SUFFICIENT CONDITION FOR THE EXISTENCE
OF SOLUTIONS OF A FREE BOUNDARY PROBLEM

By

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Abstract. In this paper we study the existence of critical points of a functional depending on $\Omega \subset \mathbb{R}^2$ through its perimeter and the solution of the Dirichlet problem in Ω, under the constraint that the measure of Ω is given. We give a sufficient condition for the existence of critical points using the implicit function theorem.

1. Introduction. In this paper we present a sufficient condition for the existence of regular critical points of some energy functional associated with an interior two-dimensional electromagnetic casting problem. This problem has been studied from different points of view by some authors, for instance [1], [3], [4], [6]. However, the existence of critical points of the energy functional we consider (which depends on the solution of a Dirichlet problem and the perimeter) is still an open problem.

Let $p > 2$, $j \in \mathbb{L}^p(\mathbb{R}^2)$. For any $\Omega \subset \mathbb{R}^2$, an open bounded subset, let $m(\Omega)$ be its Lebesgue measure and u_Ω the variational solution in $H_0^1(\Omega)$ of the following interior Dirichlet problem:

$$\begin{cases} -\Delta u_\Omega & = & j & \text{in} & \Omega, \\ u_\Omega & = & 0 & \text{on} & \partial\Omega. \end{cases} \tag{1}$$

Let m_0, μ be two prescribed positive numbers. We consider the problem of finding a C^2 *bounded open subset* Ω^* of $\mathbb{R}^2$ and $\Lambda^* \in \mathbb{R}$ such that

$$\begin{cases} -\dfrac{1}{2}\left(\dfrac{\partial u_{\Omega^*}}{\partial \nu^*}\right)^2 + \mu\mathcal{H}^* + \Lambda^* = 0 \ \text{ on } \ \partial\Omega^*, \\ m(\Omega^*) - m_0 = 0, \end{cases} \tag{2}$$

where $\mathcal{H}^*$ is the curvature of $\partial\Omega^*$ and ν^* is the unit exterior normal vector to $\partial\Omega^*$.

Equations (2) are necessary conditions for a regular set Ω^* to be a solution of the following shape optimization problem (see, for example, [4]):

$$E(\Omega^*) = \min\{E(\Omega), \; \Omega \in \mathcal{O}\}, \tag{3}$$

where $\mathcal{O} = \{\Omega \subset \mathbb{R}^2, \; C^2 \text{ bounded open set and } m(\Omega) = m_0\}$,

$$e(\Omega) = -\frac{1}{2} \int_\Omega |Du_\Omega(y)|^2 dy, \qquad P(\Omega) = \int_{\partial\Omega} d\partial\Omega, \tag{4}$$

with $D = (\partial_x, \partial_y)$, and $E(\Omega) = e(\Omega) + P(\Omega)$. Recall that $e(\Omega)$ is nothing but the "energy" of the Dirichlet problem in Ω.

The regular subset Ω^* that obeys (2) is called *a critical point* of E under the constraint $m(\Omega) = m_0$.

In Theorem 1.1 we give a sufficient condition for the existence of critical points of E for μ large enough. The idea for such a condition is suggested by the following consideration. Let us assume :

(i) for any $\mu \gg 1$, there exists Ω_μ^* solution of (3);

(ii) there exists $\Omega^* \in \mathcal{O}$ such that

$e(\Omega_\mu^*) \xrightarrow[\mu\to\infty]{} e(\Omega^*)$ with $e(\Omega^*)$ finite and $P(\Omega_\mu^*) \xrightarrow[\mu\to\infty]{} P(\Omega^*)$.

Since $m(\Omega_\mu^*) = m_0$, it follows by the isoperimetric inequality that $P(\Omega_\mu^*) = 2\pi r_0 + \epsilon_\mu$ with $\epsilon_\mu \geq 0$ and $r_0 = (m_0/\pi)^{1/2}$. From (i) we have $E(\Omega_\mu^*) \leq E(B(x, r_0))$, for any $x \in \mathbb{R}^2$, where $B(x, r)$ is the ball with radius r centered at x. Thus

$$e(\Omega_\mu^*) + \mu\epsilon_\mu \leq e(B(x, r_0)), \quad \forall x \in \mathbb{R}^2. \tag{5}$$

From (ii) we deduce that $\lim\limits_{\mu\to+\infty} \epsilon_\mu = 0$ and therefore $P(\Omega^*) = 2\pi r_0$ and $m(\Omega^*) = \pi r_0^2$; this implies that $\Omega^* = B(x_0, r_0)$ for some $x_0 \in \mathbb{R}^2$. Taking $x = x_0$ in (5) and letting μ tend to infinity, we get $\lim\limits_{\mu\to\infty} \mu\epsilon_\mu = 0$. Then, from (5) we have (letting μ tend to infinity): $e(B(x_0, r_0)) \leq e(B(x, r_0))$, for $x \in \mathbb{R}^2$, which says that x_0 is a minimum of the function

$$h(x) := -\frac{1}{2} \int_{B(x, r_0)} |Du_{B(x, r_0)}(y)|^2 dy.$$

Now, since we are interested in the solutions of (2), we shall assume that there exists $x_0 \in \mathbb{R}^2$ such that

$$Dh(x_0) = (0, 0), \quad D^2 h(x_0) \text{ has nonzero eigenvalues}, \tag{6}$$

where D^2 is the matrix operator $D^t \cdot D$.

