

VORTICES AND BOUNDARIES

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Abstract. This paper develops a framework in which vortex/boundary interactions can be analysed for the cases of inviscid fluid dynamics and type II superconductors. Comments are made concerning the possibility of reconnection, defined as an interaction in which an initially single-component vortex eventually emerges with two components.

1. Introduction. There is much current interest in vortex/boundary interactions in various different continuous media. By the word vortex we mean a particular type of line singularity of the solution of the relevant field equations, which may, for example, be the Euler equations for an inviscid fluid or the London equations for a type II superconductor. We will not review the motivation for all the recent studies except to say that vortex reconnection is a commonly occurring problem in inviscid flow, relevant to aircraft trailing vortices [8], wakes containing vortex rings [3], and horseshoe vortices in environmental flows [28], while vortex nucleation/annihilation is an important ingredient in determining the response of superconductors to external fields [12].

Our aim in this paper is to present a self-contained asymptotic framework that unifies some situations where vortices interact with boundaries. We will illustrate this framework with one example from fluid dynamics and one from type II superconductivity. Both will reveal the delicate mathematical questions that may be expected to arise whenever such interactions are considered theoretically.

The first example is the classical one that arises when a vortex in an otherwise inviscid fluid encounters an impenetrable boundary (or, equivalently, when two co-rotating vortices of equal strength move so that they are separated by a symmetry plane). This problem has a long history, as discussed in [1, 13, 14, 15, 21, 23, 25] for example. The basic picture that emerges from our discussion is that the model proposed in section five of [15] describes the boundary interactions as long as the radius of the vortex core is negligibly small throughout the flow. Of course this will not be the case if and when “reconnection” actually occurs, but a study of the zero-core radius limit is helpful when considering more detailed investigations of models that take the core structure into account. This can be done either using a viscous model or an inviscid model with distributed vorticity; such

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investigations have been carried out in [1, 21, 23] and [25, 27, 30], respectively. A further mathematical motivation for studying the zero-core radius limit is that the solution of the model proposed in [15] still poses several unanswered questions in the theory of both ordinary and partial differential equations.

Our second illustration is the less well-studied problem of a superconducting vortex intersecting with a boundary between a vacuum and a superconducting material. Here our approach will lead to a simpler mathematical model about which some more precise mathematical statements can be made [19].

The conclusion that we will draw from these examples is that modelling vortices as line singularities enables some interesting conjectures to be made about how vortices interact with boundaries. These predictions need to be assessed modulo the effects of finite core radius and, for fluids, we will discuss this further in section four. No comparable body of work exists for superconductors, but core-scale effects in superfluid vortices have been analysed using a nonlinear Schrödinger model in [17].

The layout of this paper is to begin by giving a concise and self-contained presentation of models for isolated vortices in inviscid fluids and type II superconductors. Then, in section three, we consider the solutions of these models in the presence of boundaries and summarise the implications about the zero-core radius limit and some conjectures for further work in section four.

2. Isolated vortices.

2.1. *Inviscid fluids.* To set the scene, we recall that for an isolated vortex γ in an inviscid fluid, the field equation for the velocity potential ϕ is

$$\nabla^2 \phi = 0, \quad R > 0 \tag{1}$$

where (R, θ) are local polar coordinates normal to the vortex and

$$\nabla \phi = \mathbf{u} \sim \frac{\mathbf{e}_\theta}{2\pi R} \quad \text{as } R \rightarrow 0; \tag{2}$$

here \mathbf{e}_θ is a local unit azimuthal vector and we are working with variables made dimensionless with the vortex strength Γ and a typical radius of curvature¹ κ_0^{-1} . Hence,

$$\mathbf{u}(\mathbf{r}, t) = -\frac{1}{4\pi} \int_\gamma \frac{(\mathbf{r} - \mathbf{r}') \wedge d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} + \mathbf{u}_0(\mathbf{r}, t) \tag{3}$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{u}_0(\mathbf{r}, t)$ is the background (irrotational) flow due to boundaries, etc., and, as shown in [24, 29],

$$\mathbf{u} \sim \frac{\mathbf{e}_\theta}{2\pi R} + \frac{\kappa \mathbf{b}}{4\pi} (\log 2 - \log R - 1) + \frac{\kappa \cos \theta \mathbf{e}_\theta}{4\pi} + \mathbf{F} + \mathbf{u}_0|_\gamma + O(R \log R), \tag{4}$$

as $R \rightarrow 0$, where κ is the curvature of the filament, \mathbf{b} the binormal, and θ is measured from the direction of the principal normal; \mathbf{F} denotes the finite part of the Biot-Savart integral, given by

$$\mathbf{F} = \lim_{L \rightarrow 0} \left(\frac{\kappa \mathbf{b}}{4\pi} \log L - \frac{1}{4\pi} \int_{|\mathbf{r} - \mathbf{r}'| > L} \frac{(\mathbf{r} - \mathbf{r}') \wedge d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right). \tag{5}$$

