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BREAKDOWN OF SMOOTH SOLUTIONS IN ONE-DIMENSIONAL MAGNETOSTRICTION

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Abstract. We discuss a hyperbolic aspect of magnetostriction. The equations governing the longitudinal motion consist of the nonlinear wave equations and the rate equations for the motion of spin for the magnetic moment. We show that the breakdown of smooth solutions will take place in finite time even if the initial data are smooth.

1. Introduction. In this paper we study the development of singularities for a system describing the one-dimensional motion of magnetostrictive materials. The magnetostriction or magnetoelastic interaction is a subject in which we study the interrelation between the elasticity and magnetization. We will be concerned about an aspect associated with the nonlinear wave equations. We assume that the material is elastic and it is nonconducting and magnetically saturated. We also assume that the material occupies the whole space. This is to avoid the complication arising from the fact that if there is a free space, the magnetic field extends into the free space. This is in the same spirit as [11]. The system we study is given by

$$v_t = u_x, \tag{1.1}$$

$$u_t = \left\{ \frac{\partial W}{\partial v} \right\}_x, \tag{1.2}$$

$$m_t = -m \times \frac{\partial W}{\partial m} + \eta m \times \left(m \times \frac{\partial W}{\partial m} \right), \tag{1.3}$$

where $u, v, m = (m_1, m_2, m_3)^{\mathrm{T}}$ are velocity, strain, and magnetic moment, respectively, and W is the potential energy given by

$$W = W_e(v) + W_{em}(v, m) + W_m(m) + W_H(v, m),$$

where W_e is the stored elastic energy, W_{em} is the elastic-magnetic energy, W_m is the anisotropy energy, and W_H is the energy due to the magnetic induction induced by the

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magnetization. The specific form of W will be discussed in Sec. 2. The microstructure of magnetic materials is very complicated and we should regard (1.1)–(1.3) as a phenomenological description of bulk property of such materials.

The development of singularity is inherent in nonlinear hyperbolic equations due to the nonlinearity of waves. An interesting case is where there is dissipation in the system. The dissipation counteracts the nonlinearity of waves and makes it harder for the nonlinearity to develop singularities. Typical cases are where the dissipation is given by damping, fading memory, or heat conduction. Slemrod [13] discussed the case where the dissipation is given by the linear damping, Hattori [5] and Dafermos [3] discussed the case where dissipation is given by fading memory, and Dafermos and Hsiao [4] and Hrusa and Messaoudi [6] discussed the case where the dissipation is given by heat conduction. In our case the dissipation is provided by the second term in (1.3) as we discuss it later. However, the dissipation is very tame since the interaction between the elasticity and the magnetic moment is weak. Therefore, we expect the nonlinearity to dominate and the breakdown of smooth solutions to occur. This will be discussed in Secs. 3 and 4.

In hyperbolic conservation laws, if there is no dissipative mechanism, the solutions in general will develop singularities in finite time no matter how smooth the initial data are. On the other hand, if there is a dissipative mechanism, the situation becomes subtle. The solutions may or may not develop singularities depending on the initial data. In the present case, the dissipative mechanism is weak and this makes the existence of smooth solutions more interesting. Also, the dissipation affects the traveling-wave solutions. If the dissipative mechanism is not strong, the traveling-wave solutions are smooth or discontinuous depending on two end states. It should be interesting to discuss how (1.3) affects the existence of smooth and discontinuous traveling-wave solutions. These issues will be discussed in the forthcoming papers.

Recently, the microstructure of magnetostriction and magnetization has been discussed in various literature including [7], [8], [10], [12]. These approaches are based on the variational principle. The dynamical aspect of magnetization was discussed in Miranker and Willner [11] and Visintin [16]. Visintin also discussed the magnetostriction where the elasticity is given by linear elasticity. On the other hand, the dynamical aspect of magnetostriction where the elasticity is nonlinear has not yet been discussed. This paper deals with a dynamic aspect of bulk property of magnetostriction. The dynamical aspect of microstructure will be a future issue.

This paper consists of four sections. In Sec. 2 we derive the equations governing the one-dimensional motion of magnetostriction. We also discuss the form of dissipation and the equilibrium solutions. In Sec. 3 we discuss the main assumptions and state the main theorem. In Sec. 4 we discuss the lemma necessary for the proof and then give the proof of the main theorem.

2. Preliminary. In this section we derive the equations governing the one-dimensional magnetostriction. We also discuss the form of dissipation and the equilibrium solutions.

2.1. *Derivation of the equations.* We employ the summation convention and the components of a vector are denoted with a subscript and the corresponding vector is denoted without a subscript. First, we derive the governing equations in the general case and then reduce the longitudinal motion from them. The derivation is based on the results by Brown [1], [2] and Tiersten [14], [15].

Let x_i denote the Cartesian components of a material particle at some reference time t_0, and y_i the components of the same particle at some arbitrary time t. The x_i and y_i are referred to as material coordinates and spacial coordinates, respectively. The deformation of the body is the mapping

$$y_i = y_i(x_j, t),$$

which we require to be one-to-one. The deformation gradient $\frac{dy_i}{dx_j}$ is denoted by F and the strain tensor $(\frac{dy_i}{dx_j} + \frac{dy_j}{dx_i})$ is denoted by ε_{ij}. We denote by ρ_0 the mass density in the reference configuration and by ρ the mass density at t. The material velocity vector $\frac{dy_i}{dt}$ at t is denoted by V. The magnetic moment per unit mass m_i is related to the magnetization vector M_i by

$$M_i = \rho m_i.$$

We assume that the material is magnetically saturated and since the mass is conserved, we have

$$m_i m_i = \mu_0^2.$$

In what follows, without loss of generality we set $\mu_0^2 = 1$.

