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ON THE CAUCHY PROBLEM OF THE BOLTZMANN AND LANDAU EQUATIONS WITH SOFT POTENTIALS

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Abstract. Global classical solutions near Maxwellians are constructed for the Boltzmann and Landau equations with soft potentials in the whole space. The construction of global solutions is based on refined energy analysis. Our results generalize the classical results in Ukai and Asano (*Publ. Res. Inst. Math. Sci.* **18** (1982), 477–519) to the very soft potentials for the Boltzmann equation and also extend the classical results in Caflisch (*Comm. Math. Phys.* **74** (1980), 97–107), Guo (*Comm. Math. Phys.* **231** (2002), 391–434), and Guo (*Arch. Rat. Mech. Anal.* **169** (2003), 305–353) in the periodic box to the whole space for the Boltzmann equation and the Landau equation in the Coulomb interaction.

1. Introduction. We consider the Cauchy problem of the Boltzmann and Landau equations with soft potentials for dynamics of dilute particles. The equation considered in this paper reads as

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \tag{1.1}$$

where $F(t, x, v)$ is the distribution function of the particles at time $t \geq 0$, located at $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ with velocity $v = (v_1, v_2, v_3) \in \mathbf{R}^3$. The collision between particles is given by the standard Boltzmann collision operator,

$$Q(F, G)(v) = \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} [F(u')G(v') - F(u)G(v)] B(|u - v|, \vartheta) du d\omega,$$

where $F(u) = F(t, x, u)$ etc., $\omega \in \mathbf{S}^2$, the unit sphere in $\mathbf{R}^3$, and

$$v' = v - [(v - u) \cdot \omega]\, \omega, \quad u' = u + [(v - u) \cdot \omega]\, \omega \tag{1.2}$$

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denote the velocities of particles after the collision, with u and v being their velocities before the collision.

For the interaction potentials of inverse power laws, the collision kernel $B(|u-v|,\vartheta)$ takes the form [5, 10, 26]

$$B(|u-v|,\vartheta) = B(\vartheta)|u-v|^{\gamma}, \quad \cos\vartheta = ((u-v)\cdot\omega)/|u-v|, \quad -3<\gamma\le 1,$$

where it is customary to assume that $B(\vartheta)$ satisfies the Grad angular cutoff assumption, $0 < B(\vartheta) \le \text{const.}|\cos\vartheta|$. The exponent γ is related to the models of potentials of intermolecular forces, namely, the soft potentials ($-3<\gamma<0$), Maxwellian molecules ($\gamma=0$), the hard potentials ($0<\gamma<1$) and the hard sphere model ($\gamma=1$, $B(\vartheta)$ =const. $\cos\vartheta$). We are concerned with the soft potentials $-3\le\gamma<0$.

For the Coulomb interaction, the Boltzmann equation becomes inadequate because grazing collisions become preponderant over all other collisions. In 1936, Landau derived from the Boltzmann equation another equation in which only grazing collisions are taken into account written as (1.1), but the collision operator is given by

$$\begin{aligned} Q(F,G)(v) &= \nabla_v\cdot\Big\{\int_{\mathbf{R}^3}\phi(v-v')[F(v')\nabla_v G(v) - F(v)\nabla_{v'}G(v')]dv'\Big\} \\ &= \sum_{i,j=1}^{3}\partial_i\int_{\mathbf{R}^3}\phi^{ij}(v-v')[F(v')\partial_j G(v) - F(v)\partial_j G(v')]dv', \end{aligned}$$

where $\phi(w)$ is the positive-semi-definite matrix that takes the general form

$$\phi(w) = B(|w|)S(w),$$

where B is a function depending on the nature of the interaction between the particles, and $S(w)$ is the 3×3 matrix

$$S(w) = I_3 - \frac{w\otimes w}{|w|^2},$$

which is the orthogonal projection onto the orthogonal plane to w and I_3 is the identity matrix of order 3. When an interaction force $\Re$ between the particles depends on the inter-particle distance r according to an inverse power law $\Re = r^{-s}$ with $s\ge 2$, the function B takes the form $B(|w|) = |w|^{\gamma+2}$ where $\gamma = \frac{s-5}{s-1}$. This leads to the usual classification in terms of hard potentials ($\gamma>0$), Maxwellian molecules ($\gamma=0$) or soft potentials ($\gamma<0$) ([6, 18, 29]). The original Landau collision operator for the Coulomb interaction corresponds to the case $\gamma=-3$. The present study with the Landau equation is restricted to the very soft potentials $-3\le\gamma<-2$.

We denote a normalized global Maxwellian by $\mu(v)\equiv e^{-|v|^2}$. We define the standard perturbation $f(t,x,v)$ to $\mu(v)$ as $F=\mu+\sqrt{\mu}f$. We plug this perturbation into (1.1) to derive a perturbation equation for $f(t,x,v)$. Equation (1.1) for the perturbation $f(t,x,v)$ takes the form

$$\partial_t f + v\cdot\nabla_x f + Lf = \Gamma(f,f), \tag{1.3}$$

with $f(0,x,v)=f_0(x,v)$. L is the linear part and $\Gamma(f,f)$ is the nonlinear part.

For the Boltzmann equation, the linearized collision operator $Lf = \nu(v)f - Kf$ and notice that $K = K_2 - K_1$ is given by [5, 9]:

$$\nu(v) = \int_{\mathbf{R}^3\times\mathbf{S}^2} |v-u|^\gamma \mu(u) B(\vartheta) du d\omega,$$

$$[K_1 g](v) = \int_{\mathbf{R}^3\times\mathbf{S}^2} |u-v|^\gamma \mu^{1/2}(u)\mu^{1/2}(v) g(u) B(\vartheta) du d\omega,$$

$$[K_2 g](v) = \int_{\mathbf{R}^3\times\mathbf{S}^2} |u-v|^\gamma \mu^{1/2}(u)[\mu^{1/2}(u')g(v') + \mu^{1/2}(v')g(u')] B(\vartheta) du d\omega.$$

Notice that for the soft potentials, $\nu(v)$ behaves like $(1+|v|)^\gamma$. The nonlinear collision operator $\Gamma(f,g) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \sqrt{\mu} g)$.

For the Landau equation, the linearized collision operator L is defined by $Lf = -Af - Kf$, where

$$Af = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \mu), \quad Kf = \frac{1}{\sqrt{\mu}} Q(\mu, \sqrt{\mu} f)$$

and the bilinear collision operator $\Gamma(f,g)$ is given by $\Gamma(f,g) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \sqrt{\mu} g)$.

As expected from the H-theorem, L is nonnegative and for every fixed (t,x) the null space of L is given by the five-dimensional space $(1 \le j \le 3)$

$$N = \text{span}\{1, v_j\sqrt{\mu}, |v|^2\sqrt{\mu}\}. \tag{1.4}$$

We define the orthogonal projection from $L^2(\mathbf{R}^3)$ onto the null space N by $\mathbf{P}$.

Notation. For notational simplicity, we shall use $\langle \cdot, \cdot \rangle$ to denote the standard L^2 inner product in $\mathbf{R}^3$ and $(\cdot,\cdot)$ to denote the standard L^2 inner product in $\mathbf{R}^3 \times \mathbf{R}^3$. For the Boltzmann equation, we introduce a weight function of v as $w_1(v) = [1+|v|]^\gamma$, while we introduce another weight function of v as $w_2(v) = [1+|v|]^{\gamma+2}$ for the Landau equation. We always denote the two weight functions as $w = w(v)$.