¹Our results will all tend to sensible two-dimensional limits when $\kappa_0 \rightarrow 0$ even though the reference length-scale is then arbitrary.

It is impossible to determine the motion of the vortex without the introduction of a local regularising mechanism and here we could either invoke a kinematic viscosity ν or the presence of a core of vorticity distributed over a cylinder of dimensionless radius δ . It is easy to see that a remote observer will perceive the regularised vortex as a line singularity of strength Γ in either case as long as Γ and ν are related by $\delta = O((\nu/\Gamma)^{1/2})$. Then, when we carry out an inner asymptotic expansion over scales $R \sim \delta$, we can find the velocity \mathbf{v} of the vortex from the solvability condition for the second term in this expansion, as shown in [4, 11, 15, 16, 29]; the result is the local induction formula

$$\mathbf{v} \sim -\frac{\kappa \mathbf{b}}{4\pi} \log \delta + \mathbf{u}_0|_\gamma + O(1). \tag{6}$$

We see that there is a self-induced contribution in the binormal direction proportional to the local curvature as well as a background contribution due to boundary effects. The $O(1)$ term here includes the contribution from the remote part of the vortex (i.e., (5)), as well as a contribution due to the internal structure of the vortex core. Throughout this paper we will be considering flows in which the term $\mathbf{u}_0|_\gamma$ is larger than $O(1)$ as $\delta \rightarrow 0$.

2.2. *Type II superconductors.* An exactly similar analysis can be performed for type II superconductors. Here the field equation is the London equation

$$\nabla^2 \mathbf{H} - \mathbf{H} = \mathbf{0}, \quad R > 0, \tag{7}$$

where \mathbf{H} is the magnetic field. In contrast to the case of fluid vortices, the equations already have a natural length-scale. In dimensional terms this length-scale is the penetration depth λ and lengths in (7) have been scaled with λ . When we introduce a vortex into Eq. (7) we have a second length-scale, namely the radius of curvature of the vortex κ_0^{-1} and it is natural to consider configurations in which $\kappa_0^{-1} \sim \lambda$. The variable analogous to the fluid velocity \mathbf{u} is the electric current $\mathbf{j} = \nabla \wedge \mathbf{H}$, and we have

$$\mathbf{j} \sim \frac{\mathbf{e}_\theta}{2\pi R}, \quad \text{as } R \rightarrow 0. \tag{8}$$

Hence,

$$\mathbf{j}(\mathbf{r}, t) = -\frac{1}{4\pi} \int_\gamma \nabla \left(\frac{e^{-|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) \wedge d\mathbf{r}' + \mathbf{j}_0(\mathbf{r}, t), \tag{9}$$

where \mathbf{j}_0 is the background current due to boundary interactions, applied fields, etc. The regularisation of the vortex core is given by the Ginzburg-Landau equations, which play a role analogous to the Navier-Stokes equations for fluid vortices, the coherence length being analogous to δ above.² We find that the vortex velocity is given by

$$\mathbf{v} \sim -\frac{\kappa \mathbf{n}}{4\pi} \log \delta + \mathbf{j}_0|_\gamma \wedge \mathbf{t} + O(1) \tag{10}$$

where \mathbf{t} is the tangent and \mathbf{n} is the principal normal to the vortex [5]. Here the self-induced contribution is still proportional to curvature but is now in the normal direction; hence it is possible for a twisted superconducting vortex to remain in a plane, which a fluid vortex cannot unless it is circular or straight. As in the case of fluid vortices, we will consider cases where $\mathbf{j}_0|_\gamma \wedge \mathbf{t}$ is greater than $O(1)$ as $\delta \rightarrow 0$.