The equations we discuss consist of the conservation of mass, the equation for the linear momentum, the Maxwell equations, and the equations for the magnetic moments. The conservation of mass takes the form

$$\rho J = \rho_0,$$

where

$$J = \det F.$$

In the spacial coordinates, the conservation of linear momentum has the form

$$\rho \frac{dV}{Dt} = \nabla_y \cdot \tau + h,$$

where

$$\frac{Dg}{Dt} = \frac{\partial g}{\partial t} + V \cdot \nabla_y g,$$

τ is the stress tensor, and h is the body force. The stress tensor τ_{ij} is given by

$$\tau_{ij} = \rho \frac{\partial y_i}{\partial x_m} \frac{\partial U}{\partial(\partial y_j / \partial x_m)},$$

where U is the internal energy. In this paper, we assume that it is given by

$$U = W_e(F) + W_{em}(F, m) + W_m(m),$$

where W_e is the stored elastic energy, W_{em} is the elastic-magnetic energy, and W_m is the anisotropy energy. A typical example of W_{em} and W_m is given by

$$\begin{aligned} W_{em}(F,m) &= \sum b_{ij}\varepsilon_{ij}m_i m_j, \\ W_m(m) &= \frac{1}{2}\sum a_i m_i^2, \end{aligned} \tag{2.4}$$

where a_i are positive constants and b_{ij} is a constant and symmetric matrix; see [12], [16]. The conditions for W_e are given later. The body force h is the force that the magnetic field H^M exerts on a magnetic dipole and we assume that it has the following form:

$$h = M \cdot \nabla_y H^M. \tag{2.5}$$

For Maxwell's equations we use the quasistationary approximation. In this case, the magnetic field H^M and the magnetic induction B satisfy

$$\begin{aligned} H^M &= -\nabla_y \varphi, \\ \nabla_y \cdot B = \nabla_y &\cdot (H^M + M) = 0, \end{aligned} \tag{2.6}$$

where φ is the axial magnetic scalar potential. The equation of the conservation of angular momentum takes the form

$$\rho \frac{Dm}{Dt} = m \times W^m, \tag{2.7}$$

where

$$W^m = -\rho \frac{\partial U}{\partial m_j} + \rho\lambda m_j - \rho \frac{\partial \varphi}{\partial y_j}, \tag{2.8}$$

$$\lambda = \frac{\partial U}{\partial m_k} m_k.$$

Note that there is no dissipation in (2.7). Landau and Lifshitz [9] considered the case where the equations of angular momentum have dissipation. They are given by

$$\rho \frac{Dm}{Dt} = m \times W^m - \eta m \times (m \times W^m), \tag{2.9}$$

where η is a nonnegative constant and the term with η is the dissipative term. In this paper we will use (2.9). The form of the dissipation will be discussed later.

We now derive the equations governing the longitudinal motion. We assume that elastic motion is restricted in the y_1-direction and that the material is perfectly rigid in the y_2 and y_3-directions. Setting $x = x_1$, we see that

$$\begin{aligned} y_1 &= y_1(x,t), \\ y_2 &= x_2, \\ y_3 &= x_3, \end{aligned}$$

$$F = \begin{bmatrix} \frac{\partial y_1}{\partial x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\varepsilon_{ij} = \begin{bmatrix} \frac{\partial y_1}{\partial x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore,

$$W_{em}(F, m) = \frac{1}{2} b_{11} \left(\frac{\partial y_1}{\partial x} \right) m_1^2.$$

From (2.6) we see that

$$H^M = \left(-\frac{\partial \varphi}{\partial y_1}, 0, 0 \right)$$

and we have

$$\nabla_y \cdot (H_1^M + M_1) = \frac{\partial x}{\partial y_1} (H_1^M + M_1)_x = 0.$$

Therefore, we obtain

$$H_1^M = -M_1 + C,$$

where C is a constant. For simplicity we assume that $C = 0$. Now from (2.5) we see that

$$h = -\rho m_1 \left(\frac{\partial x}{\partial y_1} \right) (\rho m_1)_x = -\frac{1}{2} \left(\frac{\partial x}{\partial y_1} \right) ((\rho m_1)^2)_x.$$

Also from (2.8), we see that

$$W^m = -\rho \left(\frac{\partial U}{\partial m_j} + \delta_{1j} \rho m_1 \right),$$

where $\delta_{1j} = 1$ if $j = 1$ and $\delta_{1j} = 0$ if $j \neq 1$.

We will use the Lagrangian coordinates since they are more convenient for the one-dimensional motion. Setting $x = x_1$, $\frac{\partial y_1}{\partial x} = v$, $\frac{\partial y_1}{\partial t} = u$, and noting that $\rho = 1/v = \frac{\partial x}{\partial y_1}$, for the system of the longitudinal motion we obtain

$$v_t = u_x, \tag{2.10}$$

$$u_t = \left\{ \frac{\partial W}{\partial v} \right\}_x, \tag{2.11}$$

$$m_t = -m \times \frac{\partial W}{\partial m} + \eta m \times \left(m \times \frac{\partial W}{\partial m} \right), \tag{2.12}$$

where in the case of example (2.4),

$$W = W_e(v) + \frac{1}{2} b_{11} v m_1^2 + \frac{1}{2} \sum a_i m_i^2 + \frac{1}{2v} m_1^2.$$

