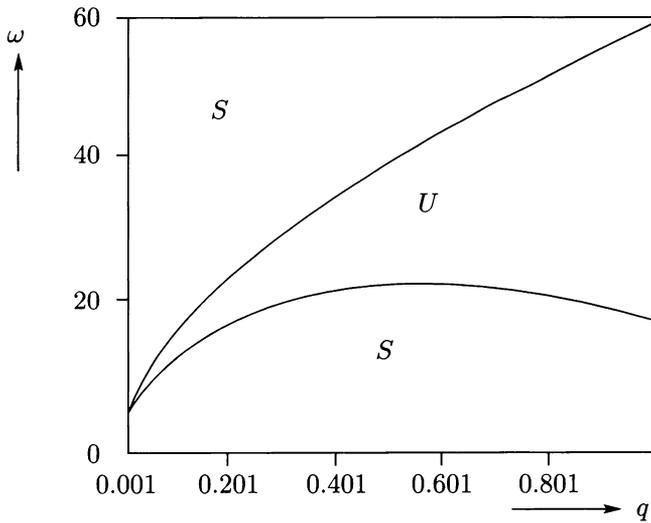
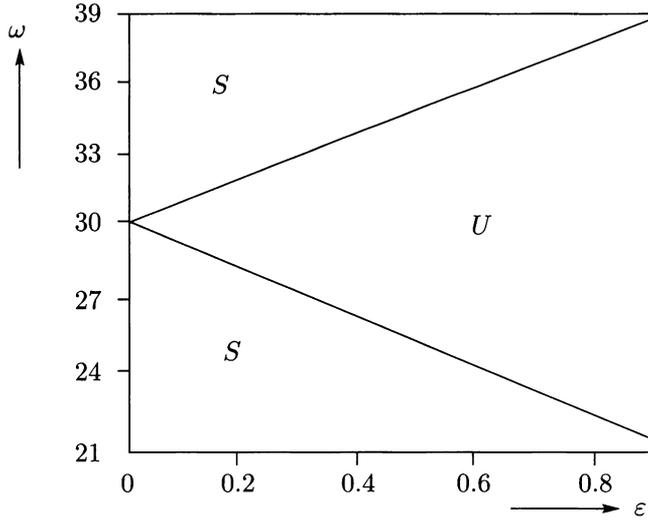


FIG. 5. For a system of a nematic PAA taking into account $\varepsilon = 1.0$


 FIG. 6. For a system of a nematic PAA taking into account $q = 08 \text{ cm}$

Appendix. In regard to Eq. (4.17), we have the following:

$$a_3 = b_9(b_6b_9 - b_5b_{10}) - b_7(b_8b_9 - b_7b_{10}),$$

$$a_2 = b_9(b_3b_9 - b_3b_{10}) - b_5(b_8b_9 - b_7b_{10}),$$

$$a_1 = b_9(b_2b_9 - b_1b_{10}) - b_3(b_8b_9 - b_7b_{10}),$$

$$a_0 = b_0b_9^2 - b_1(b_8b_9 - b_7b_{10}),$$

$$b_{10} = -\frac{\rho^6}{q^4\nu_3^2},$$

$$b_9 = \frac{4\rho^5}{q^2\nu_3} \left(1 + 2 \left[\frac{\nu_1 + \nu_2}{\nu_3} \right] \right),$$

$$b_8 = 4\rho^4 \left(1 + 6 \left[\frac{\nu_1 + \nu_2}{\nu_3} \right] + 6 \left[\frac{\nu_1 + \nu_2}{\nu_3} \right]^2 \right),$$

$$b_7 = -8\rho^3(\nu_1 + \nu_2)q^2 \left(2 + 7 \left[\frac{\nu_1 + \nu_2}{\nu_3} \right] + 4 \left[\frac{\nu_1 + \nu_2}{\nu_3} \right]^2 \right),$$

$$b_6 = \frac{2\rho^4(\sigma q^2 + \rho g)^2}{q^2\nu_3^2} - 16\rho^2[\nu_1 + \nu_2]^2 q^4 \left(2 + 4 \left[\frac{\nu_1 + \nu_2}{\nu_3} \right] + \left[\frac{\nu_1 + \nu_2}{\nu_3} \right]^2 \right),$$

$$b_5 = 4\rho[\nu_1 + \nu_2] \left(8[\nu_1 + \nu_2]^2 \left[\frac{\nu_1 + \nu_3 + \nu_2}{\nu_3} \right] q^6 + \frac{\rho}{\nu_3^2} \left[1 + \frac{2\rho}{\nu_3} \right] [\sigma q^2 + \rho g]^2 \right),$$