²An atomic length-scale regularisation has been suggested for the core of a superfluid vortex [17].

In view of the complicated asymptotics involved in going from (3) or (9) to (6) or (10), it is far more convenient mathematically to postulate a cut-off distance than it is to introduce a local regularising mechanism. This simply involves replacing \int_γ in (3) or (9) by $\int_{|\mathbf{r}-\mathbf{r}'|>\delta}$ instead of carrying out the matched asymptotic expansions referred to above. In another context, if we were to consider the motion of a glide dislocation in an elastic crystal [10] moving under a linear mobility law, we would find that the dislocation, which is constrained to lie in the slip plane, has a velocity along its normal that is proportional to its curvature to lowest order, as in (10); this is a result of the so-called ‘‘Peach-Koehler’’ force exerted on the dislocation by itself. In this case δ can be interpreted as a core radius over which molecular effects must be taken into consideration but in practice δ is defined as a cut-off.

We emphasise that laws of motion such as (6, 10) are all crucially dependent on the assumption that the core radius, over which the local regularising mechanism is assumed to act, is both nonzero and much less than the radius of curvature of the vortex, i.e., $\delta \ll 1$. This implies that the self-induced velocity of a curved vortex is dominated (by a factor of $\log \delta$) by the local vortex geometry; the self-induced motion resulting from the global vortex geometry only enters as a correction term and, in this paper, this correction term will also be small in comparison to the non-self-induced velocities $\mathbf{u}_0|_\gamma$ and $\mathbf{j}_0|_\gamma \wedge \mathbf{t}$ resulting from the presence of a boundary. Circumstances under which this can happen are described in the next section.

3. Boundary interactions.

3.1.1. *Fluid vortex interactions.* We now consider the relative importance of boundary and self-induced contributions to the vortex motion. The strongest boundary interaction occurs in a two-dimensional situation in which the vortex is parallel to the boundary. Since the presence of the boundary corresponds to an image vortex or vortex system, if a typical dimensional proximity of the vortex to the boundary is h , then the boundary-induced velocity will be $O(\Gamma/h)$ from (2). Hence, for a curved vortex with curvature of $O(\kappa_0)$ locally at its nearest proximity to the boundary, it will only be when h becomes of the same order as $(\kappa_0 \log \delta^{-1})^{-1}$ that the effect of the boundary will be noticeable to lowest order.³ Thus the presence of the $\log \delta$ term indicates that if we have a configuration in which the boundary and self-induced contributions balance, then, when we work on the scale of h , the curvature of the vortex is small and hence the vortex is approximately straight. Even though this statement is corroborated by the inviscid calculations of [25, 27], we can only have confidence in it if the predictions of any model on which it is based are stable with respect to small perturbations of arbitrary wavelength and we will make this verification at the end of this section. However, at this stage the most important configuration for us to study is that of a fluid vortex lying roughly parallel to the x -axis moving at a distance $O(h)$ from a rigid boundary $z = 0$. Indeed an impermeable boundary condition is the only one we will consider in this section because of its frequent occurrence in fluid dynamics; even free surface/vortex interactions usually occur at such small Froude numbers that the boundary is effectively undisturbed.

³This parameter regime was identified in (2.12) of [15] and in (3.17) of [11].

In the configuration that we wish to study, $(\log \delta^{-1})^{-1} = \kappa_0 h$ is a small parameter; hence, in our dimensionless coordinates, with the fluid in $z < 0$, we write the vortex position as

$$\mathbf{r} = (x, \kappa_0 h Y(x, t), \kappa_0 h Z(x, t)) \tag{11}$$

with $Y(x, t) \sim Z(x, t) \sim O(1)$, so that the lowest-order image contribution to the velocity of the vortex is

$$\left(0, \frac{1}{4\pi\kappa_0 h Z}, 0 \right).$$

Then (6) gives, to lowest order,

$$\kappa_0 h \left(0, \frac{\partial Y}{\partial y}, \frac{\partial Z}{\partial t} \right) = -\frac{\kappa_0 h \log \delta}{4\pi} \left(0, -\frac{\partial^2 Z}{\partial x^2}, \frac{\partial^2 Y}{\partial x^2} \right) + \left(0, \frac{1}{4\pi\kappa_0 h Z}, 0 \right). \tag{12}$$