We denote the weighted L^2 norm as

$$|g|_{2,\theta}^2 \equiv \int_{\mathbf{R}^3} w^{2\theta} g^2 dv, \qquad \|g\|_{2,\theta}^2 \equiv \int_{\mathbf{R}^3\times\mathbf{R}^3} w^{2\theta} g^2 dv dx.$$

For the Boltzmann equation, define the weighted dissipation norm as

$$|g|_{\nu,\theta}^2 \equiv \int_{\mathbf{R}^3} \nu(v) w^{2\theta} g^2 dv, \qquad \|g\|_{\nu,\theta}^2 \equiv \int_{\mathbf{R}^3\times\mathbf{R}^3} \nu(v) w^{2\theta} g^2 dv dx.$$

For the Landau equation, define the weighted dissipation norm as

$$|g|_{\sigma,\theta}^2 = \sum_{i,j=1}^3 \int_{\mathbf{R}^3} w^{2\theta} \Big[\sigma^{ij}\partial_i g \partial_j g + \sigma^{ij} v_i v_j g^2\Big] dv,$$

$$\|g\|_{\sigma,\theta}^2 = \sum_{i,j=1}^3 \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} w^{2\theta} \Big[\sigma^{ij}\partial_i g \partial_j g + \sigma^{ij} v_i v_j g^2\Big] dv dx,$$

where $|\cdot|_\sigma = |\cdot|_{\sigma,0}$, $\|\cdot\|_\sigma = \|\cdot\|_{\sigma,0}$ and σ^{ij} is defined as

$$\sigma^{ij} = \int_{\mathbf{R}^3} \phi^{ij}(v-v')\mu(v') dv'.$$

We will unify the notation as $\|g\|_{w,\theta}$, which denotes either $\|g\|_{\nu,\theta}$ or $\|g\|_{\sigma,\theta}$ and if $\theta = 0$, $\|g\|_{w,0} = \|g\|_w$, which denotes either $\|g\|_\nu$ or $\|g\|_\sigma$. Let α and β be $\alpha = [\alpha_0, \alpha_1, \alpha_2, \alpha_3]$ and $\beta = [\beta_1, \beta_2, \beta_3]$. We denote

$$\partial_\beta^\alpha \equiv \partial_t^{\alpha_0} \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}.$$

If each component of β is not greater than that of β_1's, we denote it by $\beta \le \beta_1$. We define $\beta < \beta_1$ if $\beta \le \beta_1$, and $|\beta| < |\beta_1|$. We denote $\binom{\beta}{\beta}$ by C_{β}^{β}. Let

$$|||f|||_{|\beta|}^2 = \sum_{|\alpha|+|\beta|\le N} \|w^{|\beta|}\partial_\beta^\alpha f(t)\|^2,$$

$$|||f|||_{w,|\beta|}^2 = \sum_{|\beta|\le N} \|w^{|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|_w^2 + \sum_{|\alpha|+|\beta|\le N, |\alpha|\ne 0} \|w^{|\beta|}\partial_\beta^\alpha f(t)\|_w^2.$$

We next define the high order energy norm as

$$\mathcal{E}(f(t)) \equiv \frac{1}{2}|||f(t)|||_{|\beta|}^2 + \int_0^t |||f(s)|||_{w,|\beta|}^2 ds,$$

with the initial energy

$$\mathcal{E}(f_0) = \mathcal{E}(f(0)) \equiv \sum_{|\alpha|+|\beta|\le N} \|w^{|\beta|}\partial_\beta^\alpha f_0\|^2.$$

Throughout this paper, $N \ge 8$. The main results are stated as follows:

THEOREM 1.1. Let $F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v) \ge 0$. There exist $C_0 > 0$ and $M > 0$ such that if $\mathcal{E}(f_0) \le M$, then there exists a unique global classical solution $f(t,x,v)$ to (1.3). Moreover, $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \ge 0$ solves equation (1.1) and $\sup_{0\le s\le\infty} \mathcal{E}(f(s)) \le C_0 \mathcal{E}(f_0)$.

Although there is extensive mathematical literature for the Boltzmann theory (see [1, 4, 8, 9, 15, 21, 31, 27, 28, 35] and their references), much less is known for soft potentials $\gamma < 0$. Global smooth small-amplitude solutions of the Vlasov-Poisson-Boltzmann system near vacuum were constructed with $-3 \le \gamma < -2$ in [11]. Caflisch [4] obtained the global solutions of the Boltzmann equation near Maxwellians with $-1 < \gamma < 0$ in a periodic box by use of spectral analysis. Recently Guo [15] generalized the results in [4] to the cases $-3 < \gamma < 0$ by an energy method developed in [13, 14]. In the whole space, it was Ukai and Asano [28] who obtained the global solutions of the Boltzmann equation near Maxwellians with $-1 < \gamma < 0$. However, it is still an open problem to extend the results in [28] to the general cases. On the other hand, there have been some investigations on the global weak solutions for the inverse power law [8, 19], some even without any angular cutoff [1, 2, 29].

Despite its physical significance for the Landau equation, few global solutions have been constructed. Desvillettes and Villani [7] proved global existence and uniqueness of classical solutions for the spatially homogeneous Landau equation for hard potentials and a large class of initial data. Degond and Lemou [6] studied the spectral properties and dispersion relation of the linearized Landau operator. For the Coulomb interaction with $\gamma = -3$, global weak solutions have been studied in [2, 29, 30], up to some defect measures, while Guo [12] constructed global classical solutions near Maxwellians for a

general Landau equation (both hard and soft potentials) in a periodic box via an energy method. There are many other studies on the Landau equation (see [28-32]).

In the present paper, we obtain global classical solutions of the Boltzmann and Landau equations near Maxwellians in the whole space. Our results generalize the classical results of Ukai and Asano [28] about the global solutions of the Boltzmann equation in the whole space to the very soft potentials $-3 < \gamma < 0$. On the other hand, our results extend the classical results of Claflisch [4] and Guo [15] about the Boltzmann equation in the periodic box to the whole space. For the Landau equation in the Coulomb interaction $\gamma = -3$, global weak solutions have been studied in [20, 29] and global classical solutions have recently been constructed in the periodic box in [12], while no global classical solutions in whole space have been known. Therefore, our global existence results are new on the Landau equation near Maxwellians in the whole space.

Since the solutions to the linearized Boltzmann equation around μ have a slow time decay rate for γ near -3, which is impossible to be used in the nonlinear problem, instead of studying the linear problem like [4, 27, 28], we directly investigate the energy estimates for the nonlinear problem. Our construction of the global solution, almost positive definite for the solution to (1.3), is based on a recent nonlinear energy method developed in [14, 16, 21] with a new a priori estimate for the linearized collision operator L itself, not its time integration [13, 15, 12].

