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ON A MODIFIED SHOCK FRONT PROBLEM FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

By

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Abstract. We discuss the possibility of considering the shock wave in a compressible viscous heat conducting gas as a strong discontinuity on which surface the generalized Rankine-Hugoniot conditions hold. The corresponding linearized stability problem for a planar shock lacks boundary conditions; i.e., the shock wave in a viscous gas viewed as a (fictitious) strong discontinuity is like undercompressive shock waves in ideal fluids and, therefore, it is unstable against small perturbations. We propose such additional jump conditions so that the stability problem becomes well-posed and its trivial solution is asymptotically stable (by Lyapunov). The choice of additional boundary conditions is motivated by a priori information about steady-state solutions of the Navier-Stokes equations which can be calculated, for example, by the stabilization method. The established asymptotic stability of the trivial solution to the modified linearized shock front problem can allow us to justify, at least on the linearized level, the stabilization method that is often used, for example, for steady-state calculations for viscous blunt body flows.

1. Introduction. As is known, one uses two main approaches for describing motions of different continuous media with shock fronts. The first one is based on representing shock waves as surfaces of *strong discontinuity*. Such an approach is usually utilized for modelling shock waves in ideal fluids for which dissipative mechanisms (e.g., viscosity or heat conduction) can be neglected. According to the second, *viscous profile* (continuous) approach, the shock "spread" by dissipation is represented by traveling wave solutions of viscous conservation laws connecting asymptotically constant states (these solutions are called viscous profiles; see, e.g., [7], [14], [18]).

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Observe that motions of ideal continuous media are generally described by hyperbolic systems of conservation laws for which the mathematical stability theory for strong discontinuities has been well discovered both on the linearized and initial nonlinear level (at least for gas dynamics; see [6], [1], [2], [3], [4], [11], [12], [13]).

However, the discontinuous approach is found to be ineligible for shock waves in continuous media with dissipation when the shock is viewed as a (fictitious) strong discontinuity on which surface the *generalized Rankine-Hugoniot conditions* derived by the usual way (by analogy with ideal fluids; see, e.g., [10]) from the system of viscous conservation laws are satisfied. Such a conclusion follows even from the linear analysis. Namely, as was shown in [5], the planar shock (with the equation $x = 0$) separating a supersonic steady viscous flow (under $x < 0$) from a subsonic one (under $x > 0$) is unstable against small perturbations (depending not on the character of linearized boundary conditions at $x = 0$). This instability is a direct consequence of the fact that the corresponding linearized stability problem proves to be *undetermined* according to the number of boundary conditions obtained by linearizing the generalized Rankine-Hugoniot relations. In other words, this problem is ill-posed (by Hadamard).

At the same time, it should be noted that there are a lot of computational works in which the shock wave in a viscous gas is considered as a fictitious surface of strong discontinuity (see, e.g., [16], [15], [17] and references therein). As a rule, such works are devoted to the numerical computation of steady viscous flows near blunt bodies. For example, in [16], to bound essentially the calculated domain, where solutions of the compressible Navier-Stokes equations are sought, one introduces a bow shock that is treated as a strong discontinuity on which surface corresponding jump conditions (generalized Rankine-Hugoniot conditions) hold. Moreover, steady-state solutions to the Navier-Stokes equations are computed there by the stabilization method; i.e., they are found as a limit of unsteady solutions under $t \to \infty$.

Although, we should observe that in the mentioned works devoted to steady-state calculations for viscous blunt body flows, the supersonic coming flow is supposed to be inviscid and not heat-conducting. Therefore, as in gas dynamics (see, e.g., [13]), the linearized system (the acoustic system) ahead of the planar shock does not need boundary conditions (there are no outgoing characteristic modes under $x < 0$). But anyway, as follows from the results in [5], the linearized (nonstationary) Navier-Stokes system behind the planar shock lacks boundary conditions: one boundary condition for the case of one space dimension (1-D) and more than one boundary condition for 2-D or 3-D. In this connection, we underline once more that, for example, in [16], [15], [17], one considers the stationary Navier-Stokes (or simplified Navier-Stokes) equations, but their solutions are calculated there by the stabilization method. Hence, it is the *nonstationary* linearized shock front problem that should be correctly posed according to the number of boundary conditions.

Thus, one can conclude the groundlessness of the discontinuous approach applied for steady-state calculations for viscous blunt body flows if the stabilization method is used. On the other hand, accounting for some advantages of the discontinuous approach (especially for numerical calculations), it would be advantageous to modify this approach so that it might be applied (together with the stabilization method) with a mathematical ground for steady-state calculations for blunt body flows with dissipation. In this work, on the example of the linearized stability problem for (discontinuous) shock wave in a compressible viscous heat conducting gas, we propose an idea of such a modification. The essence of this idea is that for the initial shock front problem we write *additional boundary conditions* so that for the modified problem the steady flow regime with a shock wave described above becomes asymptotically stable (by Lyapunov). So, at least on the linearized level it might justify the stabilization method which can now be applied for finding (e.g., numerically) the steady flow regime for a viscous gas with a shock wave. The mentioned additional boundary conditions are suggested to be written with regard to a priori information about steady-state solutions of the Navier-Stokes equations.

2. Generalized Rankine-Hugoniot conditions. We consider the motion of a compressible viscous heat conducting gas. As is known, it is governed by the Navier-Stokes equations (see, e.g., [10]):

$$\begin{gathered}\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{u})=0\,,\quad \frac{\partial(\rho\mathbf{u})}{\partial t}+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}-\mathsf{P})=0\,,\\ \frac{\partial}{\partial t}\left(\rho\Big(e_0+\frac{1}{2}|\mathbf{u}|^2\Big)\right)+\operatorname{div}\left(\rho\Big(e_0+\frac{1}{2}|\mathbf{u}|^2+pV\Big)\mathbf{u}-\boldsymbol{\xi}-\varkappa\nabla T\right)=0\,.\end{gathered}\tag{2.1}$$

Here ρ denotes the density and $\mathbf{u}=(u_1,u_2,u_3)$ the velocity of the gas, P the stress tensor with the components $P_{ik}=-p\delta_{ik}+\sigma_{ik}$, $\sigma_{ik}=\eta(\partial u_i/\partial x_k+\partial u_k/\partial x_i-(2/3)\delta_{ik}\operatorname{div}\mathbf{u})+\zeta\,\delta_{ik}\operatorname{div}\mathbf{u}$, p the pressure, η and ζ are first and second viscosity coefficients, $\varkappa$ is the heat conductivity (η, ζ, and $\varkappa$ are usually assumed to be functions of ρ and s), s the entropy, e_0 the internal energy, $V=1/\rho$, $\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3)$, $\xi_i=\sum_{i=1}^{3}\sigma_{ik}u_k$, and T denotes the temperature. With regard to the state equation $e_0=e_0(\rho,s)$, (2.1) is a close system for finding components of the vector $(p,s,\mathbf{u})$ (it follows from the Gibbs relation $TdS=de_0+pdV$ that $T=(e_0)_s$, $p=\rho^2(e_0)_\rho$).