Hence, we achieve a balance of these terms when we scale x with $(\log \delta^{-1})^{-1/2}$ and t with $(\log \delta^{-1})^{-2}$, giving the model of Zakharov [31], namely

$$\begin{aligned} \frac{\partial Y}{\partial t} &= -\frac{1}{4\pi} \frac{\partial^2 Z}{\partial x^2} + \frac{1}{4\pi Z}, \\ \frac{\partial Z}{\partial t} &= \frac{1}{4\pi} \frac{\partial^2 Y}{\partial x^2}. \end{aligned} \tag{13}$$

3.1.2. *Reconnection of fluid vortices.* We now study the prediction of (13) for the phenomenon of reconnection, in which an initially-connected vortex meets the boundary in finite time and then splits into two or more components in physical space. Reconnection can also be interpreted as the climax of the Crow instability of two equal and opposite parallel vortices [8]. If such a reconnection is to occur, it must, as explained above, be described by (13), in which the closest part of the vortex that responds to the boundary is nearly straight and parallel to it.

We note that the principal part of (13) describes two-dimensional wave motion on an elastic beam or isolated vortex filament. In the latter case, it is well known that weakly nonlinear modulation of these waves leads to envelope soliton behaviour [9]. A similar analysis can be carried out in the presence of the image contribution by perturbing about a two-dimensional solution such as $Y = -t/4\pi$, $Z = -1$ [6]. It is found that envelope solitons only emerge when the carrier wave has sufficiently small wavelength; above a certain cut-off, instabilities grow into a fully nonlinear regime. This reinforces our belief, mentioned before (11), that reconnection cannot occur as a result of high frequency oscillations developing on the vortex but rather as a long wavelength motion on a nearly straight filament. However, the strongest evidence we can adduce for reconnection comes from a study of similarity solutions of (13) and such a study has been carried out in [15]. There it has been pointed out that, assuming the vortex is symmetric in x and asymptotically straight and non-oscillating far from the boundary, and that reconnection only occurs at $x = t = 0$, we can write

$$Y = x\bar{Y}(\eta), \quad Z = x\bar{Z}(\eta) \tag{14}$$

where $\eta = x(-4\pi t)^{-1/2}$ so that

$$\frac{\eta^3}{2} \bar{Y}' = -(\eta^2 \bar{Z}')' + \frac{1}{\bar{Z}}, \quad (15)$$

$$\frac{\eta^3}{2} \bar{Z}' = (\eta^2 \bar{Y}')'. \quad (16)$$

The assumptions imply that $\bar{Z} \neq 0$ for $t < 0$ and, since $Z(0, t) \sim O(\sqrt{-t})$ as $t \nearrow 0$, $\bar{Z} = O(\eta^{-1})$ as $\eta \rightarrow 0$. Thus, writing $\bar{Z} = f(\eta)/\eta$, we obtain

$$4f''' + \frac{4f'}{f^2} + \eta^2 f' - \eta f = 0, \quad (17)$$

with

$$f(0) = a, \quad (18)$$

$$f'(0) = 0, \quad (19)$$

$$f(\eta) \sim b\eta \quad \text{as } \eta \rightarrow \infty, \quad (20)$$

where a and b are negative constants to be determined. This is a difficult boundary value problem to analyse rigorously but we note that, as $\eta \rightarrow 0$, the only asymptotic solutions relevant to our discussion satisfy the three-parameter asymptotic expansion

$$f \sim A_0 + B_0\eta + C_0\eta^2 + \dots, \quad A_0 \neq 0. \quad (21)$$

Moreover, linearising about the asymptotic behaviour (20) as $\eta \rightarrow \infty$ gives the three-parameter asymptotic expansion

$$f - b\eta \sim C_\infty\eta + \frac{A_\infty}{\eta^2} \cos\left(\frac{\eta^2}{4}\right) + \frac{B_\infty}{\eta^2} \sin\left(\frac{\eta^2}{4}\right) + \dots. \quad (22)$$