Although it has the same framework as [14, 16, 17], there are several major new difficulties in this paper. First, it is impossible to directly control v-derivatives ∂_β of linear streaming term $v_j\partial^j f$ in terms of the weaker $\|\partial_\beta f\|_w$. Thus we introduce the additional weighted function w in the energy norm like [12, 15] to overcome this difficulty. Since $\gamma < 0$, the estimates of the nonlinear term $\Gamma(g_1, g_2)$ in terms of the weighted norms for g_1 and g_2 are particularly delicate. To get rid of the singularity of $\Gamma(f, f)$, we first decompose $\Gamma(f, f)$ into

$$\Gamma(\mathbf{P}f, \mathbf{P}f) + \Gamma(\{\mathbf{I} - \mathbf{P}\}f, \mathbf{P}f) + \Gamma(\mathbf{P}f, \{\mathbf{I} - \mathbf{P}\}f) + \Gamma(\{\mathbf{I} - \mathbf{P}\}f, \{\mathbf{I} - \mathbf{P}\}f),$$

carefully estimate each part by using the exponential decay of the hydrodynamic part $\mathbf{P}f$ about the velocity variable v and we obtain the new estimates (Theorem 2.1 and Theorem 3.1), which are crucial to obtain the positivity of the linearized collision operator L and the global existence of solutions. A sequence of carefully chosen decompositions of various integration regions in [12, 15] is also used to overcome the singularity in the collision kernels.

Since the Landau collision operator contains the derivative of v, and the estimates of the nonlinear collision term $\Gamma(g_1, g_2)$ indeed need the differentiability of v, we are forced to take the v-derivative of (1.3). Unfortunately, the zero-th order hydrodynamic part $\|\partial_\beta \mathbf{P}f\|_\sigma$ appears, which is difficult to be controlled. Therefore, we first introduce the weighted norm $|||f|||_{w,|\beta|}$ which only excludes the $L^2(\mathbf{R}^3)$ norm of the zero-th order hydrodynamic part, compared to the norm $|||f|||_{|\beta|}$. Then we turn equation (1.3) into a microscopic-type equation (5.4) and use the technique of [21, 32] to overcome these difficulties. On the other hand, unlike the periodic box, the crucial Poincaré inequality does not hold for the problem in the whole space and we have to refine the energy analysis to obtain our results.

2. Boltzmann estimates. In this section, we will give some basic estimates used to obtain global existence of solutions for the Boltzmann equation. This section is mainly devoted to the estimates for the nonlinear collision term $\Gamma(g_1, g_2)$ with $g_i(x, v)$ $(i = 1, 2)$. In (1.3), the bilinear form $\Gamma(g_1, g_2)$ in the Boltzmann equation is

$$\begin{aligned}\Gamma(g_1, g_2) &= \mu^{-1/2}(v)Q(\mu^{1/2}g_1, \mu^{1/2}g_2) \equiv \Gamma_{gain}(g_1, g_2) - \Gamma_{loss}(g_1, g_2)\\ &= \int_{\mathbf{R}^3}\int_{\mathbf{S}^2} |u-v|^\gamma \mu^{1/2}(u) g_1(u') g_2(v') B(\vartheta) du d\omega\\ &\quad -\Big[\int_{\mathbf{R}^3}\int_{\mathbf{S}^2} |u-v|^\gamma \mu^{1/2}(u) g_1(u) B(\vartheta) du d\omega\Big] g_2(v).\end{aligned}$$

Notice that the change of variables $u - v \to u$ gives

$$\begin{aligned}\partial^\alpha_\beta \Gamma(g_1, g_2) &= \partial^\alpha_\beta\Big[\int_{\mathbf{R}^3}\int_{\mathbf{S}^2} |u|^\gamma \mu^{1/2}(u+v) g_1(v+u_\parallel) g_2(v+u_\perp) B(\vartheta) du d\omega\Big]\\ &\quad -\partial^\alpha_\beta\Big[\int_{\mathbf{R}^3}\int_{\mathbf{S}^2} |u|^\gamma \mu^{1/2}(u+v) g_1(v+u) g_2(v) B(\vartheta) du d\omega\Big]\\ &\equiv \sum C^{\beta_0\beta_1\beta_2}_\beta C^{\alpha_1\alpha_2}_\alpha \Gamma^0(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_1),\end{aligned}$$

where the summation is over $\beta_0 + \beta_1 + \beta_2 = \beta$ and $\alpha_1 + \alpha_2 = \alpha$. Here $u_\parallel = (u \cdot \omega)\omega$ and $u_\perp = u - u_\parallel$. By the product rule and a reverse change of variables, we have

$$\begin{aligned}\Gamma^0(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2) &= \int_{\mathbf{R}^3}\int_{\mathbf{S}^2} |u-v|^\gamma \partial_{\beta_0}[\mu^{1/2}(u)] \partial^{\alpha_1}_{\beta_1} g_1(u') \partial^{\alpha_2}_{\beta_2} g_2(v') B(\vartheta) du d\omega\\ &\quad -\partial^{\alpha_2}_{\beta_2} g_2(v) \int_{\mathbf{R}^3}\int_{\mathbf{S}^2} |u-v|^\gamma \partial_{\beta_0}[\mu^{1/2}(u)] \partial^{\alpha_1}_{\beta_1} g_1(u) B(\vartheta) du d\omega\\ &\equiv \Gamma^0_{gain}(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2) - \Gamma^0_{loss}(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2). \qquad (2.1)\end{aligned}$$

THEOREM 2.1. Recall (2.1) and let $\beta_0 + \beta_1 + \beta_2 = \beta$, $\alpha_1 + \alpha_2 = \alpha$ and $|\beta| \le \theta$. If $g_1(x, v) = a(x)\mu^{1/4}(v)$, then

$$|(w^{2\theta}\Gamma^0(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2), \partial^\alpha_\beta g_3)| \le C\int_{\mathbf{R}^3} |\partial^{\alpha_1} a(x)| |\partial^{\alpha_2}_{\beta_2} g_2(x)|_{\nu,\theta} |\partial^\alpha_\beta g_3(x)|_{\nu,\theta} dx. \qquad (2.2)$$

If $g_2(x, v) = a(x)\mu^{1/4}(v)$, then

$$|(w^{2\theta}\Gamma^0(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2), \partial^\alpha_\beta g_3)| \le C\int_{\mathbf{R}^3} |\partial^{\alpha_2} a(x)| |\partial^{\alpha_1}_{\beta_1} g_1(x)|_{\nu,\theta} |\partial^\alpha_\beta g_3(x)|_{\nu,\theta} dx. \qquad (2.3)$$

Proof. First consider the second term Γ^0_{loss} in (2.2). Note that $g_1(x, v) = a(x)\mu^{1/4}(v)$. Since $|\partial_{\beta_0}[\mu^{1/2}(u)]| \le Ce^{-|u|^2/4}$, then we have

$$\begin{aligned}&\int_{\mathbf{R}^3} |u-v|^\gamma \partial_{\beta_0}[\mu^{1/2}(u)] \partial^{\alpha_1}_{\beta_1} g_1(x, u) du\\ \le\ & C\Big\{\int_{\mathbf{R}^3} e^{-|u|^2/4} |\partial^{\alpha_1}_{\beta_1} g_1(x, u)|^2 du\Big\}^{1/2} \Big\{\int_{\mathbf{R}^3} |u-v|^{2\gamma} e^{-|u|^2/4} du\Big\}^{1/2}\\ \le\ & C\Big\{\int_{\mathbf{R}^3} e^{-|u|^2/4} |\partial_{\beta_1} e^{-|u|^2/4}|^2 |\partial^{\alpha_1} a(x)|^2 du\Big\}^{1/2} \times [1 + |v|]^\gamma\\ \le\ & C|\partial^{\alpha_1} a(x)| \times [1 + |v|]^\gamma.\end{aligned}$$