By the usual way (see, e.g., [10]) we write out for the viscous conservation laws (2.1) the following jump conditions (generalized Rankine-Hugoniot conditions):

$$\begin{gathered}[j]=0,\quad [u_{\mathrm{n}}]j+[\mathcal{P}]=0\,,\quad j[u_{\tau_{1,2}}]=\left[\sum_{i,k=1}^{3}\sigma_{ik}n_im_k^{1,2}\right],\\ \left[e_0+\frac{1}{2}|\mathbf{u}|^2\right]j+\left[pu_{\mathrm{n}}-\sum_{i,k=1}^{3}\sigma_{ik}n_iu_k-\varkappa\frac{\partial T}{\partial\mathbf{n}}\right]=0\,.\end{gathered}\tag{2.2}$$

Here the equation $f(t,x_2,x_3)-x_1=0$ represents a surface of propagating strong discontinuity, $[g]=g-g_\infty=g|_{f(t,x_2,x_3)-x_1\to-0}-g|_{f(t,x_2,x_3)-x_1\to+0}$ denotes the jump for every regularly discontinuous function g (here and below the subindex $_\infty$ stands for boundary values ahead of the shock front), $\mathbf{n}=(n_1,n_2,n_3)=(1+f_{x_2}^2+f_{x_3}^2)^{-1/2}(-1,f_{x_2},f_{x_3})$ the unit normal to the discontinuity front, $j=\rho(u_{\mathrm{n}}-D_{\mathrm{n}})$ the mass transfer flux across the discontinuity surface, $D_{\mathrm{n}}=-(1+f_{x_2}^2+f_{x_3}^2)^{-1/2}f_t$ the discontinuity speed in the normal direction, $\boldsymbol{\tau}_1=(m_1^1,m_2^1,m_3^1)=(f_{x_2},1,0)$, $\boldsymbol{\tau}_2=(m_1^2,m_2^2,m_3^2)=(f_{x_3},0,1)$, $u_{\mathrm{n}}=(\mathbf{u},\mathbf{n})$, $u_{\tau_i}=(\mathbf{u},\boldsymbol{\tau}_i)$, $i=1,2$, and $\mathcal{P}=p-\sum_{i,k=1}^{3}\sigma_{ik}n_in_k$. Note that for the case of shock wave,

$[\rho] \neq 0$, $[j] \neq 0$, one can reduce the last condition in (2.2) to the following form of a generalized Hugoniot adiabat:

$$[e_0] + \frac{\mathcal{P} + \mathcal{P}_\infty}{2}[V] = \frac{1}{2j^2}\left[\sum_{i=1}^{3}\Big(\sum_{k=1}^{3}\sigma_{ik}n_k\Big)^2 - \Big(\sum_{i,k=1}^{3}\sigma_{ik}n_in_k\Big)^2\right] + \frac{1}{j}\left[\varkappa\frac{\partial T}{\partial \mathbf{n}}\right].$$

3. The modified shock front problem. The linearized stability problem for a planar shock wave in a viscous gas was formulated in [5]. Let us, following [5], write out this problem here. For this purpose one considers a planar steady strong discontinuity (stepshock) with the equation $x_1 = 0$ and the piecewise constant solution

$$\begin{aligned} &\mathbf{u} = (\hat{u}_{1\infty}, 0, 0), \quad \rho = \hat{\rho}_\infty, \quad \text{and} \quad s = \hat{s}_\infty, \quad \text{if} \quad x_1 < 0, \\ &\mathbf{u} = (\hat{u}_1, 0, 0), \quad \rho = \hat{\rho}, \quad \text{and} \quad s = \hat{s}, \quad \text{if} \quad x_1 > 0, \end{aligned} \tag{3.1}$$

to system (2.1) which satisfies the jump conditions (2.2) on the plane $x_1 = 0$:

$$\begin{aligned} &\hat{\rho}\hat{u}_1 = \hat{\rho}_\infty\hat{u}_{1\infty}, \quad (\hat{u}_1 - \hat{u}_{1\infty})^2 + (\hat{p} - \hat{p}_\infty)\left(\widehat{V} - \widehat{V}_\infty\right) = 0, \\ &(\hat{e}_0 - \hat{e}_{0\infty}) + \frac{(\hat{p} + \hat{p}_\infty)}{2}\left(\widehat{V} - \widehat{V}_\infty\right) = 0. \end{aligned} \tag{3.2}$$

Here the constants $\hat{u}_{1\infty}$, $\hat{\rho}_\infty$, $\hat{s}_\infty$, $\hat{u}_1$, $\hat{\rho}$, and $\hat{s}$ are parameters of the steady viscous flow ahead of and behind the stepshock ($\hat{\rho} > 0$, $\hat{\rho}_\infty > 0$); moreover,

$$\hat{u}_{1\infty} > \hat{c}_\infty > 0, \quad 0 < \hat{u}_1 < \hat{c}, \tag{3.3}$$

$\hat{c}_\infty^2 = (\rho^2(e_0)_\rho)_\rho(\hat{\rho}_\infty, \hat{s}_\infty)$ and $\hat{c}^2 = (\rho^2(e_0)_\rho)_\rho(\hat{\rho}, \hat{s})$ the squares of the sound speed ahead of and behind the stepshock, $\hat{p}_\infty = \hat{\rho}_\infty^2(e_0)_\rho(\hat{\rho}_\infty, \hat{s}_\infty)$, $\widehat{V}_\infty = 1/\hat{\rho}_\infty$, $\hat{e}_{0\infty} = e_0(\hat{\rho}_\infty, \hat{s}_\infty)$, $\hat{p} = \hat{\rho}^2(e_0)_\rho(\hat{\rho}, \hat{s})$, $\widehat{V} = 1/\hat{\rho}$, $\hat{e}_0 = e_0(\hat{\rho}, \hat{s})$. We will also suppose that the state equation $e_0 = e_0(\rho, s)$ satisfies the requirements for a so-called *normal gas* (see, e.g., [14]). In that case, as is known (see, e.g., [14], [10]), inequalities (3.3), the entropy increase assumption $\hat{s} > \hat{s}_\infty$, and the compressibility conditions $\hat{p} > \hat{p}_\infty$, $\hat{\rho} > \hat{\rho}_\infty$, $\hat{u}_{1\infty} > \hat{u}_1$ are equivalent to each other (conditions (3.2) coincide with the corresponding ones for inviscid gas dynamic flows without heat conduction).