Thus, we think of the solutions Ω^* of (2) as perturbations of the domain $\Omega_0 := B(x_0, r_0)$ (letting Γ_0 denote the boundary of Ω_0). Since $\mu \gg 1$, we set $\mu = \sigma^{-2}$ and introduce

$$T : \begin{cases} \mathbb{R} \times C^2(\Gamma_0) & \longrightarrow \quad C^2(\Gamma_0; \mathbb{R}^2), \\ (\sigma, \psi) & \longmapsto \quad I + U(\sigma, \psi) := I + t(\psi) + \sigma\psi\nu_0, \end{cases}$$

where I is the identity on $\mathbb{R}^2$, ν_0 is the unit exterior normal vector to Γ_0 and $t(\psi) = \fint_{\Gamma_0} \psi \cdot \nu_0 \, ds_0 := \frac{1}{|\Gamma_0|} \int_{\Gamma_0} \psi \cdot \nu_0 \, ds_0$, with s_0 the arclength of Γ_0. We let $\Gamma(\sigma, \psi) = \{T(\sigma, \psi)(x), x \in \Gamma_0\}$, where $\Omega(\sigma, \psi)$ denotes the interior domain defined by $\Gamma(\sigma, \psi)$, $\nu(\sigma, \psi)$ denotes

the unit exterior normal vector to $\Gamma(\sigma, \psi)$, $\mathcal{H}(\sigma, \psi)$ denotes the curvature of $\Gamma(\sigma, \psi)$, $\mathcal{H}_0$ denotes the curvature of Γ_0, $\mathbf{u}(\sigma, \psi) = u_{\Omega(\sigma,\psi)}$ and $\mathbf{m}(\sigma, \psi) = m(\Omega(\sigma, \psi))$.

The main result we prove is:

THEOREM 1.1. Under the hypothesis (6) there exist $\sigma^* > 0$ and a unique C^1 map $\Sigma :$ $(-\sigma^*, \sigma^*) \longmapsto C^2(\Gamma_0) \times \mathbb{R}$ such that for any $\sigma \in (-\sigma^*, \sigma^*) \backslash \{0\}$ if $\Sigma(\sigma) = (\psi, \Lambda)$ we have

$$-\frac{1}{2}\left(\frac{\partial \mathbf{u}(\sigma, \psi)}{\partial \nu(\sigma, \psi)}\right)^2 + \frac{1}{\sigma^2}\mathcal{H}(\sigma, \psi) + \left(\frac{\Lambda}{\sigma} - \frac{\mathcal{H}_0}{\sigma^2}\right) = 0 \text{ on } \Gamma(\sigma, \psi), \tag{7}$$

$$\mathbf{m}(\sigma, \psi) - m_0 = 0. \tag{8}$$

REMARK 1.2. The main tool we use to prove existence of critical points of E is the implicit function theorem applied near $(0, 0, 0)$ to some function F of (σ, ψ, Λ) (see (17), (18)), which involves the left-hand side of (7).

The hypothesis (6) is not explicit. However, in the case of radial functions, this hypothesis is equivalent to another explicit condition.

PROPOSITION 1.3. Assume that $j \in \mathbb{L}^p(\mathbb{R}^2)$ is radial and $u_{\Omega_0} \in W^{3,1}(\Omega_0)$. Then for $x_0 = (0, 0)$, the condition (6) is equivalent to

$$\left(j(r_0) - \frac{2}{r_0^2}\int_0^{r_0} j(r)r\,dr\right)\int_0^{r_0} j(r)r\,dr \neq 0.$$

Thus, in particular, for any radial function j with $\mathrm{supp}(j) \subset B(0, r_0)$ and integral nonzero (with respect to the measure $r\,dr$) the hypotheses (6) are always fulfilled.

Proof. Using classical results concerning the differentiability with respect to the shape of an integral which depend on the solution u_Ω (see, for example, [7], [8]) we can prove that

$$Dh(0) = -\frac{1}{2}\int_{\Gamma_0}\left(\frac{\partial u_{\Omega_0}}{\partial \nu_0}\right)^2 \nu_0 \, ds_0,$$

$$D_{i,j}^2 h(0) = -\int_{\Gamma_0}\frac{\partial u_{\Omega_0}}{\partial \nu_0}\frac{\partial u'_{\Omega_0}(\nu_0^j)}{\partial \nu_0}\nu_0^i + \left(\frac{\partial u_{\Omega_0}}{\partial \nu_0}\right)^2\frac{2\nu_0^i\nu_0^j - \delta_{ij}}{2r_0} + \frac{\partial u_{\Omega_0}}{\partial \nu_0}\frac{\partial^2 u_{\Omega_0}}{\partial \nu_0^2}\nu_0^i\nu_0^j,$$

where $u'_{\Omega_0}(\nu_0^j)$ is the shape derivative of u_Ω at Ω_0 in the direction ν_0^j, given by

$$\begin{cases} -\Delta u'_{\Omega_0}(\nu_0^j) &= \quad 0 \quad \text{in} \quad \Omega_0, \\ u'_{\Omega_0}(\nu_0^j) &= -\nu_0^j\dfrac{\partial u_{\Omega_0}}{\partial \nu_0} \quad \text{on} \quad \Gamma_0, \end{cases}$$

with $i, j = 1, 2$ and $\nu_0 = (\nu_0^1, \nu_0^2)$. We remark that, with the hypothesis of the proposition, the condition of the proposition, $Dh(0)$ and $D^2h(0, 0)$ are well defined.