Therefore, by $0 < B(\vartheta) \leq C$, we have

$$\begin{aligned}
& |(w^{2\theta}\Gamma^0_{loss}(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2), \partial^{\alpha}_{\beta} g_3)| \\
\leq\ & C\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}[1+|v|]^{\gamma} w^{2\theta}|\partial^{\alpha_1} a(x)||\partial^{\alpha_2}_{\beta_2} g_2(x,v)\partial^{\alpha}_{\beta} g_3(x,v)|dxdv \\
\leq\ & C\int_{\mathbf{R}^3}|\partial^{\alpha_1} a(x)||w^{\theta}\partial^{\alpha_2}_{\beta_2} g_2(x)|_{\nu}|w^{\theta}\partial^{\alpha}_{\beta} g_3(x)|_{\nu}dx.
\end{aligned}$$

Then we consider the second term Γ^0_{loss} in (2.3). Since $|\partial_{\beta_0}[\mu^{1/2}(u)]| \leq Ce^{-|u|^2/4}$, then we have

$$\begin{aligned}
& \int_{\mathbf{R}^3}|u-v|^{\gamma}\partial_{\beta_0}[\mu^{1/2}(u)]\partial^{\alpha_1}_{\beta_1} g_1(x,u)du \\
\leq\ & C\Big\{\int_{\mathbf{R}^3} e^{-|u|^2/4}|\partial^{\alpha_1}_{\beta_1} g_1(x,u)|^2 du\Big\}^{1/2}\Big\{\int_{\mathbf{R}^3}|u-v|^{2\gamma}e^{-|u|^2/4}du\Big\}^{1/2} \\
\leq\ & C|e^{-|u|^2/8}\partial^{\alpha_1}_{\beta_1} g_1(x)|_2 \times [1+|v|]^{\gamma}.
\end{aligned}$$

Note that $g_2(x,v) = a(x)\mu^{1/4}(v)$. By $|\partial_{\beta_2}[\mu^{1/4}(v)]| \leq Ce^{-|v|^2/8}$, we have

$$\begin{aligned}
& |(w^{2\theta}\Gamma^0_{loss}(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2), \partial^{\alpha}_{\beta} g_3)| \\
\leq\ & C\int_{\mathbf{R}^3}|\partial^{\alpha_2} a(x)||e^{-|u|^2/8}\partial^{\alpha_1}_{\beta_1} g_1(x)|_2 dx\int_{\mathbf{R}^3}[1+|v|]^{\gamma}w^{2\theta}e^{-|v|^2/8}|\partial^{\alpha}_{\beta} g_3(x,v)|dv \\
\leq\ & C\int_{\mathbf{R}^3}|\partial^{\alpha_2} a(x)||e^{-|u|^2/8}\partial^{\alpha_1}_{\beta_1} g_1(x)|_2|w^{\theta}\partial^{\alpha}_{\beta} g_3(x)|_{\nu}dx \\
\leq\ & C\int_{\mathbf{R}^3}|\partial^{\alpha_2} a(x)|w^{\theta}\partial^{\alpha_1}_{\beta_1} g_1(x)|_{\nu}|w^{\theta}\partial^{\alpha}_{\beta} g_3(x)|_{\nu}dx,
\end{aligned}$$

where we have used $0 < B(\vartheta) \leq C$ and the exponential decay of $e^{-|u|^2/8}$.

The next step is to estimate the first term Γ^0_{gain} in (2.2). Here the (u,v) integration domain is split into two parts:

$$\{|u| \geq |v|/2\} \cup \{|u| \leq |v|/2\}.$$

For the first region $\{|u| \geq |v|/2\}$, we have

$$e^{-|u|^2/4} \leq e^{-|u|^2/8}e^{-|v|^2/32}.$$

Then the integral of $w^{2\theta}\Gamma^0_{gain}(\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2)\partial^\alpha_\beta g_3$ over such a region is bounded by

$$\begin{aligned}
&\int_{|u|\ge|v|/2} B(\vartheta)|u-v|^\gamma e^{-|u|^2/8}e^{-|v|^2/32}w^{2\theta}|\partial^{\alpha_1}_{\beta_1}g_1(x,u')\partial^{\alpha_2}_{\beta_2}g_2(x,v')\partial^\alpha_\beta g_3(x,v)|d\omega dudv\\
&\le C\Big\{\int B(\vartheta)|u-v|^\gamma e^{-|u|^2/8}e^{-|v|^2/32}w^{2\theta}|\partial^{\alpha_1}_{\beta_1}g_1(x,u')|^2|\partial^{\alpha_2}_{\beta_2}g_2(x,v')|^2d\omega dudv\Big\}^{1/2}\\
&\quad\times\Big\{\int B(\vartheta)|u-v|^\gamma e^{-|u|^2/8}e^{-|v|^2/32}w^{2\theta}|\partial^\alpha_\beta g_3(x,v)|^2d\omega dudv\Big\}^{1/2}\\
&\le C|\partial^{\alpha_1}a||w^\theta\partial^\alpha_\beta g_3(x)|_\nu\\
&\quad\times\Big\{\int B(\vartheta)|u-v|^\gamma e^{-|u|^2/8}e^{-|v|^2/32}e^{-|u'|^2/8}w^{2\theta}|\partial^{\alpha_2}_{\beta_2}g_2(x,v')|^2d\omega dudv\Big\}^{1/2}\\
&\le C|\partial^{\alpha_1}a||w^\theta\partial^\alpha_\beta g_3(x)|_\nu\Big\{\int B(\vartheta)|u'-v'|^\gamma e^{-|u'|^2/32-|v'|^2/32}|\partial^{\alpha_2}_{\beta_2}g_2(x,v')|^2d\omega dudv\Big\}^{1/2}\\
&\le C|\partial^{\alpha_1}a||w^\theta\partial^\alpha_\beta g_3(x)|_\nu\Big\{\int B(\vartheta)|u-v|^\gamma e^{-|u|^2/32-|v|^2/32}|\partial^{\alpha_2}_{\beta_2}g_2(x,v)|^2d\omega dudv\Big\}^{1/2}\\
&\le C|\partial^{\alpha_1}a||w^\theta\partial^{\alpha_2}_{\beta_2}g_2(x)|_\nu|w^\theta\partial^\alpha_\beta g_3(x)|_\nu,
\end{aligned}$$

where we have used the change of variables $(u,v)\to(u',v')$, $B(\vartheta)\le C$ and $w^{2\theta}\le 1$. Similar arguments imply that if $g_2(x,v)=a(x)\mu^{1/4}(v)$, $\langle w^{2\theta}\Gamma^0_{gain}(\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2),\partial^\alpha_\beta g_3\rangle$ over such a region is bounded by

$$C|\partial^{\alpha_2}a(x)||w^\theta\partial^{\alpha_1}_{\beta_1}g_1(x)|_\nu|w^\theta\partial^\alpha_\beta g_3(x)|_\nu.$$