Linearizing system (2.1) and the jump conditions (2.2) about the piecewise constant solution (3.1), we obtain the stability problem (in a dimensionless form, see below) for determining the small perturbations δp, $\delta\mathbf{u}$, δs, and the small disturbance of discontinuity surface $\delta f = F = F(t, x_2, x_3)$ (in order to simplify the notation we indicate perturbations again by p, $\mathbf{u}$, and s). Its 1-D variant looks as follows ($x := x_1$, $u := u_1$, $F = F(t)$).

We seek solutions to the system

$$M^2Lp + u_x = \widehat{N}\left(\hat{\alpha}p_{xx} + \hat{\beta}s_{xx}\right), \quad M^2Lu + p_x = rM^2u_{xx}, \quad Ls = \hat{\alpha}p_{xx} + \hat{\beta}s_{xx}, \tag{3.4}$$

for $x > 0$ and the system

$$\begin{aligned} &L_\infty p_\infty + (u_\infty)_x = \widehat{N}_\infty\left(\hat{\alpha}_\infty(p_\infty)_{xx} + \hat{\beta}_\infty(s_\infty)_{xx}\right), \\ &M_\infty^2L_\infty u_\infty + (p_\infty)_x = r_\infty M_\infty^2(u_\infty)_{xx}, \quad L_\infty s_\infty = \hat{\alpha}_\infty(p_\infty)_{xx} + \hat{\beta}_\infty(s_\infty)_{xx}, \end{aligned} \tag{3.5}$$

for $x<0$ satisfying the boundary conditions

$$u + dp + d_0\left(\hat{\alpha}p_x + \hat{\beta}s_x\right) + d_1u_x$$

$$= \hat{u}u_\infty + d_2p_\infty + d_3(u_\infty)_x + d_4\left(\hat{\alpha}_\infty(p_\infty)_x + \hat{\beta}_\infty(s_\infty)_x\right) + d_5s_\infty\,, \tag{3.6}$$

$$s + \nu p + \nu_0\left(\hat{\alpha}p_x + \hat{\beta}s_x\right) + \nu_1u_x$$

$$= \nu_2p_\infty + \nu_3(u_\infty)_x + \nu_4\left(\hat{\alpha}_\infty(p_\infty)_x + \hat{\beta}_\infty(s_\infty)_x\right) + \nu_5s_\infty\,, \tag{3.7}$$

$$F_t = \mu\left(u + p - u_\infty - p_\infty - \widehat{N}s + \widehat{N}_\infty s_\infty\right) \tag{3.8}$$

at $x=0$ and corresponding initial data for $t=0$.

Here the coordinate x, time t, and small perturbations p, u, s $(x>0)$, p_∞, u_∞, s_∞ $(x<0)$ are related to the following characteristic values: $\hat{l}$ (characteristic length), $\hat{l}/\hat{u}_1$ (time), $\hat{\rho}\hat{c}^2$ (pressure for $x>0$), $\hat{\rho}_\infty\hat{c}_\infty^2$ (pressure for $x<0$), $\hat{u}_1$ (velocity for $x>0$), $\hat{u}_{1\infty}$ (velocity for $x<0$), $\hat{s}$ (entropy for $x>0$), $\hat{s}_\infty$ (entropy for $x<0$);

$$L = \frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\,,\quad L_\infty = \frac{1}{\hat{u}}\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\,,\quad \hat{u} = \frac{\hat{u}_{1\infty}}{\hat{u}_1} > 1\,,\quad M = \frac{\hat{u}_1}{\hat{c}} < 1,\quad M_\infty = \frac{\hat{u}_{1\infty}}{\hat{c}_\infty} > 1\,,$$

$$\widehat{N} = -\frac{\hat{s}(e_0)_{Vs}(\hat{\rho},\hat{s})}{\widehat{V}(e_0)_{VV}(\hat{\rho},\hat{s})}\,,\quad \hat{\alpha} = -\frac{\hat{\varkappa}\widehat{V}^2(e_0)_{Vs}(\hat{\rho},\hat{s})}{\widehat{T}\hat{s}\hat{u}_1\hat{l}}\,,\quad \hat{\beta} = \frac{\hat{\varkappa}\widehat{V}((e_0)_{VV}(e_0)_{ss} - (e_0)^2_{Vs})(\hat{\rho},\hat{s})}{(e_0)_{VV}(\hat{\rho},\hat{s})\widehat{T}\hat{u}_1\hat{l}}\,,$$

$$\widehat{T} = (e_0)_s(\hat{\rho},\hat{s})\,,\quad r = \frac{4}{3R_1} + \frac{1}{R_2}\,,\quad r_\infty = \frac{4}{3R_{1\infty}} + \frac{1}{R_{2\infty}}\,,\quad R_1 = \frac{\hat{\rho}\hat{u}_1\hat{l}}{\hat{\eta}}\,,\quad R_2 = \frac{\hat{\rho}\hat{u}_1\hat{l}}{\hat{\zeta}}$$

(the values $\hat{N}_\infty$, $\hat{\alpha}_\infty$, $\hat{\beta}_\infty$, and $R_{1\infty,2\infty}$ have an analogous form), M, M_∞, $R_{1,2}$, and $R_{1\infty,2\infty}$ are respectively Mach and Reynolds numbers behind and ahead of the stepshock, $\hat{\varkappa} = \varkappa(\hat{\rho},\hat{s})$, $\hat{\eta} = \eta(\hat{\rho},\hat{s})$, $\hat{\zeta} = \zeta(\hat{\rho},\hat{s})$. The coefficients of boundary conditions μ, d, and ν completely coincide with the corresponding ones in the linearized stability problem for gas dynamical shocks ([6], [3], [13]) and have the following form:

$$\mu = \frac{\hat{u}}{\hat{u}-1}\,,\quad d = \frac{1+M^2+\beta^2\omega}{2M^2}\,,\quad \nu = \frac{\beta^2}{M^2\widehat{N}}\,\omega\,,\quad \omega = \frac{1}{1-\widehat{D}}\,,\quad \widehat{D} = \frac{2\widehat{T}\hat{s}}{\widehat{N}(\hat{u}-1)\hat{u}_1^2}\,,$$