It is easy to see that when j is a radial function we have that

$$u_{\Omega_0}(y) = -\int_0^{|y|}\frac{1}{s}\int_0^s j(r)r\,dr + \int_0^{r_0}\frac{1}{s}\int_0^s j(r)r\,dr, \quad u'_{\Omega_0}(\nu_0^i)(y) = -\frac{y_i}{r_0}\frac{\partial u_{\Omega_0}}{\partial \nu_0}.$$

It is clear that for $|y| = r_0$ we also have

$$\frac{\partial u_{\Omega_0}}{\partial \nu_0}(y) = -\frac{1}{r_0}\int_0^{r_0} j(r)r\,dr, \qquad \frac{\partial^2 u_{\Omega_0}}{\partial \nu_0^2}(y) = \frac{1}{r_0^2}\int_0^{r_0} j(r)r\,dr - j(r_0).$$

Then we obtain these expressions for $Dh(0)$ and $D^2h(0)$:

$$Dh(0) = (0,0),$$

$$D^2h(0) = \pi \left[\left(\frac{\partial u_{\Omega_0}}{\partial \nu_0} \right)^2 - r_0 \frac{\partial u_{\Omega_0}}{\partial \nu_0} \frac{\partial^2 u_{\Omega_0}}{\partial \nu_0^2} \right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \pi \left[\frac{2}{r_0^2} \left(\int_0^{r_0} j(r) r \, dr \right)^2 - j(r_0) \int_0^{r_0} j(r) r \, dr \right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which prove the first part of the proposition. The second part follows immediately because when $\sup(j) \subset B(0, r_0)$, we have $j(r_0) = 0$. $\qquad \square$

REMARK 1.4. When the conditions of Proposition 1.3 are not satisfied, typically when $D^2h(0,0)$ is not invertible, the result of Theorem 1.1 fails. However, this does not mean that there are not optimal shapes but, probably, there is no uniqueness.

For example, let us consider $j(y) = 1$ in $D_\epsilon := \{|y| \leq r_0 + \epsilon\}$. The condition of Proposition 1.3 fails ($D^2h(0)$ is not invertible). However, the disk $B(0, r_0)$, which obviously is an optimal shape, is not unique. Indeed, for any $x \in D_\epsilon$ we have that $u_{B(x,r_0)}$ is equal to $u_{B(0,r_0)}$, modulo one translation. Thus, the optimal shape $B(0, r_0)$ is not unique.

2. Preliminary results.

Let τ_0 (resp. $\tau(\sigma, \psi)$) denote the unit tangent vector to Γ_0 (resp. $\Gamma(\sigma, \psi)$) such that $\nu_0 \wedge \tau_0 = 1$ (resp. $\nu(\sigma, \psi) \wedge \tau(\sigma, \psi) = 1$) where $\wedge$ is the product operator given by

$$x \wedge y = x_1 y_2 - x_2 y_1, \text{ where } x = (x_1, x_2), \ y = (y_1, y_2).$$

Let $s(\sigma, \psi)$ be the arclength of $\Gamma(\sigma, \psi)$. We recall that $\Gamma(\sigma, \psi) = \{T(\sigma, \psi)(x), \ x \in \Gamma_0\}$. Then we have

$$\frac{d}{ds_0} [s(\sigma, \psi) \circ T(\sigma, \psi)] = \left| \frac{d}{ds_0} T(\sigma, \psi) \right| = \left| \tau_0 + \sigma \frac{d(\psi \nu_0)}{ds_0} \right|$$

$$= \left(1 + 2\sigma \mathcal{H}_0 \psi + \sigma^2 \left[\mathcal{H}_0^2 \psi^2 + \left(\frac{d\psi}{ds_0} \right)^2 \right] \right)^{1/2}, \qquad (9)$$

where $d(\cdot)/ds_0$ is the derivative of $(\cdot)$ with respect to the arclength of Γ_0.

In order to simplify the notation when there is no confusion, we do not specify the dependence on (σ, ψ). Thus, we will write just Γ, Ω, ν, τ, $\mathcal{H}$, $\mathbf{u}$, $\mathbf{m}$, T, U and s instead of $\Gamma(\sigma, \psi)$, $\Omega(\sigma, \psi)$, $\nu(\sigma, \psi)$, $\tau(\sigma, \psi)$, $\mathcal{H}(\sigma, \psi)$, $\mathbf{u}(\sigma, \psi)$, $\mathbf{m}(\sigma, \psi)$, $T(\sigma, \psi)$, $U(\sigma, \psi)$ and $s(\sigma, \psi)$.

LEMMA 2.1. The application $(\sigma, \psi) \longmapsto (\nu \circ T, \mathcal{H} \circ T)$ from $\mathbb{R} \times C^2(\Gamma_0)$ into $C^0(\Gamma_0; \mathbb{R}^2) \times C^0(\Gamma_0)$ is of class C^1 near $(0,0)$.

Moreover, the map $(\sigma, \psi) \longmapsto \dfrac{\mathcal{H}(\sigma, \psi) \circ T(\sigma, \psi) - \mathcal{H}_0}{\sigma}$ from $\mathbb{R} \times C^2(\Gamma_0)$ into $C^0(\Gamma_0)$ is of class C^1 near $(0,0)$ and

$$D_\psi \frac{\mathcal{H}(\sigma, \psi) \circ T(\sigma, \psi) - \mathcal{H}_0}{\sigma}(0,0)(\phi) = -\frac{d^2\phi}{ds_0^2} - \mathcal{H}_0^2 \phi. \qquad (10)$$

Proof. To prove the C^1 regularity of $\nu \circ T$, it suffices to prove the C^1 regularity of $\tau \circ T$, because $\nu \circ T$ is obtained by a rotation of $\tau \circ T$. It is clear that the vector dT/ds_0 is tangent to Γ. Then

$$
\begin{aligned}
\tau \circ T &= \frac{dT}{ds_0} \left| \frac{dT}{ds_0} \right|^{-1} = \frac{d(I + t(\psi) + \sigma\psi\nu_0)}{ds_0} \left| \frac{d(I + t(\psi) + \sigma\psi\nu_0)}{ds_0} \right|^{-1} \\
&= \left(\tau_0 + \sigma\frac{d(\psi\nu_0)}{ds_0} \right) \left(\frac{d(s \circ T)}{ds_0} \right)^{-1}.
\end{aligned}
\tag{11}
$$

The C^1 regularity of $\tau \circ T$ near $(0,0)$ is clear, because $\tau_0 + \sigma d(\psi\nu_0)/ds_0$ is of class C^1 from $\mathbb{R} \times C^2(\Gamma_0)$ into $C^0(\Gamma_0)$. From (9) one can see that $\left(\frac{d(s \circ T)}{ds_0} \right)^{-1}$ is of class C^1 in a small neighbourhood of $(0,0)$ in $\mathbb{R} \times C^2(\Gamma_0)$.