In the component form the terms in (2.12) become

$$m \times \frac{\partial W}{\partial m} = \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix} \frac{\partial W}{\partial m}, \tag{2.13}$$

$$\eta m \times \left(m \times \frac{\partial W}{\partial m} \right) = \eta \begin{bmatrix} -(m_2^2 + m_3^2) & m_1 m_2 & m_1 m_3 \\ m_1 m_2 & -(m_1^2 + m_3^2) & m_2 m_3 \\ m_1 m_3 & m_2 m_3 & -(m_1^2 + m_2^2) \end{bmatrix} \frac{\partial W}{\partial m}, \tag{2.14}$$

where

$$\frac{\partial W}{\partial m} = \begin{bmatrix} (a_1 + b_{11}v + \frac{1}{v})m_1 \\ a_2 m_2 \\ a_3 m_3 \end{bmatrix}.$$

In what follows we denote $W_e'(v)$ by $f(v)$. We require that $v > 0$. Note that the equations for the magnetization are redundant due to the fact that m_i satisfies $\sum m_i^2 = 1$.

2.2. *The form of dissipation.* It is interesting to see how the energy for the system is obtained from (2.10)–(2.12). Multiplying (2.11) by u and integrating it over x and t, we see

$$\int_{-\infty}^{\infty} \frac{1}{2}u^2(x,t)dx + \int_0^t \int_{-\infty}^{\infty} \frac{\partial W}{\partial v} v_t(x,s)dxds = \int_{-\infty}^{\infty} \frac{1}{2}u^2(x,0)dx. \tag{2.15}$$

Now applying the cross product with m to (2.12) and making the dot product with m_t, we have

$$m_t \cdot m \times m_t = -m_t \cdot m \times \left(m \times \frac{\partial W}{\partial m}\right) + \eta m_t \cdot m \times \left(m \times \left(m \times \frac{\partial W}{\partial m}\right)\right). \tag{2.16}$$

The left-hand side is zero and the second term on the right-hand side is rewritten as

$$\begin{aligned} \eta m_t \cdot m \times \left(m \times \left(m \times \frac{\partial W}{\partial m}\right)\right) &= -\eta m_t \cdot m \times \frac{\partial W}{\partial m} \\ &= \eta m_t \cdot \left(m_t - \eta m \times \left(m \times \frac{\partial W}{\partial m}\right)\right). \end{aligned}$$

Consider the second term on the right-hand side. Using (2.14), we have

$$-m_t \cdot m \times \left(m \times \frac{\partial W}{\partial m}\right) = \frac{\partial W}{\partial m_i} m_{it} - \left(\sum_{i=1}^{3} m_i\right)(m_i m_{it}) = \frac{\partial W}{\partial m_i} m_{it},$$

where $m_i m_{it} = 0$ has been used. Integrating (2.16) over x and t, we obtain

$$\int_0^t \int_{-\infty}^{\infty} \left\{ \frac{\partial W}{\partial m} \cdot m_t + \frac{\eta}{1+\eta^2} m_t^2 \right\}(x,s)dxds = 0. \tag{2.17}$$

Combining (2.15) and (2.17), we have

$$\begin{aligned} \int_{-\infty}^{\infty} \left\{\frac{1}{2}u^2 + W\right\}(x,t)dx + \int_0^t \int_{-\infty}^{\infty} \frac{\eta}{1+\eta^2} m_t^2(x,s)dxds \\ = \int_{-\infty}^{\infty} \left\{\frac{1}{2}u^2 + W\right\}(x,0)dx. \end{aligned}$$

As we see, the η term in (2.12) gives the dissipation for the system.

2.3. *The equilibrium solutions.* Next, we discuss the constant equilibrium solutions of (2.10) and (2.12). In order for a constant state to be an equilibrium, the right-hand side of (2.12) has to be zero. There are three possibilities for it. They are

(i) $$\frac{\partial W}{\partial m} = 0,$$

(ii) $$m \times \frac{\partial W}{\partial m} = 0,$$

(iii)
$$m \times \frac{\partial W}{\partial m} + \eta m \times \left(m \times \frac{\partial W}{\partial m} \right) = 0.$$

In the case of the example, case (i) is impossible. We have case (ii) if

$$\frac{\partial W}{\partial m} = \alpha m,$$

where α is a proportional constant. In the case of the example, m can be one of $(\pm 1, 0, 0)$ or $(0, \pm 1, 0)$ or $(0, 0, \pm 1)$. In case (iii) we have either

(iii)-1
$$\frac{\partial W}{\partial m} = \eta \left(m \times \frac{\partial W}{\partial m} \right),$$

or

(iii)-2
$$\frac{\partial W}{\partial m} - \eta \left(m \times \frac{\partial W}{\partial m} \right) = \beta m,$$

where β is a proportional constant. Case (iii)-1 is impossible since $m \times \frac{\partial W}{\partial m}$ is perpendicular to $\frac{\partial W}{\partial m}$. Case (iii)-2 is rewritten as follows:

$$\eta \left(m \times \frac{\partial W}{\partial m} \right) = \frac{\partial W}{\partial m} - \beta m.$$

This says that the vector $m \times \frac{\partial W}{\partial m}$ is spanned by $\frac{\partial W}{\partial m}$ and m. This is possible when $\frac{\partial W}{\partial m}$ and m are parallel and $\frac{\partial W}{\partial m} = \beta m$, which reduces to case (ii).

3. Main result. In this section we discuss the main assumptions on the system (2.10)–(2.12) and on the initial data. Then, we state the main theorem of this paper. The proof of the theorem will be given in the next section.