Since we are seeking reconnection in the absence of rapid oscillations of the vortex, we require $A_\infty = B_\infty = 0$ as well as $B_0 = 0$ and we might therefore expect that these three conditions yield a solution for f , a , and b . Indeed, a rough numerical experiment (using Mathematica) suggests that these conditions are satisfied for at least one value of (a, b) , somewhere near $(-0.757, -0.750)$. The corresponding $f(\eta)$ is shown in Fig. 1 and hence the vortex configuration is as in Figs. 2, 3, and 4. Our procedure was simply to shoot from $\eta = 0$ by searching the region $0.1 < A_0, C_0 < 1$ and our results are certainly no more accurate than $O(10^{-3})$. That the problem may possess subtleties that are unrevealed by numerical shooting can be seen by mathematically shooting from $\eta = \infty$ using varying values of the parameter b . Assuming that for each b there exists a unique solution to (17, 20) over the whole range $0 \leq \eta < \infty$, we note that for sufficiently large negative b , f will be approximately $b\eta$ and hence $f'(0) < 0$. In order to discover if any values of b give $f'(0) > 0$, we study (17) as $b \nearrow 0$, and write $f = (-b)^{1/2}F$, $\eta = (-b)^{-1/2}\xi$ to give

$$4b^2 \frac{d^3 F}{d\xi^3} + \left(\frac{4}{F} + \xi^2\right) \frac{dF}{d\xi} - \xi F = 0$$

with $\frac{dF}{d\xi} \rightarrow -1$ as $\xi \rightarrow \infty$. Thus in the absence of rapid oscillations,