Now we turn to the gain term over $\{|u|\le|v|/2\}$. Since $|u|\le|v|/2$ implies $|v-u|\ge|v|-|u|\ge|v|/2$, we have, from $\gamma<0$, that

$$|u-v|^\gamma\le 2^{-\gamma}|v|^\gamma.$$

Further assume $|v|\le 1$; then $|u|\le 1/2$ and the gain term is bounded by

$$\begin{aligned}
&\int_{|v|\le1,|u|\le|v|/2} B(\vartheta)|u-v|^\gamma e^{-|u|^2/4}w^{2\theta}|\partial^{\alpha_1}_{\beta_1}g_1(x,u')\partial^{\alpha_2}_{\beta_2}g_2(x,v')\partial^\alpha_\beta g_3(x,v)|d\omega dudv\\
\le\ & C\int_{|v|\le1} w^{2\theta}|\partial^\alpha_\beta g_3(x,v)|dv\\
&\times\Big\{\int_{|u|\le1/2} B(\vartheta)|u-v|^\gamma e^{-|u|^2/4}|\partial^{\alpha_1}_{\beta_1}g_1(x,u')\partial^{\alpha_2}_{\beta_2}g_2(x,v')|d\omega du\Big\}\\
\le\ & C\int_{|v|\le1} w^{2\theta}|\partial^\alpha_\beta g_3(x,v)|dv\Big\{\int_{|u|\le1/2}|u-v|^\gamma e^{-|u|^2/2}du\Big\}^{1/2}\\
&\times\Big\{\int_{|u|\le1/2} B(\vartheta)|v|^\gamma|\partial^{\alpha_1}_{\beta_1}g_1(x,u')|^2|\partial^{\alpha_2}_{\beta_2}g_2(x,v')|^2dud\omega\Big\}^{1/2}\\
\le\ & C\Big\{\int_{|u|\le1/2,|v|\le1} B(\vartheta)|v|^\gamma|\partial^{\alpha_1}_{\beta_1}g_1(x,u')|^2|\partial^{\alpha_2}_{\beta_2}g_2(x,v')|^2dudvd\omega\Big\}^{1/2}|\omega^\theta\partial^\alpha_\beta g_3(x)|_\nu.
\end{aligned}$$

We now estimate the first factor. Since $|u|\le|v|/2$, we have

$$|u'|+|v'|\le 2[|u|+|v|]\le 3|v|.$$

Since $\gamma < 0$, this implies that

$$|v|^\gamma \le 3^{-\gamma}|u'|^\gamma, \quad |v|^\gamma \le 3^{-\gamma}|v'|^\gamma. \tag{2.4}$$

$$\int_{|u|\le 1/2, |v|\le 1} B(\vartheta)|v|^\gamma |\partial^{\alpha_1}_{\beta_1} g_1(x,u')|^2 |\partial^{\alpha_2}_{\beta_2} g_2(x,v')|^2 dudvd\omega$$
$$\le C\int_{|u'|\le 3, |v'|\le 3} B(\vartheta)\min[|u'|^\gamma, |v'|^\gamma]|\partial^{\alpha_1}_{\beta_1} g_1(x,u')|^2 |\partial^{\alpha_2}_{\beta_2} g_2(x,v')|^2 dudvd\omega.$$

Now change variables $(v,u)\to(v',u')$ so that the above is bounded by

$$C\int_{|u|\le 3, |v|\le 3} B(\vartheta)\min[|u|^\gamma, |v|^\gamma]e^{-|u|^2/8}|\partial^{\alpha_1} a(x)|^2 |\partial^{\alpha_2}_{\beta_2} g_2(x,v)|^2 dudvd\omega$$
$$\le C|\partial^{\alpha_1} a(x)|^2 \int_{|v|\le 3} |\partial^{\alpha_2}_{\beta_2} g_2(x,v)|^2 dv \le C|\partial^{\alpha_1} a(x)|^2 |w^\theta \partial^{\alpha_2}_{\beta_2} g_2(x)|^2_\nu.$$

Thus, $\langle w^{2\theta}\Gamma^0_{gain}(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2), \partial^\alpha_\beta g_3\rangle$ over $\{|u|\le |v|/2, |v|\le 1\}$ is bounded by

$$C|\partial^{\alpha_1} a(x)||w^\theta \partial^{\alpha_2}_{\beta_2} g_2(x)|_\nu |\omega^\theta \partial^\alpha_\beta g_3(x)|_\nu.$$

If $g_2(x,v) = a(x)\mu^{1/4}(v)$, by similar arguments, $\langle w^{2\theta}\Gamma^0_{gain}(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2), \partial^\alpha_\beta g_3\rangle$ over such a region is bounded by

$$C|\partial^{\alpha_2} a(x)||w^\theta \partial^{\alpha_1}_{\beta_1} g_1(x)|_\nu |w^\theta \partial^\alpha_\beta g_3(x)|_\nu.$$

The last case is the gain term over the region $\{|u|\le |v|/2, |v|\ge 1\}$. The integral of $\langle w^{2\theta}\Gamma^0_{gain}(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2), \partial^\alpha_\beta g_3\rangle$ over such a region is bounded by

$$\int_{|v|\ge 1, |u|\le |v|/2} B(\vartheta)|u-v|^\gamma e^{-|u|^2/4} w^{2\theta}|\partial^{\alpha_1}_{\beta_1} g_1(x,u')\partial^{\alpha_2}_{\beta_2} g_2(x,v')\partial^\alpha_\beta g_3(x,v)| d\omega dudv$$
$$\le C\Big\{\int_{|v|\ge 1, |u|\le |v|/2} B(\vartheta)[1+|v|]^\gamma e^{-|u|^2/4} w^{2\theta}|\partial^{\alpha_1}_{\beta_1} g_1(x,u')|^2 |\partial^{\alpha_2}_{\beta_2} g_2(x,v')|^2 dudvd\omega\Big\}^{1/2}$$
$$\times\Big\{\int_{|v|\ge 1, |u|\le |v|/2} B(\vartheta)[1+|v|]^\gamma e^{-|u|^2/4} w^{2\theta}|\partial^\alpha_\beta g_3(x,v)|^2 d\omega dudv\Big\}^{1/2}.$$

Here we have used $|u-v|^\gamma \le 4^{-\gamma}[1+|v|]^\gamma$ in the above inequality.

Recalling (2.4), $w^{2\theta}(v) \le C\min[w^{2\theta}(u'), w^{2\theta}(v')]$ for $\theta \ge 0$. Using the change of variables $(v,u)\to(v',u')$, the first factor is bounded by

$$\begin{aligned}
&\int_{|v|\ge 1, |u|\le |v|/2} B(\vartheta)[1+|v|]^\gamma e^{-|u|^2/4} w^{2\theta}|\partial^{\alpha_1}_{\beta_1} g_1(x,u')|^2 |\partial^{\alpha_2}_{\beta_2} g_2(x,v')|^2 dudvd\omega \\
\le\ & C\int B(\vartheta)[1+|v'|]^\gamma \min[w^{2\theta}(u'), w^{2\theta}(v')]|\partial^{\alpha_1}_{\beta_1} g_1(x,u')|^2 |\partial^{\alpha_2}_{\beta_2} g_2(x,v')|^2 dudvd\omega \\
\le\ & C\int B(\vartheta)[1+|v|]^\gamma \min[w^{2\theta}(u), w^{2\theta}(v)]|\partial^{\alpha_1}_{\beta_1} g_1(x,u)|^2 |\partial^{\alpha_2}_{\beta_2} g_2(x,v)|^2 dudvd\omega \\
\le\ & C|\partial^{\alpha_1} a(x)|^2 |w^\theta \partial^{\alpha_2}_{\beta_2} g_2|^2_\nu.
\end{aligned}$$

Therefore, the integral of $\langle w^{2\theta}\Gamma^0_{gain}[\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2], \partial^\alpha_\beta g_3\rangle$ over $\{|u|\le |v|/2, |v|\ge 1\}$ is bounded by

$$C|\partial^{\alpha_1} a(x)||w^\theta \partial^{\alpha_2}_{\beta_2} g_2|_\nu |w^\theta \partial^\alpha_\beta g_3|_\nu.$$

If $g_2(x,v)=a(x)\mu^{1/4}(v)$, by similar arguments, $\langle w^{2\theta}\Gamma^0_{gain}(\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2),\partial^\alpha_\beta g_3\rangle$ over such a region is bounded by

$$C|\partial^{\alpha_2}a||w^\theta\partial^{\alpha_1}_{\beta_1}g_1(x)|_\nu|w^\theta\partial^\alpha_\beta g_3(x)|_\nu.$$

We deduce (2.2) and (2.3) by further integrating over $\mathbf{R}^3$ about the x variable.