On the other hand, the curvature of a parametric curve is given by

$$
\mathcal{H} \circ T = \frac{dT}{ds_0} \wedge \frac{d^2T}{ds_0^2} \left(\frac{d(s \circ T)}{ds_0} \right)^{-3}.
\tag{12}
$$

This formula shows the C^1 regularity of $\mathcal{H} \circ T$ near $(0,0)$, because dT/ds_0 and d^2T/ds_0^2 are of class C^1 from $\mathbb{R} \times C^2(\Gamma_0)$ into $C^0(\Gamma_0)$ and, thanks to (9), $\left(\frac{d(s \circ T)}{ds_0} \right)^{-3}$ is of class C^1 in a small neighbourhood of $(0,0)$ in $\mathbb{R} \times C^2(\Gamma_0)$. This proves the first statement of the lemma.

On the other hand, using (12) we can write

$$
\begin{aligned}
\frac{\mathcal{H}(\sigma,\psi) \circ T(\sigma,\psi) - \mathcal{H}_0}{\sigma} &= \frac{1}{\sigma} \left(\frac{dT}{ds_0} \wedge \frac{d^2T}{ds_0^2} - \mathcal{H}_0 \right) \left(\frac{d(s \circ T)}{ds_0} \right)^{-3} \\
&\quad + \frac{\mathcal{H}_0}{\sigma} \left(\left(\frac{d(s \circ T)}{ds_0} \right)^{-3} - 1 \right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\sigma} \left(\frac{dT}{ds_0} \wedge \frac{d^2T}{ds_0^2} - \mathcal{H}_0 \right) &= \mathcal{H}_0^2 \psi(2 + \sigma\mathcal{H}_0\psi) \\
&\quad + 2\sigma\mathcal{H}_0 \left(\frac{d\psi}{ds_0} \right)^2 - (1 + \sigma\mathcal{H}_0\psi)\frac{d^2\psi}{ds_0^2}.
\end{aligned}
$$

Then, the second statement of the lemma follows easily using (9). $\quad\square$

For $(\sigma,\psi) \in \mathbb{R} \times C^2(\Gamma_0)$, let $\hat{U}(\sigma,\psi)$ be a C^2 extension with compact support of $U(\sigma,\psi)$ given, for example, by Lemma 6.38 in [2], and $\hat{T}(\sigma,\psi) = I + \hat{U}(\sigma,\psi)$. Let $D\hat{T}(\sigma,\psi)$ (resp. $[D\hat{T}(\sigma,\psi)]^{-1}$) denote the Jacobian matrix of $\hat{T}(\sigma,\psi)$ (resp. the inverse matrix of $D\hat{T}(\sigma,\psi)$). From [8] we have that

$$(\sigma,\psi) \longmapsto (D\hat{T}(\sigma,\psi), [D\hat{T}(\sigma,\psi)]^{-1})$$

from $\mathbb{R} \times C^2(\Gamma_0)$ into $C^0(\mathbb{R}^2; \mathbb{R}^4) \times C^0(\mathbb{R}^2; \mathbb{R}^4)$ is of class C^1 near $(0,0)$.

From [8], essentially using the implicit function theorem, we also have

LEMMA 2.2. *The function* $(\sigma,\psi) \longmapsto \mathbf{u} \circ \hat{T}$ *from* $\mathbb{R} \times C^2(\Gamma_0)$ *into* $W^{2,p}(\Omega_0)$ *is of class* C^1 *near* $(0,0)$.

COROLLARY 2.3. For $p > 2$ the function $(\sigma, \psi) \longmapsto \dfrac{\partial \mathbf{u}}{\partial \nu} \circ T$ from $\mathbb{R} \times C^2(\Gamma_0)$ into $C^0(\Gamma_0)$ is of class C^1 near $(0,0)$.

Proof. For $p > 2$ the inclusion $W^{2,p}(\Omega_0) \subset C^1(\Omega_0)$ is continuous. It follows from Lemma 2.2 that $(\sigma, \psi) \longmapsto \mathbf{u} \circ \hat{T}$ is of class C^1 near $(0,0)$, from $\mathbb{R} \times C^2(\Gamma_0)$ into $C^1(\Omega_0)$. Thus, $(\sigma, \psi) \longmapsto D(\mathbf{u} \circ \hat{T})$ is of class C^1 near $(0,0)$ from $\mathbb{R} \times C^2(\Gamma_0)$ into $(C^0(\Omega_0))^3$. But

$$\frac{\partial \mathbf{u}}{\partial \nu} \circ T = \nu \circ T \cdot [\hat{T}]^{-1} \cdot D(\mathbf{u} \circ \hat{T}) \ \text{ on } \Gamma_0.$$

This equality and the continuity of the trace operator from $C^0(\Omega_0)$ into $C^0(\Gamma_0)$ prove the corollary. $\qquad\square$

Set $\mathbf{e}(\sigma, \psi) = e(\Omega(\sigma, \psi))$. We give the next lemma without proof, but we note that it follows from the differentiability of $(\sigma, \psi) \longmapsto \hat{T}$ from $\mathbb{R} \times C^2(\Gamma_0)$ into $C^2(\Gamma_0; \mathbb{R}^2)$ and Lemma 2.2.