Equations (2.10) can be rewritten as follows:

$$v_t = u_x, \tag{3.18}$$

$$u_t = \left(f'(v) + \frac{m_1^2}{v^3} \right) v_x + \left(b_{11} - \frac{1}{v^2} \right) m_1 m_{1x}. \tag{3.19}$$

For the hyperbolicity we assume that $f'(v) > 0$ so that

$$f'(v) + \frac{m_1^2}{v^3} > 0. \tag{3.20}$$

The characteristics are denoted by

$$\frac{dx}{dt} = \pm \lambda(v, m_1), \qquad \lambda(v, m_1) = \sqrt{f'(v) + \frac{m_1^2}{v^3}}.$$

Also, for the nonlinearity of waves we assume that

$$D_v \lambda = \frac{1}{2\lambda} \left(f''(v) - 3\frac{m_1^2}{v^4} \right) > 0, \qquad |v - \overline{v}| \le \delta, \quad -1 \le m_1 \le 1, \tag{3.21}$$

where δ is a positive constant such that $\overline{v} - \delta > 0$. This condition is related to the genuine nonlinearity.

As far as the initial data are concerned, we assume that the strain and the magnetic moment are constants, and apply the impulse at $t = 0$. Using the result of the previous section, we give the following initial data:

$$v(x,0) = \overline{v}, \quad u(x,0) = u_0(x), \quad m(x,0) = m_0(x) = (1,0,0)^{\mathrm{T}}, \tag{3.22}$$

where u_0 is a C^2 function with compact support satisfying

$$\begin{aligned} |u_0(x)| &< \varepsilon, \\ -u_{0x}(x) &< N, \\ \max_{x\in R} u_{0x}(x) &> M. \end{aligned} \tag{3.23}$$

Since u_0 has compact support, the initial data are in equilibrium outside of this compact support. Since the characteristics are bounded, this will ensure that the solution of (2.10)–(2.12) will be in equilibrium outside a compact interval. Here, ε, N, M are positive numbers that will be discussed in the theorem and in the proof. Since we assume that the solution is C^2, for a given positive δ, there is a T such that

$$|v(x,t) - \overline{v}| \le \delta, \quad x \in R,\ 0 \le t \le T.$$

The value of T will be obtained during the course of the proof.

Now the main theorem is stated as follows:

THEOREM 3.1. *Assume that the inequalities (3.20) and (3.21) are satisfied. Assume that u_0, v_0, and m_0 are C^2 functions on $(-\infty,\infty)$. Then, for any N and δ, there exists a positive T such that we can choose a positive ε that depends on δ and $\overline{v}$, and a positive M that depends on $\delta, \overline{v}, \varepsilon, N$, and T so that if the initial data satisfy (3.22), (3.23), the length of the maximal time interval of existence of a C^2 solution of (2.10)–(2.12) cannot exceed T.*

4. Proof of Theorem. In this section we prove Theorem 3.1. For this purpose we need the estimates of v_x and m_{1x} in terms of u_t. This is given in the following

LEMMA 4.1. *As far as $|v - \overline{v}| < \delta$, we have for any $t \in [0,T]$,*

$$\begin{aligned} |v_x| &\le K\left\{|u_t| + \int_0^t e^{(K_4-K_1)(t-s)}|u_t|ds\right\} \\ &\le K\left\{|u_t| + e^{KT}\int_0^T |u_t|ds\right\}, \end{aligned} \tag{4.24}$$

$$\begin{aligned} \sum |m_{ix}| &\le K_3\int_0^t |u_t|ds + K_4\int_0^t\int_0^s e^{(K_4-K_1)(s-\tau)}|u_t|d\tau ds \\ &\le K(1+Te^{KT})\int_0^T |u_t|ds, \end{aligned} \tag{4.25}$$

where K is a positive constant depending on $\overline{v}$ and δ.

Proof. In what follows, the K_i's are positive generic constants depending on $\overline{v}$ and δ. Differentiating (2.12) with respect to x, we have

$$m_{tx} = M_1 m_x + M_2 v_x,$$

where M_1 and M_2 are 3×3 matrices whose components are functions of v and m. Note that $-1 \leq m_i \leq 1$. Since m is C^2,

$$|m_{ix}|_t \leq |m_{itx}|, \quad \text{a.e.}$$

Therefore, we obtain

$$\frac{\partial}{\partial t}\left\{\sum |m_{ix}|\right\} \leq K_1 \sum |m_{ix}| + K_2 |v_x|,$$

from which we deduce

$$\sum |m_{ix}| \leq \sum |m_{iox}| + K_2 \int_0^t e^{K_1(t-s)} |v_x| ds.$$

Since the initial data satisfies $m_{iox} = 0$, we have

$$\sum |m_{ix}| \leq K_2 \int_0^t e^{K_1(t-s)} |v_x| ds. \tag{4.26}$$

We need an estimate of $|v_x|$ in terms of $|u_t|$. Using (3.18) and (4.26), we obtain

$$|v_x| \leq K_3 |u_t| + K_4 \int_0^t e^{K_1(t-s)} |v_x| ds. \tag{4.27}$$

Multiplying (4.27) by $e^{-K_1 t}$, we see that

$$e^{-K_1 t} |v_x| \leq K_3 e^{-K_1 t} |u_t| + K_4 \int_0^t e^{-K_1 s} |v_x| ds.$$

The generalized Gronwall inequality yields

$$e^{-K_1 t} |v_x| \leq K_3 e^{-K_1 t} |u_t| + K_4 \int_0^t e^{K_4(t-s)} e^{-K_1 s} |u_t| ds, \tag{4.28}$$

from which we obtain (4.24). Therefore, combining (4.26) and (4.28), we obtain

$$\sum |m_{ix}| \leq K_3 \int_0^t |u_t| ds + K_4 \int_0^t \int_0^s e^{(K_4 - K_1)(s-\tau)} |u_t| d\tau ds.$$