$$F^2 \sim \frac{1}{2} \left(\xi^2 + \sqrt{8 + \xi^4} \right)$$

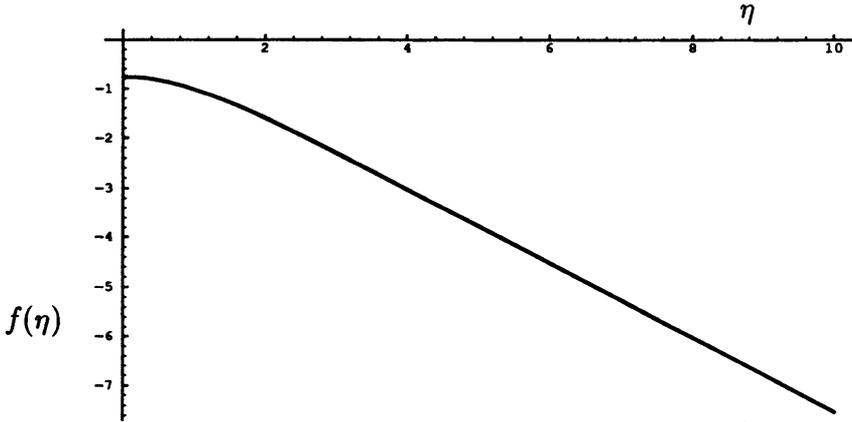


FIG. 1. Similarity solution for $(a, b) = (-0.757, -0.750)$

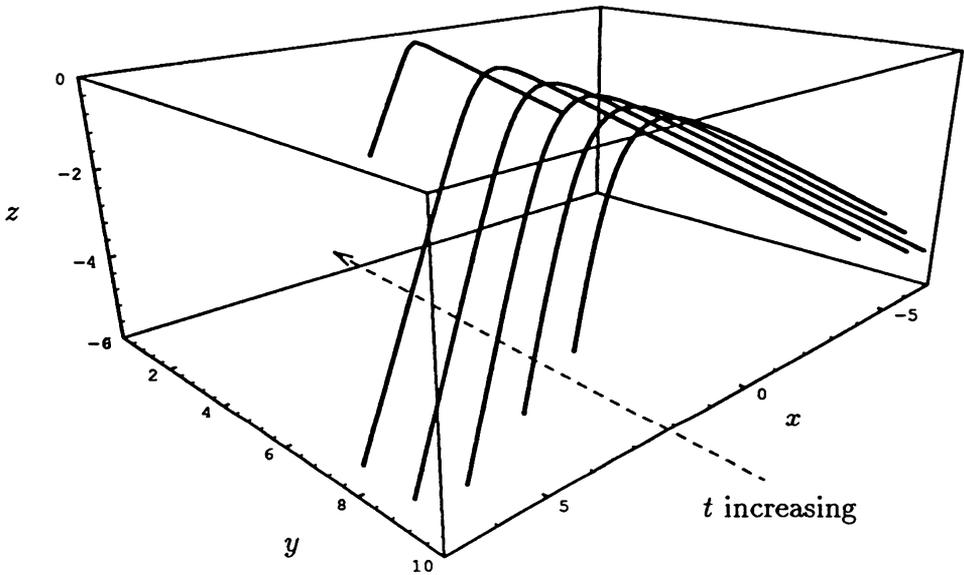


FIG. 2. Vortex geometry for increasing t

to lowest order, giving $F'(0) = 0$. In fact, when we proceed to higher order, we find $F'(0) = O(b^n)$ for all positive integers n and we thus have a problem in exponential asymptotics, reminiscent of the so-called “crystal growth” problem [18]. Applying techniques developed for that problem gives that $F'(0)$ is proportional to some power of b multiplied by $e^{\sigma/b}$ where the constant σ is complex. This means that $F'(0)$ changes sign infinitely often as $b \nearrow 0$ and hence that (17–20) has solutions for an infinite sequence $\{b_n\}$ where $b_n \nearrow 0$ as $n \rightarrow \infty$.

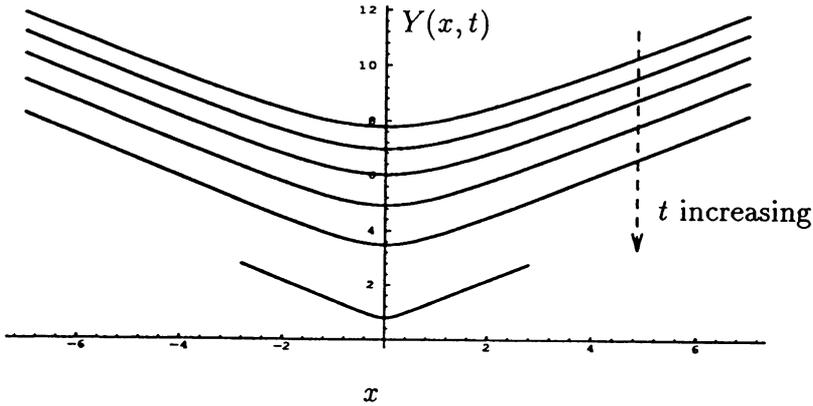


FIG. 3. Graph of $Y(x, t)$ for increasing t

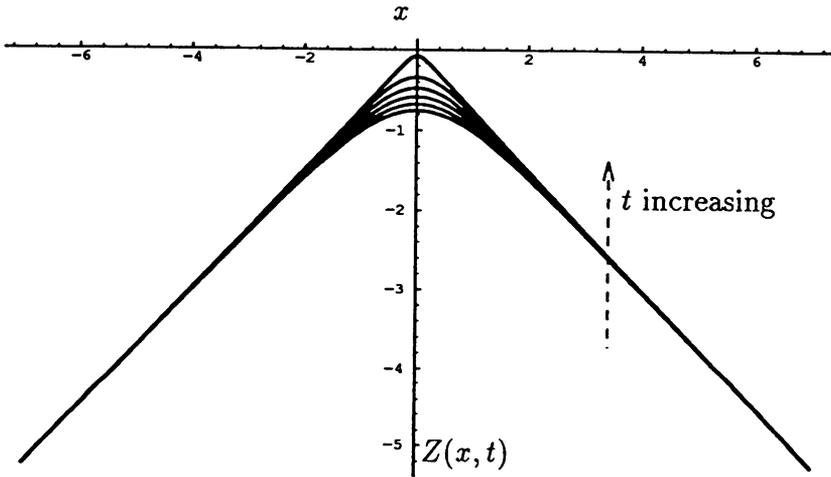


FIG. 4. Graph of $Z(x, t)$ for increasing t

3.2.1. *Superconducting vortex interactions.* As in the case of fluid vortices, there can only be a balance between the image and the self-induced motion for a superconducting vortex when the distance from the vortex to the boundary between the superconductor and the vacuum is $O((\kappa_0 \log \delta^{-1})^{-1})$. Then, with \mathbf{r} given by (11) and the same scalings that were used to derive (12),

$$\kappa \mathbf{n} \log \delta = \left(0, 0, \frac{\partial^2 Z}{\partial x^2} \right).$$

The boundary condition appropriate to (7) is the continuity of \mathbf{H} at $z = 0$, with the Maxwell equations

$$\begin{aligned} \nabla \wedge \mathbf{H} &= \mathbf{0}, \\ \nabla \cdot \mathbf{H} &= 0 \end{aligned} \tag{23}$$

that there are straight vortex solutions in which $Z \rightarrow 0$ in finite time. Similarly, if the curvature is uniformly small initially we would expect (26) still to hold, but now t_0 is a function of x and the point of reconnection is simply the point closest to the boundary initially. However, there is a scaling as $Z \rightarrow 0$ that will still have image and curvature terms in balance, and will again be of similarity form with the boundary term being negligible.