LEMMA 2.4. The application $(\sigma, \psi) \in \mathbb{R} \times C^2(\Gamma_0) \longmapsto \mathbf{e}(\sigma, \psi) \in \mathbb{R}$ is of class C^1 near $(0,0)$. Moreover, $D_\psi \mathbf{e}(\sigma, \psi)(\varphi)$, the derivative of $\mathbf{e}$ with respect to ψ at (σ, ψ) in the direction φ, is given by

$$D_\psi \mathbf{e}(\sigma, \psi)(\varphi) = -\frac{1}{2} \int_\Gamma \left(\frac{\partial \mathbf{u}}{\partial \nu} \right)^2 (U(\sigma, \varphi) \circ T^{-1} \cdot \nu) \, ds. \tag{13}$$

LEMMA 2.5. The application $(\sigma, \psi) \in \mathbb{R} \times C^2(\Gamma_0) \longmapsto \mathbf{e}(\sigma, \psi)(\varphi) \in \mathbb{R}$ is of class C^2 near $(0,0)$. Moreover, $D_\psi^2 \mathbf{e}(0,0)(\varphi, \phi)$, the second derivative of $\mathbf{e}$ with respect to ψ at $(0,0)$ in the direction (φ, ϕ), is given by

$$D_\psi^2 \mathbf{e}(0,0)(\varphi, \phi) = -\frac{1}{2} \int_{\Gamma_0} (t(\varphi) \cdot \nu_0) D_\psi \left[\left(\frac{\partial \mathbf{u}}{\partial \nu} \right)^2 \circ T \right] (0,0)(\phi) \, ds_0. \tag{14}$$

Proof. We change the variable in the expression of $D_\psi \mathbf{e}(\sigma, \psi)(\varphi)$ in order to have an integral on Γ_0:

$$D_\psi \mathbf{e}(\sigma, \psi)(\varphi) = -\frac{1}{2} \int_{\Gamma_0} \left(\frac{\partial \mathbf{u}}{\partial \nu} \right)^2 \circ T \ (U(\sigma, \varphi) \cdot \nu \circ T) \frac{d(s \circ T)}{ds_0} ds_0.$$

But $U(\sigma, \varphi) = t(\varphi) + \sigma \varphi \nu_0$ implies

$$
\begin{aligned}
D_\psi \mathbf{e}(\sigma, \psi)(\varphi) \ = \ & -\frac{\sigma}{2} \int_{\Gamma_0} \left(\frac{\partial \mathbf{u}}{\partial \nu} \right)^2 \circ T \ (\varphi \nu_0 \cdot \nu \circ T) \frac{d(s \circ T)}{ds_0} ds_0 \\
& -\frac{1}{2} \int_{\Gamma_0} (t(\varphi) \cdot \nu \circ T) \left[\left(\frac{\partial \mathbf{u}}{\partial \nu} \right)^2 \circ T \right] \frac{d(s \circ T)}{ds_0} ds_0.
\end{aligned} \tag{15}
$$

The existence of a second-order differential of $\mathbf{e}$ near $(0,0)$ follows from Lemma 2.1 and Corollary 2.3. On the other hand, $\nu \circ T$ is obtained by a rotation of $\tau \circ T$. Then from (11):

$$D_\psi \left(\tau \circ T \frac{d(s \circ T)}{ds_0} \right)(0,0)(\phi) = D_\psi \left(\tau_0 + \sigma \frac{d(\psi \nu_0)}{ds_0} \right)(0,0)(\phi) = 0.$$

Consequently, $D_\psi (\nu \circ T \dfrac{ds \circ T}{ds_0})(0,0) = 0$ and, by differentiating (15), we get the expression (14). $\qquad\square$

REMARK 2.6. It is clear that $T(0, \psi)(\Omega_0) = (I + t(\psi))(\Omega_0) = B(x_0 + t(\psi), r_0)$. Thus, $\mathbf{e}(0, \psi) = h(x_0 + t(\psi))$ and, consequently,

$$D_\psi^2 \mathbf{e}(0, 0)(\varphi, \phi) = t(\varphi) \cdot D^2 h(x_0) \cdot t(\phi). \tag{16}$$

Let F be the following function:

$$F : \mathbb{R} \times C^2(\Gamma_0) \times \mathbb{R} \longmapsto C^0(\Gamma_0) \times \mathbb{R},$$
$$(\sigma, \psi, \Lambda) \longmapsto \big(F_1(\sigma, \psi, \Lambda), F_2(\sigma, \psi, \Lambda) \big),$$

where F_1, F_2 are given by

$$\begin{aligned}
F_1(\sigma, \psi, \Lambda) &= -\frac{\nu_0}{2|\Gamma_0|} \cdot \int_\Gamma \left(\frac{\partial \mathbf{u}(\sigma, \psi)}{\partial \nu(\sigma, \psi)} \right)^2 \nu(\sigma, \psi) ds - \frac{\sigma}{2} \left(\frac{\partial \mathbf{u}(\sigma, \psi)}{\partial \nu(\sigma, \psi)} \right)^2 \circ T \\
&\quad + \frac{\mathcal{H}(\sigma, \psi) \circ T(\sigma, \psi) - \mathcal{H}_0}{\sigma} + \Lambda, \tag{17} \\
F_2(\sigma, \psi, \Lambda) &= (\mathbf{m}(\sigma, \psi) - \mathbf{m}(0, \psi))/\sigma. \tag{18}
\end{aligned}$$

REMARK 2.7. Note that $\sigma \cdot F_2$ is the left-hand side (l.h.s.) of (8) and that F_1/σ differs from the l.h.s. of (7) by $-\dfrac{\nu_0}{2|\Gamma_0|} \cdot \displaystyle\int_\Gamma \left(\frac{\partial \mathbf{u}(\sigma, \psi)}{\partial \nu(\sigma, \psi)} \right)^2 \nu(\sigma, \psi) ds$.

LEMMA 2.8. For $\psi \in C^2(\Gamma_0)$ and $\sigma \neq 0$ sufficiently small, the equation $F(\sigma, \psi, \Lambda) = 0$ is equivalent to (7) and (8).