$$b_4 = 16[\nu_1 + \nu_2]^4 q^8 + 8\rho^2 q^2 [\sigma q^2 + \rho g]^2 \left(1 - 3 \left[\frac{\nu_1 + \nu_2}{\nu_3} \right]^2 \right),$$

$$\begin{aligned}
b_3 &= 8\rho q^4[\sigma q^2 + \rho g]^2[\nu_1 + \nu_2] \left(-2 + \left[\frac{\nu_1 + \nu_2}{\nu_3} \right] + 2 \left[\frac{\nu_1 + \nu_2}{\nu_3} \right]^2 \right), \\
b_2 &= 16[\nu_1 + \nu_2]^2 \left[\frac{\nu_1 - \nu_3 + \nu_2}{\nu_3} \right] q^6 - \frac{\rho^2(\sigma q^2 + \rho g)^4}{\nu_3^2}, \\
b_1 &= 4\rho q^2[\sigma q^2 + \rho g]^2 \left[\frac{\nu_1 - 2\nu_3 + \nu_2}{\nu_3} \right], \text{ and} \\
b_0 &= 4q^4[\sigma q^2 + \rho g]^4 \left[\frac{\nu_1 - \nu_3 + \nu_2}{\nu_3} \right]^2 - 4q^4[\sigma q^2 + \rho g]^4.
\end{aligned}$$

The value of the constant δ_1 , appearing in Eq. (5.19), is given by $\delta_1 = \rho q g (Z_1 Z_2 / Z_3)^{1/2}$, where

$$\begin{aligned}
Z_1 &= \left\{ \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} + 4 \left[\frac{\nu_1 - \nu_3 + \nu_2}{\nu_3} \right]^2 + 4 \left[1 + \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} \right]^{1/2} - 2\sqrt{2} \frac{\omega_0 \rho}{q^2 \nu_3} \left[\left[1 + \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} \right]^{1/2} \right. \right. \\
&\quad \left. \left. - 1 \right]^{1/2} + 4\sqrt{2} \left[\frac{\nu_1 - \nu_3 + \nu_2}{\nu_3} \right] \left[\left[1 + \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} \right]^{1/2} + 1 \right]^{1/2} \right\}^{1/2}, \\
Z_2 &= \left\{ \left[1 + \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} \right]^{1/2} \left[\frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} + 4 \left[\frac{\nu_1 - \nu_3 + \nu_2}{\nu_3} \right]^2 \right] + 4 \left[1 + \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} \right] \right. \\
&\quad + 2\sqrt{2} \left[\frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} + 2 \left[\frac{\nu_1 - \nu_3 + \nu_2}{\nu_3} \right] \right] \left[\left[1 + \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} \right]^{1/2} + 1 \right]^{1/2} \\
&\quad \left. - 2\sqrt{2} \frac{\omega_0 \rho}{q^2 \nu_3} \left[2 \left[\frac{\nu_1 - \nu_3 + \nu_2}{\nu_3} \right] - 1 \right] \left[\left[1 + \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} \right]^{1/2} - 1 \right]^{1/2} \right\}, \\
Z_3 &= \left[1 + \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} \right]^{1/2} [X_1^2 + Y_1^2] + [X_0^2 + Y_0^2] + \sqrt{2}[X_1 X_0 + Y_1 Y_0] \\
&\quad \times \left[\left[1 + \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} \right]^{1/2} + 1 \right]^{1/2} - \sqrt{2}[X_1 Y_0 - X_0 Y_1] \left[\left[1 + \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} \right]^{1/2} - 1 \right]^{1/2},
\end{aligned}$$

where

$$\begin{aligned}
X_0 &= 4q^2 \frac{(\nu_1 + \nu_2)^2}{\nu_3} \left[2 + \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} \right] - 8q^2(\nu_1 + \nu_2) - 7(\nu_1 + \nu_2) \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2}, \\
Y_0 &= \omega_0 \rho \left\{ -2 \frac{(\nu_1 + \nu_2)}{\nu_3} + 6 \frac{(\nu_1 + \nu_2)^2}{\nu_3^2} + \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} \left[1 + 2 \frac{(\nu_1 + \nu_2)}{\nu_3} \right] \right\}, \\
X_1 &= \omega_0^2 \rho^2 + 4q^2(\nu_1 + \nu_2) \left[2 + \frac{\omega_0^2 \rho^2}{q^4 \nu_3^2} \right],
\end{aligned}$$

and

$$Y_1 = \omega_0 \rho \frac{(\nu_1 + \nu_2)}{\nu_3}.$$

Acknowledgment. We would like to express appreciation for the assistance given by Professor Abou El Magd A. Mohamed in some important points in the analysis.

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