QUARTERLY OF APPLIED MATHEMATICS
VOLUME LVI, NUMBER 4
DECEMBER 1998, PAGES 807-812

MATHEMATICS AND THE REAL WORLD

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In tackling this rather grandiose title I intend to adopt a personal approach. Mathematics and science are much too large and complex to be adequately covered by a single point of view. They have many angles, many scales, many philosophies, rather in the way a modern atlas is constructed using many pages with different parts of the globe and different types of information (geographical, climatic, economic). Since we all have our own personal prejudices, arising from our experience and background, we bring different perspectives. This is not always openly recognised or admitted: few lecturers begin "I am a narrow-minded___, and I make no claim to being objective". So I want to emphasise that I do have a particular outlook, many others are equally valid, and I do not offer a philosophical straitjacket.

Since I am on the verge of retirement at the close of the 20th century, and since my teachers studied in the first half of the century, I now think of myself as a child of the 20th century. I have also travelled widely, worked in both Britain and the USA, and my field of interest has also moved over time. I hope this gives me a perspective suitable for the occasion.

For brevity I describe myself as a Geometer. I was always attracted by the classical beauty of the subject and it has remained at the centre of my interest although its interpretation has evolved. Broadly understood, Geometry is fairly all-encompassing. Essentially our visual insight is so powerful that we convert many things into geometrical language. For example, Hilbert space brings Analysis into Geometry, phase space provides pictures for dynamics and the Argand diagram tames complex numbers. One could also say that, in one dimension, where things are ordered, Geometry plays no real role, but as soon as we move to two or more dimensions, Geometry becomes essential.

We can contrast two different modes of thinking: one spatial and geometric, the other logical and sequential. In the latter formal reasoning, algorithms and the digital computer predominate. In the former, pictures, structure and relationships are the essential ingredients. Much of mathematics can be seen as a dichotomy between these two modes. Even the computer has had to learn how to produce pictures and how to perform parallel processes.

Received November 19, 1997.
1991 *Mathematics Subject Classification.* Primary 00A30.

William Rowan Hamilton, the great mathematical physicist, once wrote a treatise on Algebra as the study of pure time (in contrast to Geometry as the study of space). I have always felt that this was a profound observation, which the modern computer exemplifies. Algebraic operators are always thought of as performed sequentially in time, just like a computer programme.

If Geometry versus Algebra provides a dichotomy in modes of mathematical thinking then Pure versus Applied provides, on the face of it, a dichotomy of subject matter or of philosophy. But this is misleading; the terms Pure and Applied are not sharp alternatives but descriptions of different ends of a spectrum that itself changes with time. Frequently the Pure Mathematics of one century becomes the Applied Mathematics of the next.

One could describe or caricature a pure mathematician as someone working alone in a room, building elaborate constructions by himself, not looking out of the window and not exploring the rest of the building. In each room there are different mathematicians each with their own building bricks. The rooms are not locked but there is not much traffic between them.

Continuing the analogy, I started in this way but found myself wandering from room to room, then out into the garden, then into the street, down the road, etc. In this way I have drifted through much of Pure Mathematics and stumbled eventually into Theoretical Physics. On the way, I have had glimpses of other fields, particularly through the medium of differential equations.

This broadening of my mathematical interests was enhanced by, and reflected in, various changes in my career. As President of the Royal Society (1990–1995) I had to become involved in all of science. I was the first mathematician in a hundred years to hold this position, the previous one being Stokes. Of course, centuries earlier, Isaac Newton had been President for over 20 years. I had to reflect on the role of mathematics in science and then on the role of science in society.

Simultaneously, I became the first Director of the Isaac Newton Institute for Mathematical Sciences in Cambridge, and I had to develop its policy and philosophy. This is particularly relevant for this meeting in Providence, celebrating the 50th anniversary of the Division of Applied Mathematics at Brown University, and I will return to this later. Finally, as Master at Trinity College Cambridge (1990–1997), with its great mathematical traditions going back again to Newton, I became aware of past history and the role of mathematicians in the development of science.