With $Z = O(\varepsilon)$, $t = O(\varepsilon^2)$, $x = O(\varepsilon)$ we have

$$\frac{\partial Z}{\partial t} = \frac{1}{4\pi} \frac{\partial^2 Z}{\partial x^2} - \frac{1}{4\pi Z}. \quad (27)$$

As before, we seek a similarity solution of the form

$$Z = x\bar{Z} \left(\frac{x}{2\sqrt{-\pi t}} \right) = x\bar{Z}(\eta), \quad (28)$$

so that

$$\frac{\eta^3 \bar{Z}'}{2} = (\eta^2 \bar{Z}')' - \frac{1}{\bar{Z}}. \quad (29)$$

We again write $\bar{Z} = f(\eta)/\eta$ to find that

$$f'' - \frac{\eta f'}{2} + \frac{f}{2} - \frac{1}{f} = 0, \quad (30)$$

and our previous assumptions of asymptotic straightness and monotonicity imply that (18) to (20) still hold. However, instead of (21) and (22) we have that, as $\eta \rightarrow 0$,

$$f \sim A_0 + B_0\eta + \dots, \quad A_0 \neq 0 \quad (31)$$

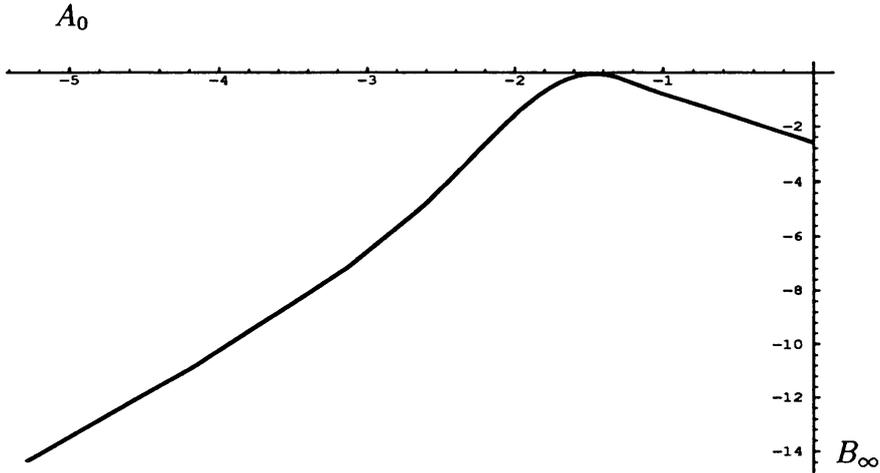
and linearising about (20) now gives the two-parameter expansion

$$f - b\eta \sim A_\infty\eta + \frac{B_\infty}{\eta^2} e^{\eta^2/4} + \dots \quad (32a)$$

as $\eta \rightarrow \infty$ unless $b = 0$ when the expansion is

$$f - \sqrt{2} \sim A'_\infty\eta^2 + \frac{B'_\infty}{\eta^2} e^{\eta^2/4} + \dots \quad (32b)$$

Clearly the straight vortex $f \equiv \sqrt{2}$ is one candidate solution, and we expect it to be the only solution in which f is bounded at infinity because the three conditions $A'_\infty = B'_\infty = B_0$ would force any deviation from $\sqrt{2}$ of a solution of (30) to be zero. However, if $b \neq 0$, there are only two conditions to be satisfied, namely $B_\infty = B_0 = 0$, which suggests that there may also be a non-straight solution. Nevertheless, numerical evidence suggests that such a solution cannot exist, because when we shoot from $\eta = 0$ for values of A_0 in $-10 < A_0 < -0.01$, with $B_0 = 0$, we obtain the function B_∞ shown in Fig. 5. McLeod [19] has proved that this is indeed the case and so $f \equiv \sqrt{2}$ is the only solution of (19, 30) that is $O(\eta)$ as $\eta \rightarrow \infty$. Thus a much longer segment of the vortex remains straight to lowest order as it approaches the boundary than in the case of fluids and the contact is much more oscillatory than in Fig. 4.