Changing the order of integration in the second term on the right-hand side, we obtain (4.25). □

Proof of Theorem 3.1. In the proof, Γ_1 is a generic positive constant that depends on δ and $\overline{v}$ and Γ_2 is another positive generic constant depending on N, δ, and $\overline{v}$. We introduce the Riemann invariants by

$$r = u + \int^v \lambda(w, m_1) dw,$$

$$s = -u + \int^v \lambda(w, m_1) dw.$$

In terms of the Riemann invariants, (3.18) is expressed as follows:

$$r_t - \lambda r_x = A(v, m_1)(m_{1t} - \lambda m_{1x}) + \left(b_{11} - \frac{1}{v^2}\right) m_1 m_{1x}, \tag{4.29}$$

$$s_t + \lambda s_x = A(v, m_1)(m_{1t} + \lambda m_{1x}) - \left(b_{11} - \frac{1}{v^2}\right) m_1 m_{1x}, \tag{4.30}$$

where

$$A(v, m_1) = \frac{\partial}{\partial m_1}\left(\int^v \lambda(w, m_1)dw\right).$$

Using the Riemann invariants, we see that

$$u_t = \frac{1}{2}(r_t - s_t), \qquad v_t = \frac{1}{2\lambda}(r_t + s_t) - \frac{A}{\lambda}m_{1t}. \tag{4.31}$$

We use the following notation for the derivatives along the characteristics:

$$\begin{aligned} {}^{\backprime} &= \partial_t - \lambda\partial_x, \\ {}' &= \partial_t + \lambda\partial_x. \end{aligned} \tag{4.32}$$

Then, differentiating (4.29) and (4.30) with respect to t, we see that

$$\begin{aligned} (r_t)^{\backprime} = &\left\{(D_v\lambda)\left(\frac{1}{2\lambda}(r_t + s_t) - \frac{A}{\lambda}m_{1t}\right) + (D_{m_1}\lambda)m_{1t}\right\} \\ &\times \frac{1}{\lambda}\left\{r_t - A(m_{1t} - \lambda m_{1x}) - \left(b_{11} - \frac{1}{v^2}\right) m_1 m_{1x}\right\} \\ &+ (D_vA)\left(\frac{1}{2\lambda}(r_t + s_t) - \frac{A}{\lambda}m_{1t}\right)(m_{1t} - \lambda m_{1x}) \\ &+ (D_{m_1}A)m_{1t}(m_{1t} - \lambda m_{1x}) + A(m_{1tt} - \lambda m_{1tx}) \\ &- Am_{1x}\left\{(D_v\lambda)\left(\frac{1}{2\lambda}(r_t + s_t) - \frac{A}{\lambda}m_{1t}\right) + (D_{m_1}\lambda)m_{1t}\right\} \\ &+ \frac{3m_1}{v^3}m_{1x}\left(\frac{1}{2\lambda}(r_t + s_t) - \frac{A}{\lambda}m_{1t}\right) + \left(b_{11} - \frac{1}{v^2}\right)(m_{1t}m_{1x} + m_1 m_{1tx}), \end{aligned} \tag{4.33}$$

$$\begin{aligned} (s_t)' = &-\left\{(D_v\lambda)\left(\frac{1}{2\lambda}(r_t + s_t) - \frac{A}{\lambda}m_{1t}\right) + (D_{m_1}\lambda)m_{1t}\right\} \\ &\times \frac{1}{\lambda}\left\{-s_t + A(m_{1t} + \lambda m_{1x}) - \left(b_{11} - \frac{1}{v^2}\right) m_1 m_{1x}\right\} \\ &+ (D_vA)\left(\frac{1}{2\lambda}(r_t + s_t) - \frac{A}{\lambda}m_{1t}\right)(m_{1t} + \lambda m_{1x}) \\ &+ (D_{m_1}A)m_{1t}(m_{1t} - \lambda m_{1x}) + A(m_{1tt} + \lambda m_{1tx}) \\ &+ Am_{1x}\left\{(D_v\lambda)\left(\frac{1}{2\lambda}(r_t + s_t) - \frac{A}{\lambda}m_{1t}\right) + (D_{m_1}\lambda)m_{1t}\right\} \\ &- \frac{3m_1}{v^3}m_{1x}\left(\frac{1}{2\lambda}(r_t + s_t) - \frac{A}{\lambda}m_{1t}\right) - \left(b_{11} - \frac{1}{v^2}\right)(m_{1t}m_{1x} + m_1 m_{1tx}). \end{aligned} \tag{4.34}$$

We need to handle $r_t s_t$ terms in the above equations. From (4.29) and (4.32) we see that

$$
\begin{aligned}
r_t &= \frac{1}{2}(r' + s') - A\lambda m_{1x} + \left(b_{11} - \frac{1}{v^2}\right) m_1 m_{1x},\\
s_t &= \frac{1}{2}(r^{\backprime} + s^{\backprime}) + A\lambda m_{1x} - \left(b_{11} - \frac{1}{v^2}\right) m_1 m_{1x},
\end{aligned}
$$

and

$$
\begin{aligned}
v^{\backprime} &= \frac{1}{2\lambda}(r^{\backprime} + s^{\backprime}) - \frac{A}{\lambda} m_1{}^{\backprime},\\
(\ln\lambda)^{\backprime} &= \frac{1}{\lambda}\{(D_v\lambda)v^{\backprime} + (D_{m_1}\lambda)m_1{}^{\backprime}\}.
\end{aligned}
$$