$\beta^2 = 1 - M^2$. In particular, for a polytropic gas with the adiabatic index γ, one has: $\omega = -(\gamma-1)\beta^2/(2+(\gamma-1)M^2)$, $\hat{\alpha}\widehat{N}/\hat{\beta} = \gamma - 1$. Other coefficients ($d_j$, ν_j, $j=\overline{0,5}$) can be also easily written out, for example, $d_0 = \omega\widehat{D}\widehat{N}/2$, $\nu_0 = \omega\widehat{D}$ (below we will need the concrete form only of these coefficients). Observe that for a normal gas the state equation is convex, $(e_0)_V < 0$, $(e_0)_s > 0$, $(e_0)_{VV} > 0$, $(e_0)_{VV}(e_0)_{ss} - (e_0)^2_{Vs} > 0$, that implies the inequalities $\hat{\beta} > 0$, $\hat{\alpha}\widehat{N} > 0$, $\hat{\beta}_\infty > 0$, and $\hat{\alpha}_\infty\widehat{N}_\infty > 0$. We finally note that the boundary condition (3.8) is the equation for determining the function F and can be considered separately from problem (3.4)–(3.7).

Main ideas of the work will be explained below on the example of a non-heat-conducting gas, i.e., when $\varkappa = 0$ (the situation for a heat-conducting gas is briefly described at the end of the paper). For this case the linearized stability problem is obtained from problem (3.4)–(3.7) by setting $\hat{\varkappa} = 0$, $\hat{\varkappa}_\infty = 0$ (i.e., $\hat{\alpha} = \hat{\beta} = 0$ and $\hat{\alpha}_\infty = \hat{\beta}_\infty = 0$) and has the following form.

One seeks solutions to the system

$$M^2 Lp + u_x = 0\,, \quad M^2 Lu + p_x = rM^2 u_{xx}\,, \tag{3.9}$$

$$Ls = 0\,, \tag{3.10}$$

for $x > 0$ and the system

$$L_\infty p_\infty + (u_\infty)_x = 0\,, \quad M_\infty^2 L_\infty u_\infty + (p_\infty)_x = r_\infty M_\infty^2 (u_\infty)_{xx}\,, \tag{3.11}$$

for $x < 0$ satisfying the boundary conditions

$$u + dp + d_1 u_x = \hat{u} u_\infty + d_2 p_\infty + d_3 (u_\infty)_x, \tag{3.12}$$

$$s + \nu p + \nu_1 u_x = \nu_2 p_\infty + \nu_3 (u_\infty)_x \tag{3.13}$$

at $x = 0$ and corresponding initial data for $t = 0$.

Here, without loss of generality, the small entropy perturbation s_∞ for $x < 0$ is supposed to be equal to zero. And from problem (3.9)–(3.13) one can naturally separate subproblem (3.10), (3.13) for finding the function s.

As was already noted in Sec. 1, the ill-posedness of the linearized stability problem for shock waves in a viscous gas has been proved in [5]. This ill-posedness is a direct consequence of the fact that the number of independent parameters determining an arbitrary perturbation of the shock front is greater than that of boundary conditions (the linearized generalized Rankine-Hugoniot conditions). The ill-posedness example of Hadamard type for problem (3.9)–(3.13) constructed in [5] looks as follows:

$$\begin{pmatrix} p \\ u \\ s \end{pmatrix} = \sum_{k=1}^{3} \begin{pmatrix} p^{(k)} \\ u^{(k)} \\ s^{(k)} \end{pmatrix} \mathrm{e}^{n(\hat{\tau}t + \hat{\xi}_k x)}\,, \quad x > 0\,, \qquad \begin{pmatrix} p_\infty \\ u_\infty \end{pmatrix} = \begin{pmatrix} p_\infty^{(0)} \\ u_\infty^{(0)} \end{pmatrix} \mathrm{e}^{n(\hat{\tau}t + \hat{\xi}_\infty x)}\,, \quad x < 0\,,$$

where $p^{(k)}$, $u^{(k)}$, $s^{(k)}$, $p_\infty^{(0)}$, and $u_\infty^{(0)}$ are some constants, $n = 1, 2, 3, \ldots$, $\operatorname{Re}\hat{\tau} > 0$, $\operatorname{Re}\hat{\xi}_k < 0$, $\operatorname{Re}\hat{\xi}_\infty > 0$. The values $\hat{\xi}_k$ $(k = 1, 2, 3)$ and $\hat{\xi}_\infty$ are the roots of corresponding dispersion relations (ahead of and behind the discontinuity) following from (3.9), (3.10), and (3.11); moreover (see [5]),

$$\hat{\xi}_1 = -\hat{\tau}^{(0)} - \hat{\tau}^{(1)}\varepsilon + \mathcal{O}(\varepsilon^2)\,, \quad \hat{\xi}_2 = -\sqrt{\frac{\hat{\tau}^{(0)}}{r}}\varepsilon + \mathcal{O}(\varepsilon^2)\,, \quad \hat{\xi}_3 = -\hat{\tau}\,, \quad \hat{\xi}_\infty = \sqrt{\frac{\hat{\tau}^{(0)}}{\hat{u} r_\infty}}\varepsilon + \mathcal{O}(\varepsilon^2)\,,$$

$\varepsilon = n^{-1/2}$ is a small parameter $(n \gg 1)$, $\hat{\tau} = \hat{\tau}^{(0)} + \hat{\tau}^{(1)}\varepsilon + \mathcal{O}(\varepsilon^2)$. So, for finding 11 constants, $p^{(k)}$, $u^{(k)}$, $s^{(k)}$, $p_\infty^{(0)}$, $u_\infty^{(0)}$, one has only 9 relations (two of them follow from the boundary conditions (3.12), (3.13), and the others from Eqs. (3.9)–(3.11); see [5]. Thus, problem (3.9)–(3.13) is *underdetermined*; namely, it lacks two boundary conditions at $x = 0$.

Let us now consider the question of additional boundary conditions for problem (3.9)–(3.13). We note that the piecewise constant solution (3.1) described above satisfies, in particular, the conditions

$$u_x = 0\,, \quad (u_\infty)_x = 0 \tag{3.14}$$

at $x = 0$. Starting from this *a priori* information, we add (3.14) to the boundary conditions (3.12), (3.13) (observe that conditions (3.14) are already linear). Moreover, as will be shown below, precisely such additional boundary conditions ensure the *asymptotic stability* (by Lyapunov) of the trivial solution to the *modified stability problem.* This problem is finally the linear initial boundary value problem (IBVP) for systems (3.9), (3.11) with the boundary conditions

$$u + dp = \hat{u}u_\infty + d_2 p_\infty\,, \quad u_x = 0\,, \quad (u_\infty)_x = 0 \tag{3.15}$$

at $x = 0$. And, with regard to (3.14), the boundary condition (3.13) being necessary for finding the function s is naturally simplified.