COROLLARY 2.2. Let $|\alpha|+|\beta|\le N$ and $|\beta|\le\theta$. If $g_1(x,v)$ belongs to the null space N of L, namely, $g_1(x,v)=a(x)\sqrt{\mu}+\sum_{j=1}^3 b_j(x)v_j\sqrt{\mu}+c(x)|v|^2\sqrt{\mu}$, then

$$\begin{aligned}|(w^{2\theta}\partial^\alpha_\beta\Gamma(g_1,g_2),\partial^\alpha_\beta g_3)|\le C\sum\int_{\mathbf{R}^3}&\Big[|\partial^{\alpha_1}a(x)|+|\partial^{\alpha_1}b_j(x)|+|\partial^{\alpha_1}c(x)|\Big]\\&\times|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(x)|_{\nu,\theta}|\partial^\alpha_\beta g_3(x)|_{\nu,\theta}dx,\end{aligned}\tag{2.5}$$

and if $g_2(x,v)=a(x)\sqrt{\mu}+\sum_{j=1}^3 b_j(x)v_j\sqrt{\mu}+c(x)|v|^2\sqrt{\mu}$, then

$$\begin{aligned}|(w^{2\theta}\partial^\alpha_\beta\Gamma(g_1,g_2),\partial^\alpha_\beta g_3)|\le C\sum\int_{\mathbf{R}^3}&\Big[|\partial^{\alpha-\alpha_1}a(x)|+|\partial^{\alpha-\alpha_1}b_j(x)|+|\partial^{\alpha-\alpha_1}c(x)|\Big]\\&\times|\partial^{\alpha_1}_{\beta_1}g_1(x)|_{\nu,\theta}|\partial^\alpha_\beta g_3(x)|_{\nu,\theta}dx,\end{aligned}\tag{2.6}$$

where the summation is over j, $|\alpha_1|+|\beta_1|\le N$ and $\beta_1\le\beta$.

LEMMA 2.3. Let $0<|\alpha_1|+|\alpha_2|=|\alpha|\le N$ and $h(v)$ be a smooth function so that $|h(v)|+|\nabla h(v)|+|\nabla^2h(v)|\le C\mu^{1/4}(v)$; then we have

$$\Big|\langle\Gamma(g_1,g_2),h\rangle\Big|_2\le\begin{cases}C\Big[\sum_{|\alpha_i|+|\beta_i|\le N}\|w^{|\beta_i|}\partial^{\alpha_i}_{\beta_i}g_1\|\Big]\|g_2\|_\nu,\\ C\Big[\sum_{|\alpha_i|+|\beta_i|\le N}\|w^{|\beta_i|}\partial^{\alpha_i}_{\beta_i}g_2\|\Big]\|g_1\|_\nu,\end{cases}\tag{2.7}$$

$$\begin{aligned}&\Big|\langle\Gamma(\partial^{\alpha_1}g_1,\partial^{\alpha_2}g_2),h\rangle\Big|_2\\ &\le\begin{cases}C\Big[\sum_{|\alpha_i|+|\beta_i|\le N}\|w^{|\beta_i|}\partial^{\alpha_i}_{\beta_i}g_1\|\Big]\|\partial^{\alpha_2}g_2\|_\nu, & \text{if } |\alpha_1|\le\frac{N}{2};\\ C\Big[\sum_{|\alpha_i|+|\beta_i|\le N}\|w^{|\beta_i|}\partial^{\alpha_i}_{\beta_i}g_2\|\Big]\|\partial^{\alpha_1}g_1\|_\nu, & \text{if } |\alpha_1|>\frac{N}{2}.\end{cases}\end{aligned}\tag{2.8}$$

Proof. Notice that for the Boltzmann equation,

$$\begin{aligned}\Gamma(g_1,g_2)&=\int_{\mathbf{R}^3\times\mathbf{S}^2}|u-v|^\gamma\mu^{1/2}(u)g_1(u')g_2(v')B(\vartheta)dud\omega\\&\quad-\Big[\int_{\mathbf{R}^3\times\mathbf{S}^2}|u-v|^\gamma\mu^{1/2}(u)g_1(u)B(\vartheta)dud\omega\Big]g_2(v)\\&\equiv\Gamma^0_{gain}(g_1,g_2)-\Gamma^0_{loss}(g_1,g_2).\end{aligned}$$

We first estimate the loss term $\Gamma_{loss}(g_1, g_2)$. The term $\Gamma_{loss}(g_1, g_2)$ is estimated as

$$
\begin{aligned}
&|\langle \Gamma_{loss}(g_1, g_2), h\rangle| \\
&\le C \int_{\mathbf{R}^3} |g_2(v)h(v)| \int_{\mathbf{R}^3} |u-v|^\gamma e^{-|u|^2/2} |g_1(u)| du dv \\
&\le C \int_{\mathbf{R}^3} |g_2(v)h(v)| dv \Big\{ \int_{\mathbf{R}^3} |u-v|^{2\gamma} e^{-|u|^2/2} du \Big\}^{1/2} \Big\{ \int_{\mathbf{R}^3} e^{-|u|^2/2} |g_1(u)|^2 du \Big\}^{1/2} \\
&\le C \Big\{ \int_{\mathbf{R}^3} e^{-|u|^2/2} |g_1(u)|^2 du \Big\}^{1/2} \int_{\mathbf{R}^3} [1+|v|]^\gamma |g_2(v)| e^{-|v|^2/4} dv \\
&\le C \Big\{ \int_{\mathbf{R}^3} e^{-|u|^2/2} |g_1(u)|^2 du \Big\}^{1/2} \Big\{ \int_{\mathbf{R}^3} e^{-|v|^2/4} |g_2(v)|^2 dv \Big\}^{1/2},
\end{aligned} \tag{2.9}
$$

where we have used the Cauchy-Schwartz inequality and $h(v) \le Ce^{-|v|^2/4}$.

By $H^4(\mathbf{R}^3 \times \mathbf{R}^3) \subset L^\infty(\mathbf{R}^3 \times \mathbf{R}^3)$, it follows that

$$
\sup_{x,u} \Big\{ |e^{-|u|^2/8} |g_1(x,u)|^2 \Big\} \le C \sum_{|\alpha_i|+|\beta_i| \le N} \|w^{|\beta_i|} \partial^{\alpha_i}_{\beta_i} g_1\|^2. \tag{2.10}
$$

Using the exponential factor, integrating over u and taking the L^2 norm in x for the last factor of (2.9) yields the first inequality of $\Gamma_{loss}(g_1, g_2)$ of (2.7). By a similar argument, we can obtain the second inequality of $\Gamma_{loss}(g_1, g_2)$ of (2.7).