Proof. From the above remark, to prove the lemma it suffices to show that $F_1(\sigma, \psi, \Lambda) = 0$ is equivalent to (7).

Assume that (7) is satisfied and $\nu = (\nu^1, \nu^2)$. For $\varphi \in C^2(\Gamma_0)$ it is easy to show that

$$\int_\Gamma \mathcal{H}(\sigma, \psi) \nu^i \, ds = 0 \quad \text{and} \quad \int_\Gamma \nu^i \, ds = 0, \quad i = 1, 2. \tag{19}$$

Note that the first equality above is obtained from the formula of integration by parts on the boundary Γ_0 (see, for instance, [2]).

Then, multiplying (7) by $t(\varphi) \cdot \nu$ and integrating on Γ, we get

$$0 = \int_\Gamma \left(\frac{\partial \mathbf{u}}{\partial \nu} \right)^2 t(\varphi) \cdot \nu \, ds = \fint_{\Gamma_0} ds_0 \; \varphi \nu_0 \cdot \int_\Gamma \left(\frac{\partial \mathbf{u}}{\partial \nu} \right)^2 \nu \, ds,$$

which shows that $\nu_0 \cdot \displaystyle\int_\Gamma \left(\frac{\partial \mathbf{u}}{\partial \nu} \right)^2 \nu ds = 0$. Thus, $F_1(\sigma, \psi, \Lambda) = 0$.

Now let us assume $F_1(\sigma, \psi, \Lambda) = 0$. Multiplying both sides by ν^i, $i = 1, 2$, integrating on Γ and using (19), we get

$$\int_\Gamma \left(\frac{\partial \mathbf{u}}{\partial \nu} \right)^2 (\nu \cdot t(\nu^i \circ T) + \sigma \nu^i) ds = 0, \quad i = 1, 2.$$

Thus, we have the system of equations:

$$\begin{pmatrix} t_1(\nu^1 \circ T) + \sigma, & t_2(\nu^1 \circ T) \\ t_1(\nu^2) \circ T, & t_2(\nu^2 \circ T) + \sigma \end{pmatrix} \cdot \left(\int_\Gamma \left(\frac{\partial \mathbf{u}}{\partial \nu} \right)^2 \nu \, ds \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{20}$$

But the matrix of the above system is invertible for $\psi \in C^2(\Gamma_0)$ sufficiently small, because for such ψ, $\nu \circ T$ is near $\nu_0 = (\nu_0^1, \nu_0^2)$, and thus the matrix of the above system is near

$$\begin{pmatrix} t_1(\nu_0^1) + \sigma, & t_2(\nu_0^1) \\ t_1(\nu_0^2), & t_2(\nu_0^2) + \sigma \end{pmatrix} = \begin{pmatrix} 1/2 + \sigma, & 0 \\ 0, & 1/2 + \sigma \end{pmatrix},$$

which is invertible. This proves that the unique solution of the system (20) is the trivial solution. Then, since $\sigma \neq 0$, dividing $F_1 = 0$ by σ we obtain (7), which proves the lemma. $\qquad\square$

Now, we are able to prove Theorem 1.1. The essential tool we use is the implicit function theorem applied to the function F at $(0,0,0)$. Thus, we need to prove that F is of class C^1 near $(0,0,0)$ and its derivative with respect to (ψ, Λ) defines an isomorphism thanks to the additional term mentioned in Remark 2.7.

3. **Proof of the main result.** We prove first the following lemma:

LEMMA 3.1. The function $F : \mathbb{R} \times C^2(\Gamma_0) \times \mathbb{R} \to C^0(\Gamma_0) \times \mathbb{R}$ is of class C^1 near $(0,0,0)$. Moreover, the derivative of F with respect to (ψ, Λ) at $(0,0,0)$ is given by

$$D_{\psi,\Lambda} F_1(0,0,0)(\phi, \lambda) = -\frac{d^2\phi}{ds_0^2} - \mathcal{H}_0^2 \phi + \lambda + \frac{1}{|\Gamma_0|} \nu_0 \cdot D^2 h(x_0) \cdot t(\phi), \qquad (21)$$

$$D_{\psi,\Lambda} F_2(0,0,0)(\phi, \lambda) = \int_{\Gamma_0} \phi \, ds_0. \qquad (22)$$

Proof. Let us first prove the C^1 regularity of F. For the term F_1 this can be proved using (17). Indeed, using Lemma 2.4, we have

$$F_1(\sigma, \psi, \Lambda) = \frac{\mathcal{H} \circ T - \mathcal{H}_0}{\sigma} + \Lambda + \frac{2}{|\Gamma_0|} \nu_0 \cdot D_\psi \mathbf{e}(0, \psi)(\nu_0)$$
$$- \frac{\sigma}{2} \left(\frac{\partial \mathbf{u}}{\partial \nu}\right)^2 \circ T, \qquad (23)$$

where $D_\psi \mathbf{e}(0, \psi)(\nu_0) = \left(D_\psi \mathbf{e}(0, \psi)(\nu_0^1), D_\psi \mathbf{e}(0, \psi)(\nu_0^2)\right)$. Then, the C^1 regularity of F_1 follows from Lemmas 2.1, 2.5 and Corollary 2.3.