REMARK 3.1. Recall that a traveling wave solution of the 1-D Navier-Stokes equations describing the viscous profile (see, e.g., [7], [14], [10]) of a shock wave is a stationary profile closely approximating (under $x \to \pm\infty$) the piecewise constant solution (3.1). Such a "spread" (viscous) shock wave does not have an exact width, but on conditional boundaries of the shock zone (for which one can determine a so-called effective width [14], [10]), the values of traveling wave and piecewise constant solutions are close to each other. In this connection, we remark on the remarkable fact that it is the piecewise constant solution (3.1) that is the *unique* piecewise smooth solution to the 1-D Navier-Stokes equations (we consider the case $\varkappa = 0$) satisfying at $x = 0$ the generalized Rankine-Hugoniot relations and the additional conditions (3.14). Indeed, the mentioned piecewise smooth steady solution should satisfy the equalities

$$\rho u = \rho_\infty u_\infty = C_1\,, \tag{3.16}$$

$$p + C_1 u - \epsilon \frac{du}{dx} = p_\infty + C_1 u_\infty - \epsilon_\infty \frac{du_\infty}{dx} = C_2\,, \tag{3.17}$$

$$C_1\left(e_0 - \frac{u^2}{2}\right) + C_2 u = C_1\left(e_{0\infty} - \frac{u_\infty^2}{2}\right) + C_2 u_\infty = C_3\,, \tag{3.18}$$

where $C_{1,2,3}$ are some constants, $\epsilon = (4/3)\eta + \zeta$, $\epsilon_\infty = \epsilon(\rho_\infty, s_\infty)$. By (3.14), it follows from (3.16) that $\rho'(0) = \rho'_\infty(0) = 0$. Then, accounting for such evident equalities as $e_0' = (e_0)_\rho \rho' + (e_0)_p p'$, $\epsilon' = \epsilon_\rho \rho' + \epsilon_p p'$, etc., we obtain from (3.17), (3.18) that $p'(0) = p'_\infty(0) = 0$ and $u''(0) = u''_\infty(0) = 0$. One analogously concludes that $\rho''(0) = \rho''_\infty(0) = 0$, $p''(0) = p''_\infty(0) = 0$, $u'''(0) = u'''_\infty(0) = 0$, etc. As a result, supposing the functions $\rho(x)$, $u(x)$, and $p(x)$ under $x > 0$ and the functions $\rho_\infty(x)$, $u_\infty(x)$, and $p_\infty(x)$ under $x < 0$ to be infinitely smooth, we conclude that all of their derivatives are equal to zero. Hence, the piecewise smooth solution under consideration is none other than the piecewise constant solution (3.1) satisfying relations (3.2). And the asymptotic stability (by Lyapunov) of the trivial solution to the linear IBVP (3.9), (3.11), (3.15) being proved in this work points indirectly to the following fact. Piecewise smooth *unsteady* solutions of the 1-D Navier-Stokes equations satisfying the generalized Rankine-Hugoniot relations and the additional conditions (3.14) on a propagating discontinuity (free boundary) with the equation $x = f(t)$ should stabilize (converge under $t \to \infty$) to the piecewise constant solution (3.1), which satisfies the classical Rankine-Hugoniot relations (3.2) at $x = 0$.

4. The L_2-well-posedness of the modified stability problem. At the beginning, we prove the well-posedness of problem (3.9), (3.11), (3.15). For this purpose we rewrite systems (3.9) and (3.11) in the matrix form

$$A\mathbf{U}_t + B\mathbf{U}_x = C\mathbf{U}_{xx}\,, \tag{4.1}$$

$$A_\infty(\mathbf{U}_\infty)_t + B_\infty(\mathbf{U}_\infty)_x = C_\infty(\mathbf{U}_\infty)_{xx}\,, \tag{4.2}$$

where $\mathbf{U} = (u, p)$, $\mathbf{U}_\infty = (u_\infty, p_\infty)$,

$$A = \begin{pmatrix} M^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_\infty = \frac{1}{\hat{u}}\begin{pmatrix} M_\infty^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} M^2 & 1 \\ 1 & 1 \end{pmatrix},$$

$$B_\infty = \begin{pmatrix} M_\infty^2 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} rM^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_\infty = \begin{pmatrix} r_\infty M_\infty^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

In view of the positive definiteness of the matrix B_∞, we at once note that (4.2) yields the following estimate for the vector of perturbations ahead of the shock:

$$I_\infty(t) \le I_\infty(0)\,, \tag{4.3}$$

with $I_\infty(t) = \int\limits_{\mathbb{R}_-} (A_\infty\mathbf{U}_\infty, \mathbf{U}_\infty)dx > 0$ $(\mathbb{R}_\pm = \{\, x \mid x \gtrless 0\,\})$. Indeed, multiplying system (4.2) scalar-wise by the vector $2\mathbf{U}_\infty$, integrating the obtained equality with respect to $x \in \mathbb{R}_-$, and accounting for the boundary conditions (3.15), one gets the energetic identity

$$\frac{d}{dt}I_\infty(t) + (B_\infty\mathbf{U}_\infty, \mathbf{U}_\infty)|_{x=0} + 2r_\infty M_\infty^2 \int\limits_{\mathbb{R}_-} (u_\infty)_x^2 dx = 0, \tag{4.4}$$

which implies, with regard to the remark about the matrix B_∞, the a priori estimate (4.3) (when deducing (4.4) we assume that $|\mathbf{U}_\infty| \to 0$ for $x \to -\infty$).

Thus, if initial data for the functions u_∞ and p_∞ are zero, then, in view of (4.3), $u_\infty \equiv 0$ and $p_\infty \equiv 0$ for all $t > 0$. Therefore, as in gas dynamics (see, e.g., [6], [13], [3], [10]), without loss of generality one can assume that there are no perturbations ahead of the shock wave: $u_\infty \equiv 0$, $p_\infty \equiv 0$. Moreover, in case of need, with regard to the positive definiteness of the quadratic form $(B_\infty\mathbf{U}_\infty, \mathbf{U}_\infty)|_{x=0}$, one can always include the estimate for these perturbations into the general a priori estimate containing also the perturbations u and p behind the shock wave (for this purpose, it is enough to multiply equality (4.4) by a rather big positive constant and sum it with the corresponding energetic identity for u and p (see below)).