Mathematicians are notoriously addicted to definitions. This sometimes acts as a barrier to communication: "if you cannot define it we cannot discuss it". Ironically it is extremely difficult to give an acceptable definition of mathematics itself. We can say it is the study of numbers, space, patterns, structure, order (and disorder), change and so on. But any attempt to put limits on mathematics is likely to fail: there are always new realms for mathematicians to explore. Perhaps it is simplest to say that mathematics is a tool for precise thinking. The subject matter is secondary, the way it is tackled is primary.

Pure mathematicians frequently ignore or despise applied mathematics on the grounds that it is dull, ugly, and not intellectually challenging. They are, I believe, wrong to adopt such a negative attitude. There are several reasons why they should change their views.

(1) Applications provide sources of new problems and concepts. This has been the case historically and it is still true today.
(2) Some new exciting mathematical phenomena have been discovered through applications. Solitons, in their various forms, provide a recent example.
(3) Pure mathematicians need powerful allies to help argue their case to the public. Applied mathematicians (and other scientists) understand the strength and power of mathematics and can put this across.
(4) They need outlets for their students, most of whom will have to apply their mathematics in one way or another in their future employment.

The Newton Institute. The general philosophy in the operation of the Newton Institute, in Cambridge, is that modern science has many new areas involving complex systems and that these require new techniques and ideas. Some will generate new mathematical developments, and the Institute's task is to facilitate this process.

In broad terms we might say that linear phenomena and those that are approximately linear have been well worked out by a combination of classical mathematics and modern computers. Fully nonlinear phenomena provide totally new challenges and require much greater mathematical sophistication. Solitons are an example of such a new phenomenon and they have widespread applications in engineering, biology, and physics.

In selecting our programmes at the Newton Institute we ignored disciplinary boundaries and rigid terminology. The only important factor was new interesting and important science with some high-level mathematical aspects. We were at times criticised by our fellow pure mathematicians, but I think their criticisms were misguided. Most of our programmes involved extremely sophisticated and interesting mathematics of great benefit both to pure mathematicians and to the scientists.

To indicate the range of topics that we covered, I list in an Appendix all the programmes carried out or planned since the opening of the Institute.

Quantum physics. To conclude let me describe in more detail some of the developments of recent years in quantum field theory and their impact on pure mathematics, which has been quite dramatic. This material figured in several of the Newton Institute programmes and it is also my own area of interest; so I can report on it with greater confidence.

High Energy Physics is concerned with delving deeply into the ultimate nature of matter and forces. Because this involves experiments at increasingly high energy that are either prohibitively expensive, or even impossible on an earthly scale, such physics is dismissed in certain quarters. It is viewed as too theoretical, of no practical use, and not testable in the laboratory. Superficially these criticisms have some validity, but in defence we should

(a) remember the past,
(b) note the ability of theoretical models to jump the "species barrier" (to use a fashionable biological term).

Under (a) we have to recall that early atomic physics was not considered of practical relevance, while electromagnetic theory was similarly dismissed. It would be rash for anyone to predict that a further understanding of matter and force would not have important pay-offs. Under (b) I have in mind the fact that complicated mathematical models, developed for one physical problem, frequently have important applications in quite different situations. At the present time, nonlinear gauge theory models are being used in solid-state physics, and conformal field theory is applied in statistical mechanics.

But perhaps I can leave to physicists themselves the internal arguments about what is or is not "real physics". Let me turn instead to the relation between physics and mathematics. Traditionally the relation has frequently been one where the mathematicians came in to consolidate and establish firm foundations after the physicists had moved on to higher things. In other words the mathematicians worked on questions that physicists thought they already understood.

Recent developments, specifically between quantum field theory and geometry have been on a quite different plane. Because physicists are exploring at scales that are beyond experimental verification, they have little to guide them except the internal consistency of their theoretical models. Here, surprisingly, mathematics has come to their help. Much to the surprise of pure mathematicians, current models in quantum field theory have been turning up new and unexpected results in pure geometry. These results have been quite sensational in mathematical terms and, although not rigorously proved by physicists, they have stood up to all the mathematical checks that have been possible so far. This has given physicists the conviction that "they are on the right track" and it has also perplexed and amazed pure mathematicians.