For the gain term, since $|h(v)| \le Ce^{-|v|^2/4}$ and $0 < B(\vartheta) \le C$, we have

$$
\begin{aligned}
&|\langle \Gamma_{gain}(g_1, g_2), h\rangle| \\
&\le C \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} B(\vartheta) |u-v|^\gamma |g_1(u')| |g_2(v')h(v)| e^{-|u|^2/2} du dv d\omega \\
&\le C \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} \Big\{ \int_{\mathbf{R}^3} |u-v|^\gamma e^{-|u|^2/2} |g_1(u')|^2 |g_1(v')|^2 du \Big\}^{1/2} \\
&\qquad \times \Big\{ \int_{\mathbf{R}^3} |u-v|^\gamma e^{-|u|^2/2} du \Big\}^{1/2} B(\vartheta) |h(v)| dv d\omega \\
&\le C \Big\{ \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} B(\vartheta) |u-v|^\gamma e^{-|u|^2/2} |h(v)| |g_1(u')|^2 |g_2(v')|^2 du dv d\omega \Big\}^{1/2} \\
&\qquad \times \Big\{ \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} B(\vartheta) |h(v)| [1+|v|]^\gamma dv d\omega \Big\}^{1/2} \\
&\le C \Big\{ \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} B(\vartheta) |u-v|^\gamma e^{-|u'|^2/4 - |v'|^2/4} |g_1(u')|^2 |g_2(v')|^2 du dv d\omega \Big\}^{1/2} \\
&\le C \Big\{ \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} B(\vartheta) |u-v|^\gamma e^{-|u|^2/4 - |v|^2/4} |g_1(u)|^2 |g_2(v)|^2 du dv d\omega \Big\}^{1/2},
\end{aligned}
$$

where we have used the change of variables $(u', v') \to (u, v)$. To establish the first inequality of (2.7), we have, from the above inequality, that

$$\begin{aligned}
\Big|\langle \Gamma_{gain}(g_1, g_2), h\rangle\Big| &\le C \sup_{x,u} \Big\{ |e^{-|u|^2/8} |g_1(x,u)|^2 \Big\}^{1/2} \\
&\quad \times \Big\{ \int_{\mathbf{R}^3} e^{-|v|^2/4} |g_2(v)|^2 dv \int_{\mathbf{R}^3} |u-v|^\gamma e^{-|u|^2/8} du \Big\}^{1/2} \\
&\le C \sup_{x,u} \Big\{ |e^{-|u|^2/8} |g_1(x,u)|^2 \Big\}^{1/2} \Big\{ \int_{\mathbf{R}^3} [1+|v|]^\gamma |g_2(v)|^2 dv dx \Big\}^{1/2} \\
&\le C \sum_{|\alpha_i|+|\beta_i| \le N} \|w^{|\beta_i|} \partial^{\alpha_i}_{\beta_i} g_1\| |g_2|_\nu,
\end{aligned}$$

where we have used (2.10). We deduce the first inequality of (2.7) by taking the L^2 norm in x. Similar arguments to the above imply that the second inequality of (2.7) holds. The proof of (2.8) can be found in [15].

Next, we borrow two lemmas from [15] for the completeness of the paper.

LEMMA 2.4. There is some constant $C > 0$ such that

$$|\langle K g_1, g_2\rangle| \le C |g_1|_\nu |g_2|_\nu. \tag{2.11}$$

Let $|\beta| > 0$. For any $\eta > 0$, there exists $C_\eta > 0$ such that

$$\langle w^{2\theta} \partial_\beta [Lg], \partial_\beta g\rangle \ge |w^\theta \partial_\beta g|^2_\nu - \eta \sum_{|\beta_1| \le |\beta|} |w^\theta \partial_{\beta_1} g|^2_\nu - C_\eta |w^\theta g|^2_\nu. \tag{2.12}$$

LEMMA 2.5. Recall (2.1) and let $\beta_0 + \beta_1 + \beta_2 = \beta$, $\alpha_1 + \alpha_2 = \alpha$ and $|\beta| \le \theta$. If $|\alpha_1| + |\beta_1| \le N/2$, then

$$\begin{aligned}
&|(w^{2\theta} \Gamma^0 (\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2), \partial^\alpha_\beta g_3)| \\
\le\ & C \Big[\sum_{|\alpha_i|+|\beta_i| \le N} \|w^{|\beta_i|} \partial^{\alpha_i}_{\beta_i} g_1\| \Big] \|w^\theta \partial^{\alpha_2}_{\beta_2} g_2\|_\nu \|w^\theta \partial^\alpha_\beta g_3\|_\nu \\
&+ C \Big[\sum_{|\alpha_i|+|\beta_1| \le N} \|w^{|\beta_1|} \partial^{\alpha_i}_{\beta_1} g_1\| \Big] \|w^{\theta - |\beta_1|} \partial^{\alpha_2}_{\beta_2} g_2\|_\nu \|w^\theta \partial^\alpha_\beta g_3\|_\nu.
\end{aligned} \tag{2.13}$$

If $|\alpha_1| + |\beta_1| \ge N/2$, then

$$\begin{aligned}
&|(w^{2\theta} \Gamma^0 (\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2), \partial^\alpha_\beta g_3)| \\
\le\ & C \Big[\sum_{|\alpha_i|+|\beta_i| \le N} \|w^{|\beta_i|} \partial^{\alpha_i}_{\beta_i} g_2\| \Big] \|w^\theta \partial^{\alpha_1}_{\beta_1} g_1\|_\nu \|w^\theta \partial^\alpha_\beta g_3\|_\nu \\
&+ C \Big[\sum_{|\alpha_i|+|\beta_2| \le N} \|w^{|\beta_2|} \partial^{\alpha_i}_{\beta_2} g_2\| \Big] \|w^{\theta - |\beta_2|} \partial^{\alpha_1}_{\beta_1} g_1\|_\nu \|w^\theta \partial^\alpha_\beta g_3\|_\nu.
\end{aligned} \tag{2.14}$$

3. Landau estimates. In this section, we will prove the basic estimates used to obtain the positivity of the linearized Landau operator and global existence of solutions for the Landau equation. In this case, we need to exercise more care in estimating the nonlinear collision term $\Gamma(f, g)$, which is crucial in the later energy analysis.

The operators A, K and Γ from (1.3) in the Landau equation are defined in [12] as the following representations:

$$Ag_1 = \partial_i\Big[\sigma^{ij}\partial_j g_1\Big] - \sigma^{ij}v_iv_jg_1 + \partial_i\Big[v_j\sigma^{ij}\Big]g_1, \tag{3.1}$$

$$Kg_2 = -\mu^{-1/2}\partial_i\Big\{\mu\Big[\phi^{ij} * \Big\{\mu^{1/2}[\partial_j g_2 + v_j g_2]\Big\}\Big]\Big\}, \tag{3.2}$$

$$\begin{aligned}\Gamma(g_1,g_2) = {}& \partial_i\Big[\Big\{\phi^{ij} * \Big[\mu^{1/2}g_1\Big]\Big\}\partial_j g_2\Big] - \Big\{\phi^{ij} * \Big[v_i\mu^{1/2}g_1\Big]\Big\}\partial_j g_2 \\ & -\partial_i\Big[\Big\{\phi^{ij} * \Big[\mu^{1/2}\partial_j g_1\Big]\Big\}g_2\Big] + \Big\{\phi^{ij} * \Big[v_i\mu^{1/2}\partial_j g_1\Big]\Big\}g_2.\end{aligned} \tag{3.3}$$