As a result, instead of problem (3.9), (3.11), (3.15), we will analyze the IBVP for system (3.9) with the boundary conditions

$$u + dp = 0\,, \quad u_x = 0 \tag{4.5}$$

at $x = 0$. And the function s is found as a solution to Eq. (3.10) with the boundary condition $s = -\nu p$ at $x = 0$. The function $F = F(t)$ is determined from the relation

$$F_t = \mu(1 - d + \widehat{N}\nu)p(t, 0) \tag{4.6}$$

following from the boundary condition (3.8).

Let us now deduce an a priori estimate for solutions of the IBVP (3.9), (4.5) (with corresponding initial data for $t = 0$). With regard to the boundary conditions (4.5), we easily obtain from system (4.1) the following energetic identity (being analogous to (4.4)) for the vector $\mathbf{U}$:

$$\frac{d}{dt}I(t) + \sigma p^2(t,0) + 2rM^2 \int\limits_{\mathbb{R}_+} u_x^2 dx = 0\,, \tag{4.7}$$

where $I(t) = \int\limits_{\mathbb{R}_+} (A\mathbf{U}, \mathbf{U})dx > 0$, $\sigma = 2d - 1 - M^2d^2$. When deducing (4.7) we assume that $|\mathbf{U}| \to 0$ for $x \to +\infty$. It is easily verified that for a polytropic gas the constant $\sigma > 0$. Then, (4.7) yields the inequality $dI(t)/dt \le 0$ from which we deduce the desired a priori estimate

$$I(t) \le I(0)\,. \tag{4.8}$$

In turn, estimate (4.8) is rewritten as the L_2-estimate

$$\|\mathbf{U}(t)\|^2_{L_2(\mathbb{R}_+)} \le \frac{1}{M^2}\|\mathbf{U}(0)\|^2_{L_2(\mathbb{R}_+)}\,. \tag{4.9}$$

The a priori estimate (4.9) implies the L_2-well-posedness (global) of the linear IBVP (3.9), (4.5) and the stability (by Lyapunov) of its trivial solution (we do not discuss here the existence of solutions that can be proved, for the linear problem under consideration, with the help of the a priori estimate (4.9) by standard methods used, for example, in the theory of linear parabolic equations [9]).

REMARK 4.1. When obtaining estimate (4.9) we supposed the gas to be polytropic. In the general case of an arbitrary state equation (when the condition $\sigma > 0$ can be violated), for deducing a priori estimates one has to use expanded systems for (3.9) (i.e., equations obtained by differentiating (3.9) with respect to t and x). Here we only note that for problem (3.9), (4.5) one cannot construct ill-posedness examples of Hadamard type. Indeed, the dispersion relation for system (3.9) has the roots $\hat{\xi}_1$ and $\hat{\xi}_2$ described above (see Sec. 3). Then the constants $p^{(1)}$, $u^{(1)}$, $p^{(2)}$, and $u^{(2)}$ determining the exponential solution

$$p = \left\{p^{(1)}\mathrm{e}^{n\hat{\xi}_1 x} + p^{(2)}\mathrm{e}^{n\hat{\xi}_2 x}\right\}\mathrm{e}^{n\hat{\tau}t}, \quad u = \left\{u^{(1)}\mathrm{e}^{n\hat{\xi}_1 x} + u^{(2)}\mathrm{e}^{n\hat{\xi}_2 x}\right\}\mathrm{e}^{n\hat{\tau}t}$$

should satisfy an algebraic system of four equations following from the boundary conditions (4.5) and the first equation in (3.9). This algebraic system has a nontrivial solution if $(\hat{\xi}_1 - \hat{\xi}_2)(M^2(\hat{\tau} + \hat{\xi}_1)(\hat{\tau} + \hat{\xi}_2) - d\hat{\xi}_1\hat{\xi}_2) = 0$. By expanding $\hat{\tau}$ and $\hat{\xi}_{1,2}$ into series in the small parameter ε, one can see that the last equality being considered as an equation for $\hat{\tau}$ has the unique solution $\hat{\tau} = 0$ (consequently, $\hat{\xi}_1 = 0$ and $\hat{\xi}_2 = 0$). That is, problem (3.9), (4.5) has no exponential solutions bounded for $t = 0$.

5. The asymptotic stability of the trivial solution. To prove the asymptotic stability of the trivial solution to problem (3.9), (4.5), we take advantage of the following simple arguments. It is convenient to demonstrate the idea of these arguments on the example of the boundary value problem for the heat equation $v_t = v_{xx}$ on the half-line $x > 0$ with the boundary condition $v|_{x=0} = 0$ (or $v_x|_{x=0} = 0$).

Let there exist a solution to this boundary value problem in the class of sufficiently smooth functions decreasing on the infinity: $v \to 0$ for $x \to +\infty$ (the same assumption has been already done and will be below valid for the functions u and p). Then one easily gets the identity

$$\frac{d}{dt}j(t) + 2\int\limits_{\mathbb{R}_+} v_x^2 dx = 0\,, \tag{5.1}$$

with $j(t) = \|v(t)\|^2_{L_2(\mathbb{R}_+)}$, which yields $j'(t) \leq 0$ and the a priori estimate

$$j(t_2) \leq j(t_1) \quad \text{for} \quad t_1 < t_2\,. \tag{5.2}$$

But, actually, without loss of generality, the function $j(t)$ can be supposed to be strictly decreasing. Indeed, assuming the contrary, i.e., that there exists a point $t = t_*$ in which $j'(t_*) = 0$ (we consider the first such a point), it follows from (5.1) that $\int\limits_{\mathbb{R}_+} v_x^2 dx\big|_{t=t_*} = 0$. Then $v(t_*, x) = 0$ (observe that a constant (nonzero) solution does not belong to the class of smooth solutions under consideration). That is, in view of (5.2), $v = 0$ for all $t \geq t_*$. Thus, the function $j(t)$ decreases up to $t = t_*$ and is equal to zero for $t \geq t_*$. This means the asymptotic stability of the trivial solution to the boundary value problem.

So, the positive function $j(t)$ is monotone decreasing. But, as follows from (5.1), in the class of functions under consideration it cannot have other horizontal asymptotes except $j = 0$; i.e., one has asymptotic stability: $j(t) \to 0$ under $t \to \infty$. Observe that in the context of the present work we are not interested in an exact estimation of the character of decrease (stabilization) of solutions under $t \to \infty$. The fact of asymptotic stability of the trivial solution itself is of importance to us. For example, it is clear that the solution of the boundary value problem considered above decreases as $t^{-1/2}$ that follows from its explicit formula.