The C^1 regularity of F_2 (even analytic regularity) follows from the fact that

$$\mathbf{m}(\sigma, \psi) = \frac{1}{2} \int_{\Gamma_0} (I + \sigma \psi \nu_0) \wedge \left(\tau_0 + \sigma \frac{d(\psi \nu_0)}{ds_0}\right) ds_0. \qquad (24)$$

Let us now prove (21) and (22). From the equality (23), Lemmas 2.1, 2.5 and Corollary 2.3,

$$D_{\psi,\Lambda} F_1(0,0,0)(\phi, \lambda) = -\frac{d^2\phi}{ds_0^2} - \mathcal{H}_0^2 \phi + \lambda + \frac{2}{|\Gamma_0|} \nu_0 \cdot D_\psi^2 \mathbf{e}(0,0)(\nu_0, \phi).$$

But from (16) we have

$$
\begin{aligned}
\nu_0 \cdot D^2 \psi \mathbf{e}(0,0)(\nu_0, \phi) &= \nu_0 \cdot \left(D_\psi^2 \mathbf{e}(0,0)(\nu_0^1, \phi), D_\psi^2 \mathbf{e}(0,0)(\nu_0^2, \phi) \right) \\
&= \nu_0 \cdot \left(t(\nu_0^1), t(\nu_0^2) \right) \cdot D^2 h(x_0) \cdot t(\phi) \\
&= \tfrac{1}{2} \nu_0 \cdot D^2 h(x_0) \cdot t(\phi),
\end{aligned}
$$

which proves (21).

For Eq. (22):

$$
\begin{aligned}
D_{\psi,\Lambda} F_2(0,0,0)(\phi, \lambda) &= \lim_{\sigma \to 0} \frac{D_{\psi,\Lambda} \mathbf{m}(\sigma, 0)(\phi, \lambda)}{\sigma} = \lim_{\sigma \to 0} \frac{D_\psi \mathbf{m}(\sigma, 0)(\phi)}{\sigma} \\
&= \lim_{\sigma \to 0} \frac{1}{\sigma} \int_{\Gamma_0} \sigma \phi \, ds_0 = \int_{\Gamma_0} \phi \, ds_0,
\end{aligned}
$$

which achieves the proof of the lemma. □

Proof of Theorem 1.1. As we mentioned in the introduction, the main tool of the proof is the implicit function theorem applied to the function F. We have

(i) $F(0,0,0) = 0$,

(ii) F *is a function of class* C^1 *from* $\mathbb{R} \times C^2(\Gamma_0) \times \mathbb{R}$ *near* $(0,0,0)$.

If we prove

(iii) $(\phi, \lambda) \in C^2(\Gamma_0) \times \mathbb{R} \to D_{\psi,\Lambda} F(0,0,0)(\phi, \lambda) \in C^0(\Gamma_0) \times \mathbb{R}$ *defines an isomorphism*,
then Theorem 1.1 is proved.

Indeed, if (i), (ii) and (iii) are satisfied, from the implicit function theorem, there exists a $\sigma^* > 0$ and a unique $(\boldsymbol{\psi}, \boldsymbol{\Lambda})$ with

$$
\boldsymbol{\psi} \in C^1((-\sigma^*, \sigma^*); C^2(\Gamma_0)), \quad \boldsymbol{\Lambda} \in C^1((-\sigma^*, \sigma^*); \mathbb{R}),
$$

such that for any $\sigma \in (-\sigma^*, \sigma^*)$ we have $F(\sigma, \boldsymbol{\psi}(\sigma), \boldsymbol{\Lambda}(\sigma)) = 0$. Then, for $\sigma \neq 0$, Lemma 2.8 shows that $\boldsymbol{\psi}(\sigma)$ and $\boldsymbol{\Lambda}(\sigma)$ satisfy Theorem 1.1.

Thus, it remains only to prove (iii).

(iii.1) *Injectivity of* $(\phi, \lambda) \to D_{\psi,\Lambda} F(0,0,0)(\phi, \lambda)$.

Let $(\phi, \lambda) \in C^2(\Gamma_0) \times \mathbb{R}$ with $D_{\psi,\Lambda} F(0,0,0)(\phi, \lambda) = 0$. Thus $\int_{\Gamma_0} \phi \, ds_0 = 0$. Integrating $D_{\psi,\Lambda} F_1(0,0,0)(\phi, \lambda)$ on Γ_0, we get $\lambda = 0$. Thus ϕ satisfies

$$
\frac{d^2 \phi}{ds_0^2} + \mathcal{H}_0^2 \phi = \frac{\nu_0}{|\Gamma_0|} \cdot D^2 h(x_0) \cdot t(\phi) \quad \text{on } \Gamma_0. \tag{25}
$$

The only solutions of the previous equation in $\mathbb{R}$ are of the form

$$
\phi(s_0) = C_1 \cos(\mathcal{H}_0 s_0) + C_2 \sin(\mathcal{H}_0 s_0) + \frac{s_0}{4\pi} \begin{pmatrix} +\sin(\mathcal{H}_0 s_0) \\ -\cos(\mathcal{H}_0 s_0) \end{pmatrix} \cdot D^2 h(x_0) \cdot t(\phi).
$$

Thus, any solution of (25) is of the previous form and it is $2\pi r_0$-periodic. This implies that $D^2 h(x_0) \cdot t(\phi) = 0$. Since the matrix $D^2 h(x_0)$ is invertible by virtue of the hypothesis (6), it follows that $t(\phi) = (0,0)$. But $t(\phi) = (C_1, C_2)/2$, which shows that $C_1 = C_2 = 0$. Thus $(\phi, \lambda) = (0,0)$, which proves (iii.1).