We also obtain the similar relation for v' and $(\ln\lambda)'$. Therefore,

$$
\begin{aligned}
(D_v\lambda)\frac{1}{2\lambda^2} r_t s_t &= \frac{1}{2}\Bigg[(\ln\lambda)^{\backprime} + (D_v\lambda)\frac{1}{\lambda^3} A m_1{}^{\backprime} - (D_{m_1}\lambda)\frac{1}{\lambda} m_1{}^{\backprime}\\
&\qquad + (D_v\lambda)\frac{1}{\lambda} A m_{1x} - (D_v\lambda)\frac{1}{\lambda^2}\left(b_{11} - \frac{1}{v_2}\right) m_1 m_{1x}\Bigg] r_t,\\
(D_v\lambda)\frac{1}{2\lambda^2} r_t s_t &= \frac{1}{2}\Bigg[(\ln\lambda)' + (D_v\lambda)\frac{1}{\lambda^3} A m_1' - (D_{m_1}\lambda)\frac{1}{\lambda} m_1'\\
&\qquad - (D_v\lambda)\frac{1}{\lambda} A m_{1x} + (D_v\lambda)\frac{1}{\lambda^2}\left(b_{11} - \frac{1}{v_2}\right) m_1 m_{1x}\Bigg] s_t.
\end{aligned}
$$

Using the above expressions in (4.33) and (4.34), we find that

$$
\begin{aligned}
(r_t)^{\backprime} - (D_v\lambda)\frac{1}{2\lambda^2} r_t s_t &= (r_t)^{\backprime} - \frac{1}{2}(\ln\lambda)^{\backprime} r_t - \frac{1}{2}\left[(D_v\lambda)\frac{1}{\lambda^3} A m_1{}^{\backprime} - (D_{m_1}\lambda)\frac{1}{\lambda} m_1{}^{\backprime}\right] r_t\\
&\qquad + \frac{1}{2}\left[-(D_v\lambda)\frac{1}{\lambda} A m_{1x} + (D_v\lambda)\frac{1}{\lambda^2}\left(b_{11} - \frac{1}{v^2}\right) m_1 m_{1x}\right] r_t,\\
(s_t)' - (D_v\lambda)\frac{1}{2\lambda^2} r_t s_t &= (s_t)' - \frac{1}{2}(\ln\lambda)' s_t - \frac{1}{2}\left[(D_v\lambda)\frac{1}{\lambda^3} A m_1' - (D_{m_1}\lambda)\frac{1}{\lambda} m_1'\right] s_t\\
&\qquad + \frac{1}{2}\left[(D_v\lambda)\frac{1}{\lambda} A m_{1x} - (D_v\lambda)\frac{1}{\lambda^2}\left(b_{11} - \frac{1}{v^2}\right) m_1 m_{1x}\right] s_t.
\end{aligned}
$$

Therefore, multiplying (4.33) and (4.34) by the integrating factor $\lambda^{-1/2}$ and setting

$$
\phi = \lambda^{-1/2} r_t, \qquad \psi = \lambda^{-1/2} s_t,
$$

we obtain

$$
\begin{aligned}
\phi' = {} & \frac{(D_v\lambda)}{2\lambda^{3/2}}\phi^2 + \frac{1}{2}\left[(D_v\lambda)\frac{1}{\lambda^3}Am_1{}' - (D_{m_1}\lambda)\frac{1}{\lambda}m_1{}'\right]\phi \\
& + \frac{1}{2}\left[(D_v\lambda)\frac{1}{\lambda}Am_{1x} - (D_v\lambda)\frac{1}{\lambda^2}\left(b_{11} - \frac{1}{v^2}\right)m_1 m_{1x}\right]\phi \\
& + \frac{1}{\lambda}\left\{-(D_v\lambda)\frac{A}{\lambda}m_{1t} + (D_{m_1}\lambda)m_{1t}\right\}\phi \\
& - \left\{A(m_{1t} - \lambda m_{1x}) + \left(b_{11} - \frac{1}{v^2}\right)m_1 m_{1x}\right\}\frac{(D_v\lambda)}{2\lambda}(\phi + \psi) \\
& + (D_v A)\left(\frac{1}{2\lambda}(\phi + \psi) - \frac{A}{\lambda^{3/2}}m_{1t}\right)(m_{1t} - \lambda m_{1x}) \\
& + \lambda^{-1/2}\{(D_{m_1}A)m_{1t}(m_{1t} - \lambda m_{1x}) + A(m_{1tt} - \lambda m_{1tx})\} \\
& - Am_{1x}\left\{(D_v\lambda)\left(\frac{1}{2\lambda}(\phi + \psi) - \frac{A}{\lambda^{3/2}}m_{1t}\right) + \lambda^{-1/2}(D_{m_1}\lambda)m_{1t}\right\} \\
& + \frac{3m_1}{v^3}m_{1x}\left(\frac{1}{2\lambda}(\phi + \psi) - \frac{A}{\lambda^{3/2}}m_{1t}\right) + \lambda^{-1/2}\left(b_{11} - \frac{1}{v^2}\right)(m_{1t}m_{1x} + m_1 m_{1tx}),
\end{aligned}
\tag{4.35}
$$