Let us now utilize analogous arguments for problem (3.9), (4.5). For this purpose, one has to extend system (4.1); i.e., differentiating it with respect to t and x, we finally obtain:

$$\frac{d}{dt}I_1(t) + \sigma p_t^2(t,0) + 2rM^2 \int\limits_{\mathbb{R}_+} u_{tx}^2 dx = 0\,, \tag{5.3}$$

$$\frac{d}{dt}I_2(t) - p_t^2(t,0) + 2rM^2 \int\limits_{\mathbb{R}_+} u_{xx}^2 dx = 0\,, \tag{5.4}$$

with $I_1(t) = \int\limits_{\mathbb{R}_+} (A\mathbf{U}_t, \mathbf{U}_t)dx$, $I_2(t) = \int\limits_{\mathbb{R}_+} (A\mathbf{U}_x, \mathbf{U}_x)dx$. Multiplying (5.3), let us say, by $2/\sigma$ and summing with (4.7) and (5.4), one gets

$$\frac{d}{dt}J(t) + \sigma p^2(t,0) + p_t^2(t,0) + 2rM^2 \int\limits_{\mathbb{R}_+} \left(u_x^2 + \frac{2}{\sigma}u_{tx}^2 + u_{xx}^2\right) dx = 0\,, \tag{5.5}$$

with $J(t) = I(t) + (2/\sigma)I_1(t) + I_2(t)$. It follows from (5.5) that $J'(t) \leq 0$; i.e., the function $J(t)$ does not increase.

And, moreover, the function $J(t)$ is actually monotone decreasing from where we have the desired asymptotic stability of the trivial solution: $J(t) \to 0$ under $t \to \infty$. Indeed, supposing that there exists a point $t = t_*$ in which $J'(t_*) = 0$, from (5.5) we deduce, in particular, that the integrals $\int\limits_{\mathbb{R}_+} u_x^2 dx\big|_{t=t_*}$, $\int\limits_{\mathbb{R}_+} u_{tx}^2 dx\big|_{t=t_*}$, and $\int\limits_{\mathbb{R}_+} u_{xx}^2 dx\big|_{t=t_*}$

are equal to zero. Then $u(t_*,x) = u_t(t_*,x) = u_x(t_*,x) = 0$ and, by virtue of system (3.9), $p_x(t_*,x) = p_t(t_*,x) = 0$ and $p(t_*,x) = 0$. Thus, the inequality $J(t) \le J(t_*) = 0$ for $t \ge t_*$ yields $J(t) = 0$ for all $t \ge t_*$. As a result, by using arguments as above for the heat equation, one gets: $\|\mathbf{U}(t)\|^2_{W_2^1(\mathbb{R}_+)} \to 0$ under $t \to \infty$; i.e., the trivial solution to problem (3.9), (4.5) is asymptotically stable (by Lyapunov).

Let us make one more important observation.

REMARK 5.1. By virtue of the evident estimate $|\varphi(t,0)| \le \|\varphi(t)\|_{W_2^1(\mathbb{R}_+)}$ for the trace of a function in $W_2^1(\mathbb{R}_+)$ at the line $x=0$, we obtain that not only $W_2^1(\mathbb{R}_+)$-norms of the perturbations u and p but also their boundary values $u(t,0)$ and $p(t,0)$ converge to zero under $t \to \infty$. Then, it follows from relation (4.6) that the shock speed $F_t \to 0$ under $t \to \infty$.

6. The general case of heat-conducting gas. Let us briefly discuss the general case of heat-conducting gas, i.e., when in problem (3.4)–(3.8) $\hat{\varkappa} \ne 0$ and $\hat{\varkappa}_\infty \ne 0$. The ill-posedness example of Hadamard type for this case has the following form [5]:

$$\begin{pmatrix} p \\ u \\ s \end{pmatrix} = \sum_{k=1}^{3} \begin{pmatrix} p^{(k)} \\ u^{(k)} \\ s^{(k)} \end{pmatrix} \mathrm{e}^{n(\hat{\tau}t + \hat{\xi}_k x)}, \quad x > 0,$$

$$\begin{pmatrix} p_\infty \\ u_\infty \\ s_\infty \end{pmatrix} = \sum_{j=1}^{2} \begin{pmatrix} p_\infty^{(j)} \\ u_\infty^{(j)} \\ s_\infty^{(j)} \end{pmatrix} \mathrm{e}^{n(\hat{\tau}t + \hat{\xi}_\infty x)}, \quad x < 0, \quad F = F^{(0)}\mathrm{e}^{n\hat{\tau}t},$$

where $\hat{\xi}_k$ ($\mathrm{Re}\,\hat{\xi}_k < 0$, $k = 1,2,3$) and $\hat{\xi}_{j\infty}$ ($\mathrm{Re}\,\hat{\xi}_{j\infty} > 0$, $j = 1,2$) are the roots of corresponding dispersion relations behind and ahead of the discontinuity; moreover,

$$\hat{\xi}_1 = -\hat{\tau}^{(0)} - \hat{\tau}^{(1)}\varepsilon + \mathcal{O}(\varepsilon^2), \quad \hat{\xi}_2 = -\sqrt{\frac{\hat{\tau}^{(0)}}{\hat{\beta} + \widehat{N}\hat{\alpha}}}\,\varepsilon + \mathcal{O}(\varepsilon^2), \quad \hat{\xi}_3 = -\sqrt{\frac{\hat{\tau}^{(0)}}{r}}\varepsilon + \mathcal{O}(\varepsilon^2),$$

$$\hat{\xi}_{1\infty} = \sqrt{\frac{\hat{\tau}^{(0)}}{\hat{u}(\hat{\beta}_\infty + \widehat{N}_\infty\hat{\alpha}_\infty)}}\,\varepsilon + \mathcal{O}(\varepsilon^2), \quad \hat{\xi}_{2\infty} = \sqrt{\frac{\hat{\tau}^{(0)}}{\hat{u}r_\infty}}\varepsilon + \mathcal{O}(\varepsilon^2).$$

For finding 16 constants, $p^{(k)}$, $u^{(k)}$, $s^{(k)}$, $p_\infty^{(j)}$, $u_\infty^{(j)}$, $s_\infty^{(j)}$, $F^{(0)}$ ($k = 1,2,3$, $j = 1,2$), one has an algebraic system of 13 equations (three of them follow from the boundary conditions (3.6)–(3.8), and the others from systems (3.4) and (3.5)). That is, the cause of *ill-posedness* of the linearized stability problem (3.4)–(3.8) is that it lacks three boundary conditions at $x = 0$.