(iii.2) *Surjectivity of* $(\phi, \lambda) \to D_{\psi,\Lambda} F(0,0,0)(\phi, \lambda)$. Let

$$C_0^2(\Gamma_0) = \left\{ \psi \in C^2(\Gamma_0), \int_{\Gamma_0} \psi = 0 \right\}, \quad C_0^0(\Gamma_0) = \left\{ \psi \in C^0(\Gamma_0), \int_{\Gamma_0} \psi = 0 \right\}.$$

First, we prove the surjectivity of

$$\phi \in C_0^2(\Gamma_0) \to D_{\psi,\Lambda} F_1(0,0,0)(\phi, 0) \in C_0^0(\Gamma_0).$$

Indeed, let Δ^{-1} denote the operator $\Delta^{-1} : C_0^0(\Gamma_0) \to C_0^2(\Gamma_0)$, defined by $\Delta^{-1} f = \phi$, such that $\dfrac{d^2\phi}{ds_0^2} = f$, which is well defined. Then, multiplying by $-\Delta^{-1}$ the equation $D_{\psi,\Lambda} F_1(0,0,0)(\phi, 0) = f$, we get that it is equivalent to

$$\phi + \mathcal{H}_0^2 \Delta^{-1} \phi - \frac{1}{|\Gamma_0|} (\Delta^{-1}\nu_0) \cdot D^2 h(x_0) \cdot t(\phi) = -\Delta^{-1} f.$$

Let A denote the operator

$$A : C_0^2(\Gamma_0) \to C_0^2(\Gamma_0) \quad \text{with} \quad A\phi = \mathcal{H}_0^2 \Delta^{-1}\phi - \frac{1}{|\Gamma_0|}(\Delta^{-1}\nu_0) \cdot D^2 h(x_0) \cdot t(\phi).$$

It is clear that A is compact continuous, and from (iii.1) its eigenvalues are different from -1. This implies that $I + A$ is continuous invertible. Then, for any $f \in C_0^0(\Gamma_0)$, there exists $\phi \in C_0^2(\Gamma_0)$ a solution of $\phi + A\phi = -\Delta^{-1} f$, which is equivalent to $D_{\psi,\Lambda} F_1(0,0,0)(\phi, 0) = f$.

Now, let us prove (iii.2). For $(f, \mu) \in C^0(\Gamma_0) \times \mathbb{R}$ we can find $(\phi, \lambda) \in C^2(\Gamma_0) \times \mathbb{R}$ such that $D_{\psi,\Lambda} F(0,0,0)(\phi, \lambda) = (f, \mu)$. Indeed, if $f_0 = f - \fint_{\Gamma_0} f$, we have $f_0 \in C_0^0(\Gamma_0)$. Thus, there exists a function $\phi_0 \in C_0^2(\Gamma_0)$ a solution of $D_{\psi,\Lambda} F_1(0,0,0)(\phi_0, 0) = f_0$. If we note

$$\phi = \phi_0 + \frac{\mu}{|\Gamma_0|}, \qquad \lambda = \frac{\mathcal{H}_0^2}{|\Gamma_0|}\mu + \fint_{\Gamma_0} f_0 ds_0,$$

using the fact that $t(\phi) = t(\phi_0)$, we get $D_{\psi,\Lambda} F(0,0,0)(\phi, \lambda) = (f, \mu)$, which proves the assertion (iii.2).

(iii.3) *Continuity of* $(\phi, \lambda) \to D_{\psi,\Lambda} F(0,0,0)(\phi, \lambda) = (f, \mu)$. This follows by the next estimate:

$$\|D_{\psi,\Lambda} F_1(0,0,0)(\phi, \lambda)\|_{C^0(\Gamma_0)} + |D_{\psi,\Lambda} F_2(0,0,0)(\phi, \lambda)| \le C(\|\phi\|_{C^2(\Gamma_0)} + |\lambda|),$$

which is a consequence of (21) and (22). Thus (iii) is proved and the proof of the theorem is achieved. $\qquad \square$

REMARK 3.2. The basic space chosen in this paper is $C^2(\Gamma_0)$. Thus, if (ψ, Λ) is a solution of (7), (8), then $\Omega(\sigma, \psi)$ is a domain of class C^2.

In fact, in [5] it is proved that if $\Omega(\sigma, \psi)$ is of class C^2, then it is analytic.

REMARK 3.3. In this paper we have considered (2), the interior free boundary problem concerning the interior Dirichlet problem (1). Instead of (2), one can consider the following exterior free boundary problem:

$$\begin{cases} \dfrac{1}{2}\left(\dfrac{\partial u_{\Omega^*}}{\partial \nu^*}\right)^2 + \mu\mathcal{H}^* + \Lambda^* = 0 \quad \text{on} \quad \partial\Omega^*, \\ m(\Omega^*) - m_0 = 0, \end{cases} \tag{26}$$

u_{Ω^*} being the unique variational solution in $\overset{\circ}{W}{}_0^1(^e\Omega^*)$, the closure of $\mathcal{D}(^e\Omega^*)$ for the semi-norm $\|D\varphi\|_{\mathbb{L}^p(^e\Omega^*)}$, of the exterior Dirichlet problem:

$$
\begin{cases}
-\Delta u_{\Omega^*} = j & \text{in} \quad {}^e\Omega^* := \overline{\Omega^*}^{\,c}, \\
u_{\Omega^*} = 0 & \text{on} \quad \partial\Omega^*.
\end{cases}
\tag{27}
$$

Up to now, the sufficient conditions are not known for the existence of a (Ω^*, Λ^*), Ω^* regular, solution of (26).

We have found that if $j \in \mathbb{L}^p(\mathbb{R}^2)$, $p > 2$ is with compact support and

$$
h_e(x) = \frac{1}{2}\int_{^eB(x,r_0)} |Du_{B(x,r_0)}|^2 dx
$$

is such that

$$
Dh_e(0,0) = 0, \qquad D^2 h_e(0,0) \quad \text{invertible,} \tag{28}
$$

then we have an existence result for the problem (26) similar to Theorem 1.1. The proof of this result is extremely similar to the proof of Theorem 1.1; so we do not present it here. The only difference with respect to the interior free boundary problem (2) is that we need the differentiability with respect to the shape of the exterior Dirichlet problem (27) instead of (1). More precisely we need results, similar to Lemmas 2.2–2.5 and Corollary 2.3 for the problem (27). The proof of these differentiability results is more technical than the proof of Lemmas 2.2–2.5 and Corollary 2.3. We remark that these results for $p = 2$ are presented in [6].

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