$$
\begin{aligned}
\psi' = {} & \frac{(D_v\lambda)}{2\lambda^{3/2}}\phi^2 + \frac{1}{2}\left[(D_v\lambda)\frac{1}{\lambda^3}Am_1' - (D_{m_1}\lambda)\frac{1}{\lambda}m_1'\right]\psi \\
& - \frac{1}{2}\left[(D_v\lambda)\frac{1}{\lambda}Am_{1x} - (D_v\lambda)\frac{1}{\lambda^2}\left(b_{11} - \frac{1}{v^2}\right)m_1 m_{1x}\right]\psi \\
& + \frac{1}{\lambda}\left\{(D_v\lambda)\frac{A}{\lambda}m_{1t} - (D_{m_1}\lambda)m_{1t}\right\}\psi \\
& - \left\{A(m_{1t} + \lambda m_{1x}) - \left(b_{11} - \frac{1}{v^2}\right)m_1 m_{1x}\right\}\frac{(D_v\lambda)}{2\lambda}(\phi + \psi) \\
& + (D_v A)\left(\frac{1}{2\lambda}(\phi + \psi) - \frac{A}{\lambda^{3/2}}m_{1t}\right)(m_{1t} + \lambda m_{1x}) \\
& + \lambda^{-1/2}\{(D_{m_1}A)m_{1t}(m_{1t} - \lambda m_{1x}) + A(m_{1tt} + \lambda m_{1tx})\} \\
& + Am_{1x}\left\{(D_v\lambda)\left(\frac{1}{2\lambda}(\phi + \psi) - \frac{A}{\lambda^{3/2}}m_{1t}\right) + \lambda^{-1/2}(D_{m_1}\lambda)m_{1t}\right\} \\
& - \frac{3m_1}{v^3}m_{1x}\left(\frac{1}{2\lambda}(\phi + \psi) - \frac{A}{\lambda^{3/2}}m_{1t}\right) - \lambda^{-1/2}\left(b_{11} - \frac{1}{v^2}\right)(m_{1t}m_{1x} + m_1 m_{1tx}).
\end{aligned}
\tag{4.36}
$$

The estimates of m_{1tt} and m_{1tx} can be obtained from (2.12), (4.24), and (4.25) as follows:

$$
\begin{aligned}
|m_{1tt}| &\le K_5 + K_6(|\phi| + |\psi|), \\
|m_{1tx}| &\le K_7(|\phi| + |\psi|) + K_8\int_0^t (|\phi| + |\psi|)d\tau,
\end{aligned}
$$

where the inequalities hold for almost all $t \in [0, T]$. In what follows, the inequalities involving $R(t), S(t), \Phi(t), \Psi(t), \Phi^{\pm}(t)$, and $\Psi^{\pm}(t)$ hold in the almost everywhere sense.

The rest of the proof is very similar to Dafermos [3]. We show it for the completeness. We introduce the following Lipschitz continuous functions:

$$R(t) = \max_{x\in R}|r(x,t)|, \qquad S(t) = \max_{x\in R}|s(x,t)|,$$
$$\Phi(t) = \max_{x\in R}|\phi(x,t)|, \qquad \Psi(t) = \max_{x\in R}|\psi(x,t)|,$$

and define

$$I(t) = \int_0^t \{\Phi(\tau) + \Psi(\tau)\}d\tau.$$

We choose $\delta \in (0, \overline{v})$ so that for some $\alpha > 0$

$$\frac{(D_v\lambda)}{\lambda} \geq 4\alpha, \qquad |v - \overline{v}| < \delta,$$

and we assume that there exists a positive T such that

$$|v(x,t) - \overline{v}| < \delta, \quad x \in R,\ 0 \leq t \leq T, \tag{4.37}$$

and

$$\int_0^T I(t)dt \leq 1, \qquad \int_0^T I^3(t)dt \leq 1. \tag{4.38}$$

We also assume that $T \leq 1$. Although we do not know the explicit value of T, there exists such a T because we assume that the solution is C^2. We will pick up such a T in the course of the proof.

For any $t \in (0, T]$, identify points $\hat{x}$ and $\check{x}$ such that

$$R(t) = |r(\hat{x},t)|, \qquad S(t) = |s(\check{x},t)|. \tag{4.39}$$

Since for any $\Delta t > 0$ we have

$$\begin{aligned} R(t-\Delta t) &\geq |r(\hat{x} + \Delta t\lambda(v(\hat{x},t), m_1(\hat{x},t)), t - \Delta t)|, \\ S(t-\Delta t) &\geq |r(\check{x} - \Delta t\lambda(v(\check{x},t), m_1(\check{x},t)), t - \Delta t)|, \end{aligned} \tag{4.40}$$

subtracting (4.40) from (4.39) and taking the limit as $\Delta t \downarrow 0$, we obtain

$$D^- R(t) \leq |r'(\hat{x},t)|, \qquad D^- S(t) \leq |s'(\hat{x},t)|.$$

Using this and (4.29), we see that

$$\frac{d}{dt}\{R(t) + S(t)\} \leq \Gamma_1 + \Gamma_1 I(t), \quad 0 \leq t \leq T.$$

Upon integration, we have

$$R(t) + S(t) \leq 2\varepsilon + \Gamma_1 t + \Gamma_1 \int_0^t I(\tau)d\tau, \quad 0 \leq t \leq T. \tag{4.41}$$

Next, we define the following functions:

$$\Phi^-(t) = \max_{x\in R}\{-\phi(x,t)\}, \qquad \Psi^-(t) = \max_{x\in R}\{-\psi(x,t)\},$$

and identify points $\hat{y}$ and $\check{y}$ at which

$$\Phi^-(t) = -\phi(\hat{y},t), \qquad \Psi^-(t) = -\psi(\check{y},t).$$

Going through the same procedures as $D^-R(t)$ and $D^-S(t)$, we obtain

$$D^-\Phi^-(t) \le -\phi'(\hat{y},t), \qquad D^-\Psi^-(t) \le -\psi'(\check{y},t).$$