As for problem (3.9)–(3.13), additional boundary conditions for problem (3.4)–(3.8) are posed with regard to a priori information about the steady flow regime with a shock wave. In the capacity of these conditions we take relations (3.14) and also the equality

$$(T_\infty)_x = 0 \tag{6.1}$$

at $x = 0$ that is satisfied by the piecewise constant solution (3.1), where T_∞ is a small perturbation of the temperature ahead of the discontinuity. Observe that the boundary condition (6.1) is equivalent to the relation $\hat{\alpha}_\infty(p_\infty)_x + \hat{\beta}_\infty(s_\infty)_x = 0$ at $x = 0$. Moreover, if in problem (3.4), (3.6)–(3.8) we suppose $u_\infty \equiv 0$, $p_\infty \equiv 0$, and $s_\infty \equiv 0$ (below it will be shown that, as for problem (3.9), (3.11), (3.15), such a supposition does not restrict generality), then it is easy to see that this problem needs exactly one additional boundary condition in which capacity we take the relation $u_x = 0$ at $x = 0$.

Systems (3.4) and (3.5) can be presented in the matrix form (4.1), (4.2) for the vectors of perturbations $\mathbf{U} = (u, p, s)$ and $\mathbf{U}_\infty = (u_\infty, p_\infty, s_\infty)$ with the matrices

$$A = \begin{pmatrix} M^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{\widehat{N}\hat{\beta}}{\hat{\alpha}} \end{pmatrix}, \quad B = \begin{pmatrix} M^2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \dfrac{\widehat{N}\hat{\beta}}{\hat{\alpha}} \end{pmatrix}, \quad C = \begin{pmatrix} rM^2 & 0 & 0 \\ 0 & \widehat{N}\hat{\alpha} & \widehat{N}\hat{\beta} \\ 0 & \widehat{N}\hat{\beta} & \dfrac{\widehat{N}\hat{\beta}^2}{\hat{\alpha}} \end{pmatrix}$$

(the matrices A_∞, B_∞, and C_∞ have an analogous form). Note that $(C_\infty(\mathbf{U}_\infty)_x, (\mathbf{U}_\infty)_x) \geq r_\infty M_\infty^2 (u_\infty)_x^2$. Then, with regard to the boundary conditions (3.14), (6.1), and accounting for the positive definiteness of the matrix B_∞, for the perturbation vector $\mathbf{U}_\infty$ ahead of the discontinuity, one gets an a priori estimate analogous to estimate (4.3). Hence, without loss of generality we can suppose that $\mathbf{U}_\infty \equiv 0$.

As a result, we have the IBVP for system (3.4) with the boundary conditions

$$u + dp + d_0\mathcal{F} = 0, \quad s + \nu p + \nu_0 \mathcal{F} = 0, \quad u_x = 0, \tag{6.2}$$

where $\mathcal{F} = \hat{\alpha} p_x + \hat{\beta} s_x$. To obtain an a priori estimate for solutions of this problem we use the inequality

$$\frac{d}{dt} I(t) + \mathcal{A}|_{x=0} + 2rM^2 \int\limits_{\mathbb{R}_+} u_x^2 dx \leq 0, \tag{6.3}$$

that is easily deduced from system (3.4) (being written in the matrix form (4.1)), with regard to the boundary conditions (6.2) and the inequality $(C\mathbf{U}_x, \mathbf{U}_x) \geq rM^2 u_x^2$. Here $I(t) = \int\limits_{\mathbb{R}_+} (A\mathbf{U}, \mathbf{U}) dx > 0$; $\mathcal{A}$ is the quadratic form that looks as follows:

$$\mathcal{A} = \left\{ 2d - 1 - M^2 d - (\widehat{N}\hat{\beta}/\hat{\alpha})\nu^2 \right\} p^2 + \left\{ -(\widehat{N}\hat{\beta}/\hat{\alpha})\nu_0(2 + \nu_0) - M^2 d_0 \right\} \mathcal{F}^2$$
$$+ 2\left\{ d_0(1 - M^2 d) - (\widehat{N}\hat{\beta}/\hat{\alpha})\nu(\nu_0 + 1) + \widehat{N} \right\} p\mathcal{F}.$$

The quadratic form $\mathcal{A}$ is not always positive definite, but one can show that for the air (this case is most interesting for applications), i.e., for a polytropic gas with $\gamma = 7/5$, the inequality $\mathcal{A} > 0$ is certainly valid for Mach numbers of the coming flow $M_\infty > 4/3$. Then, with regard to the condition $\mathcal{A} > 0$, from the inequality (6.3) we easily deduce an L_2-estimate that is like a priori estimate (4.9). In this connection, we note that for problem (3.4), (6.2) an observation analogous to Remark 4.1 for problem (3.9), (4.5) takes place; i.e., one can show that problem (3.4), (6.2) does not tolerate the construction of Hadamard-type ill-posedness examples (the situation is like that described in Remark 4.1), and in the case of violation of the condition $\mathcal{A} > 0$, for obtaining an a priori estimate one has to use expanded systems for (3.4).

Concerning the asymptotic stability of the trivial solution to problem (3.4), (6.2), it is established in just the same way as for problem (3.9), (4.5) (see Sec. 5). For this purpose one should use the inequality (6.3) (under the condition $\mathcal{A} > 0$) and corresponding inequalities obtained by differentiating system (3.4) with respect to t and x.

7. Concluding remarks. So, the authors of the paper try to justify application of the stabilization method for steady-state calculations in viscous gas with the shock front as yet on the linear level. Clearly, in calculation practice, separation of the shock front instead of determination of viscous profile has certain advantages while organizing the calculation process. In the paper, a positive answer to the fundamental question was obtained: controlling the boundary conditions, whether is it possible to use the stabilization method for steady-state calculations in gas with shock fronts.

For setting additional boundary conditions for the *multidimensional* shock front problem for a viscous gas, one can apparently use ideas being analogous to those described in this paper. But, of course the well-posedness of the modified problem will not be established in such a simple way as for the 1-D case considered above. To prove well-posedness the authors are planning to utilize, in particular, elements of the dissipative integrals techniques worked out in [1], [2], [3], [4] for studying the stability of gas dynamical shock waves. Observe that the consideration of the multidimensional case is necessary to come naturally, for example, to investigating blunt body problems for a viscous heat-conducting gas.

At the same time, it should be noted that the considered 1-D case is of independent interest. In this connection, we refer, e.g., to the work [8] (see also references therein) devoted to the numerical simulation of the 1-D viscous air flow with a shock wave.

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