Let me just give two examples to illustrate what I have been saying. The first arises in 2-dimensional quantum field theory, where functions, describing fields on space-time, take their values in a nonlinear target manifold. When this manifold is taken to be a certain complex algebraic variety (a hypersurface of degree 5 in projective 4-space) the quantum field theory predicts the number of rational curves of each fixed degree that lie on the variety. This solves a classical type of geometric problem that was well beyond any method available to geometers. However, the physicists' arguments were quite beautiful, involving a notion of duality which replaced the target manifold by a dual or mirror manifold, so that the dual problem was easily soluble. The second example also involves a duality but, this time, an extension of the classical electric-magnetic duality of Maxwell theory in 4-dimensional space-time. The mathematical consequence of this duality was effectively to replace the Yang-Mills equations, so spectacularly used by Donaldson to study 4-manifolds, with a new set of equations now known as the Seiberg-Witten equations. While both sets of equations are nonlinear, the nonlinearity in the Seiberg-Witten equations is simpler and easier to control. Remarkably the two sets of equations look very different. The Yang-Mills equations use a nonabelian gauge group while the Seiberg-Witten equations use the circle group (effectively Maxwell theory) coupled nonlinearly to a spinor field.

The Seiberg-Witten equations have turned out to solve many outstanding problems in Geometry, including an old problem of René Thom about which real surfaces can be embedded in the complex projective plane. The duality theory also has striking implications for physics, at least if the relevant model (involving super-symmetry) turns out to be a correct description of nature. The theory would explain in a very beautiful fashion why quarks are "confined", that is, why they remain grouped in handfuls of three and cannot separate.

These spectacular applications of quantum field theory in geometry have done much to break down the barriers between pure and applied mathematicians. As we approach the 21st century, we can hope that these barriers will finally be removed and that mathematics can recapture its unity and its relevance to the real world.

Appendix. Past and Future Programmes of the New Institute

July to December 1992.
- Low-dimensional Topology and Quantum Field Theory
- Dynamo Theory

January to June 1993.
- L-functions and Arithmetic[1]
- Epidemic Models

July to December 1993.
- Computer Vision[2]
- Random Spatial Processes

January to June 1994.
- Geometry and Gravity
- Cellular Automata, Aggregation and Growth

July to December 1994.
- Topological Defects
- Symplectic Geometry

January to June 1995.
- Exponential Asymptotics
- Financial Mathematics[3]

July to December 1995.
- Semantics of Computation
- From Finite to Infinite Dimensional Dynamical Systems

January to June 1996.
- Dynamics of Complex Fluids
- Computer Security, Cryptology and Coding Theory[4]

July to December 1996.
- Mathematics of Atmosphere and Ocean Dynamics[5]
- Four-dimensional Geometry and Quantum Field Theory[6]
- Mathematical Modelling of Plankton Population Dynamics

January to June 1997.
- Representation Theory of Algebraic Groups and Related Finite Groups
- Non-Perturbative Aspects of Quantum Field Theory

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[1] This included Wiles' famous lecture on Fermat's Last Theorem.
[2] Co-organized by David Mumford.
[3] This was providentially timed just as Barings collapsed.
[4] This and the Financial Mathematics programme were well supported by the major banks.
[5] This was supported by the Meteorological Office.
[6] This programme centred around the 4-dimensional duality theory described in the text.

July to December 1997.

Disordered Systems and Quantum Chaos

Neural Networks and Machine Learning

January to June 1998.

Arithmetic Geometry

Dynamics of Astrophysical Discs

July to December 1998.

Biomolecular Function and Evolution in the Context of the Genome Project

Nonlinear Nonstationary Signal Processing

January to July 1999.

Turbulence

January to April 1999.

Mathematics and Applications of Fractals

May to August 1999.

Complexity, Computation and the Physics of Information

July to December 2000.

Singularity Theory