Adding them, we have

$$\frac{d}{dt}\{\Phi^-(t)+\Psi^-(t)\} \le \Gamma_1 + \Gamma_1 I(t) + \Gamma_1\{\Phi(t)+\Psi(t)\} + \Gamma_1 I(t)\{\Phi(t)+\Psi(t)\},$$

from which we obtain

$$\Phi^-(t)+\Psi^-(t) \le 2\lambda^{1/2}(\overline{v},1)N + \Gamma_1 t + \Gamma_1\int_0^t I(\tau)d\tau + \Gamma_1 I(t) + \Gamma_1 I^2(t).$$

Now, we define

$$\Phi^+(t) = \max_{x\in R}\phi(x,t), \qquad \Psi^+(t) = \max_{x\in R}\psi(x,t),$$

and identify points $\hat{z}$ and $\check{z}$ at which

$$\Phi^+(t) = \phi(\hat{z},t), \qquad \Psi^+(t) = \psi(\check{z},t).$$

Since for any $\Delta t \in (0, T-t)$,

$$\begin{aligned}
\Phi^+(t+\Delta t) &\ge \phi(\hat{z} - \Delta t\lambda(v(\hat{z},t), m_1(\hat{z},t)), t+\Delta t),\\
\Psi^+(t+\Delta t) &\ge \psi(\check{z} + \Delta t\lambda(v(\check{z},t), m_1(\check{z},t)), t+\Delta t),
\end{aligned}$$

we see that

$$D^+\Phi^+(t) \ge \phi'(\hat{z},t), \qquad D^+\Psi^+(t) \ge \psi'(\hat{z},t).$$

Using

$$\begin{aligned}
\Phi(t)+\Psi(t) &\ge \Phi^+(t)+\Psi^+(t),\\
\Phi(t)+\Psi(t) &\le \{\Phi^+(t)+\Psi^+(t)\} + \{\Phi^-(t)+\Psi^-(t)\},
\end{aligned}$$

we obtain

$$\frac{d}{dt}\{\Phi^+(t)+\Psi^+(t)\} \ge \alpha\{\Phi^+(t)+\Psi^+(t)\}^2 - \beta(t)\{\Phi^+(t)+\Psi^+(t)\} - \gamma(t), \tag{4.42}$$

where

$$\begin{aligned}
\beta(t) &= \Gamma_1 + \Gamma_1 I(t),\\
\gamma(t) &= \Gamma_2 + \Gamma_2 I^3(t).
\end{aligned}$$

Now define

$$X(t) = \{\Phi^+(t)+\Psi^+(t)\}\exp\left\{-\int_t^T \beta(\tau)d\tau\right\} - \int_t^T \gamma(\tau)d\tau.$$

Then, from (4.42) we see that

$$\frac{dX}{dt} \ge \alpha\left\{X(t) + \int_t^T \gamma(\tau)d\tau\right\}^2 \ge \alpha X^2(t) \tag{4.43}$$

and, integrating this from T backward in time, we have

$$X(t) \le \frac{X(T)}{1+\alpha X(T)(T-t)} \le \frac{1}{\alpha(T-t)}.$$

Therefore,

$$\Phi^+(t) + \Psi^+(t) \le \left\{ \frac{1}{\alpha(T-t)} + \int_t^T \gamma(\tau)d\tau \right\} \exp\left\{ \int_t^T \beta(\tau)d\tau \right\}$$
$$\le \left\{ \frac{1}{\alpha(T-t)} + \int_0^T \gamma(\tau)d\tau \right\} \exp\left\{ \int_0^T \beta(\tau)d\tau \right\}.$$

From (4.38) we have

$$\int_0^T \beta(\tau)d\tau \le \Gamma_1, \qquad \int_0^T \gamma(\tau)d\tau \le \Gamma_2.$$

Then, we see that

$$\int_0^t \{\Phi^+(\tau) + \Psi^+(\tau)\}d\tau \le -\Gamma_1 \ln(T-t) + \Gamma_2,$$
$$\int_0^t \{\Phi^-(\tau) + \Psi^-(\tau)\}d\tau \le \Gamma_2.$$

Adding the above inequalities, we observe

$$I(t) \le -\Gamma_1 \ln(T-t) + \Gamma_2.$$

Integrating this, we have

$$\int_0^T I(t)dt \le (-\Gamma_1 \ln T + \Gamma_2)T,$$
$$\int_0^T I^3(t)dt \le (-\Gamma_1 \ln^3 T + \Gamma_2)T,$$
$$\left| \int_{\overline{v}}^{v(x,t)} \lambda(\xi, m_1)d\xi \right| \le \varepsilon + \Gamma_1 T + (-\Gamma_1 \ln T + \Gamma_2)T.$$

The above relations and (4.41) show that if we choose ε and T so that

$$\varepsilon + \Gamma_1 T + (-\Gamma_1 \ln T + \Gamma_2)T \le \delta\lambda(\overline{v} - \delta, 0),$$
$$(-\Gamma_1 \ln T + \Gamma_2)T \le 1,$$
$$(-\Gamma_1 \ln^3 T + \Gamma_2)T \le 1,$$

then (4.37) and (4.38) are satisfied.

Now we show the contradiction. From (3.23), (4.31), and (3.18), we see that

$$\Phi^+(0) + \Psi^+(0) \ge 2\lambda^{1/2}(\overline{v}, 1)M.$$

Therefore, choosing a sufficiently large M, we have

$$X(0) \ge \frac{2}{\alpha T}$$

and from (4.43) we also have

$$X(t) \ge \frac{X(0)}{1 - \alpha X(0)t}, \quad 0 \le t \le T.$$

This shows that $X(t)$ will be infinite at t less than T. This completes the proof. □

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