FIG. 5. Graph of B_∞ versus A_0

4. Discussion. We have presented a framework in which to consider the dynamics of vortices whose self-induced motion is comparable to that resulting from the presence of a nearby boundary. Our scenario always results in a system of partial differential equations for the evolution of the vortex that are singular whenever the vortex meets the boundary. In this model, the ability of the vortex to reconnect with its image will depend on the existence of singular solutions of this system of equations. The analysis of even a special similarity class of such solutions is delicate, as we have seen in (17) and (30), although we have suggested that both systems may have at least one relevant solution. However, the morphologies are quite different in the two cases, the fluid reconnection apparently taking place at a nonzero angle while the superconducting reconnection is via an osculating cusp. We also note that in [15], more elaborate similarity solutions have been proposed to cater to the situation where two vortices of unequal strength may reconnect. Much more work is needed to ascertain whether there may be other relevant solutions; for example, we have not investigated reconnecting solutions that involve high frequency oscillations.

We emphasize that we have throughout been considering reconnection in terms of mechanisms that act over length-scales much greater than the core radius and this assumption is clearly violated when the vortex is of $O(\delta)$ from the boundary. As remarked in the introduction, this situation has been closely scrutinised in the context of fluid vortex reconnection and in this context we can make three remarks concerning our approach. First, our scenario leading to models such as (13) and (27) can be used to explain how vortices get close, but not too close, to boundaries, without recourse to numerical simulations such as the inviscid vortex calculations of [27]. Second, these models can act as a check on such simulations, even when, say, the computed inviscid cores may be

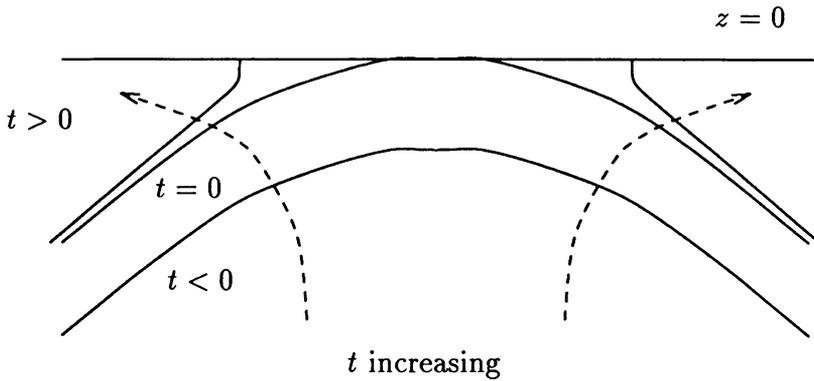


FIG. 6. Possible superconducting vortex geometry throughout reconnection

deforming quite strongly, because the models only depend on the order of magnitude of the core radius. Third, the solutions to equations such as (13) and (27) can be used as input to any *localised* numerical (such as [21]) or analytical (such as [23, 25]) reconnection model that incorporates core structure. Indeed, a numerical switch from a global inviscid model to a localised viscous one has been implemented in [1].

Our scenario would be even more useful if it could be used to describe what happens after any reconnection occurred. Immediate post-reconnection behaviour can be predicted by a calculation incorporating core structure, but, for example, we might be bold enough to use our prediction of cuspidal morphology for superconducting vortices just prior to possible reconnection to continue through $t = 0$ as in Fig. 6, with two distinct components existing for $t > 0$. Support for this conjecture comes from the fact that the tangents make an angle of $O((\log \delta^{-1})^{-1})$ with the boundary over most of the components and hence their self-induced motion will dominate their mutual interaction by a factor $O((\log \delta)^2)$. Hence we may be able to obtain an approximation to the post-reconnection flow by immediately switching off the mutual interaction as soon as reconnection has taken place.

The violent stretching caused by the binormal component of the vortex velocity makes it even more difficult to conjecture the morphology of post-reconnection flows in fluids, even if they can be described by a line vortex model. It is, nonetheless, very interesting that the viscous calculations of [21, 25] reveal a morphology that also splits into two components, but dramatic changes are also taking place in the y -direction. Indeed, the whole question of how to analyse rapid curvature changes even for more classical free boundary problems in which the free boundary has only one dimension fewer than the field equation is still largely unanswered.

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