For any vector-valued function $g(v) = (g_1, g_2, g_3)$, we define the projection to the vector v as

$$\mathbf{P}_v g_i \equiv \frac{v_i}{|v|}\sum_{j=1}^{3}\frac{v_j}{|v|}g_j, \qquad 1 \le i \le 3. \tag{3.4}$$

By Corollary 1 of [12], there exists some constant $c > 0$ such that

$$\Big|[1+|v|]^{\frac{\gamma}{2}}\{\mathbf{P}_v\partial_i g\}\Big|_{2,\theta}^2 + \Big|[1+|v|]^{\frac{\gamma+2}{2}}\{[\mathbf{I}-\mathbf{P}_v]\partial_i g\}\Big|_{2,\theta}^2 + \Big|[1+|v|]^{\frac{\gamma+2}{2}}g\Big|_{2,\theta}^2 \le c|g|_{\sigma,\theta}^2. \tag{3.5}$$

THEOREM 3.1. Recall (3.3) and let $|\alpha| + |\beta| \le N$ and $|\beta| \le \theta$. If $g_1(x,v) = a(x)\mu^{1/4}(v)$, then

$$|\langle w^{2\theta}\partial_\beta^\alpha\Gamma(g_1,g_2), \partial_\beta^\alpha g_3\rangle| \le C\sum|\partial^{\alpha_1}a(x)||w^\theta\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(x)|_\sigma|w^\theta\partial_\beta^\alpha g_3(x)|_\sigma. \tag{3.6}$$

If $g_2(x,v) = a(x)\mu^{1/4}(v)$, then

$$|\langle w^{2\theta}\partial_\beta^\alpha\Gamma(g_1,g_2), \partial_\beta^\alpha g_3\rangle| \le C\sum|\partial^{\alpha-\alpha_1}a(x)||w^\theta\partial_{\beta_1}^{\alpha_1}g_1(x)|_\sigma|w^\theta\partial_\beta^\alpha g_3(x)|_\sigma, \tag{3.7}$$

where the summation is over $|\alpha_1| + |\beta_1| \le N$, $\beta_1 \le \beta$.

Proof. By the product rule, we expand

$$\langle w^{2\theta}\partial_\beta^\alpha\Gamma(g_1,g_2), \partial_\beta^\alpha g_3\rangle = \sum C_\alpha^{\alpha_1}C_\beta^{\beta_1} \times G_{\alpha_1\beta_1},$$

where $G_{\alpha_1\beta_1}$ takes the form:

$$-\Big\langle w^{2\theta}\Big\{\phi^{ij} * \partial_{\beta_1}\Big[\mu^{1/2}\partial^{\alpha_1}g_1\Big]\Big\}\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2, \partial_i\partial_\beta^\alpha g_3\Big\rangle \tag{3.8}$$

$$-\Big\langle w^{2\theta}\Big\{\phi^{ij} * \partial_{\beta_1}\Big[v_i\mu^{1/2}\partial^{\alpha_1}g_1\Big]\Big\}\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2, \partial_\beta^\alpha g_3\Big\rangle \tag{3.9}$$

$$+\Big\langle w^{2\theta}\Big\{\phi^{ij} * \partial_{\beta_1}\Big[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1\Big]\Big\}\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2, \partial_i\partial_\beta^\alpha g_3\Big\rangle \tag{3.10}$$

$$+\Big\langle w^{2\theta}\Big\{\phi^{ij} * \partial_{\beta_1}\Big[v_i\mu^{1/2}\partial_j\partial^{\alpha_1}g_1\Big]\Big\}\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2, \partial_\beta^\alpha g_3\Big\rangle \tag{3.11}$$

$$-\Big\langle \partial_i\Big[w^{2\theta}\Big]\Big\{\phi^{ij} * \partial_{\beta_1}\Big[\mu^{1/2}\partial^{\alpha_1}g_1\Big]\Big\}\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2, \partial_\beta^\alpha g_3\Big\rangle \tag{3.12}$$

$$+\Big\langle \partial_i\Big[w^{2\theta}\Big]\Big\{\phi^{ij} * \partial_{\beta_1}\Big[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1\Big]\Big\}\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2, \partial_\beta^\alpha g_3\Big\rangle. \tag{3.13}$$

The last two terms appear when we integrate by parts over the v_i variable.

We first establish (3.7). Since $\phi^{ij} \in L^2_{loc}(\mathbf{R}^3)$ and $|\partial_{\beta_1-\overline{\beta}}[\mu^{1/2}]| \le C\mu^{1/4}$, we deduce that

$$\begin{aligned}\Big\{\phi^{ij} * \partial_{\beta_1}\Big[\mu^{1/2}\partial^{\alpha_1} g_1\Big]\Big\} &\le C\Big[|\phi^{ij}|^2 * \mu^{1/4}\Big]^{1/2} \sum_{\overline{\beta}\le\beta_1} \Big|\mu^{1/8}\partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_2 \\ &\le C[1+|v|]^{\gamma+2} \sum_{\overline{\beta}\le\beta_1} \Big|w^\theta \partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_\sigma, \end{aligned} \tag{3.14}$$

where we have used (3.5) and the exponential decay of $\mu^{1/8}(v)$.

Since $g_2(x,v) = a(x)\mu^{1/4}(v)$, (3.8) is bounded by

$$\begin{aligned} & C\sum_{\overline{\beta}\le\beta_1} \Big|w^\theta \partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_\sigma \int w^{2\theta}[1+|v|]^{\gamma+2}\Big|\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 \partial_i \partial^\alpha_\beta g_3\Big| dv \\ \le\ & C\sum_{\overline{\beta}\le\beta_1} \Big|w^\theta \partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_\sigma \Big|\partial^{\alpha-\alpha_1} a(x)\Big| \Big|w^\theta [1+|v|]^{\frac{\gamma+2}{2}} \mu^{1/16}\Big|_2 \Big|w^\theta [1+|v|]^{\frac{\gamma+2}{2}} \mu^{1/16} \partial_i \partial^\alpha_\beta g_3\Big|_2 \\ \le\ & C\sum_{\overline{\beta}\le\beta_1} \Big|\partial^{\alpha-\alpha_1} a(x)\Big| \Big|w^\theta \partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_\sigma \Big|w^\theta \partial^\alpha_\beta g_3\Big|_\sigma, \end{aligned} \tag{3.15}$$

where we have used (3.5) and the fact that $|\partial_j \partial_{\beta-\beta_1}\mu^{1/4}| \le C\mu^{1/8}$.

Since $\partial_{\beta_1-\overline{\beta}}[v_i\mu^{1/2}] \le C\mu^{1/4}$, then we have that

$$\Big\{\phi^{ij} * \partial_{\beta_1}\Big[v_i\mu^{1/2}\partial^{\alpha_1} g_1\Big]\Big\} \le C[1+|v|]^{\gamma+2} \sum_{\overline{\beta}\le\beta_1} \Big|w^\theta \partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_\sigma. \tag{3.16}$$

By (3.5) and the fact that $g_2(x,v) = a(x)\mu^{1/4}(v)$, (3.9) is bounded by

$$\begin{aligned} & C\sum_{\overline{\beta}\le\beta_1} \Big|w^\theta \partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_\sigma \int w^{2\theta}[1+|v|]^{\gamma+2}\Big|\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 \partial^\alpha_\beta g_3\Big| dv \\ \le\ & C\sum_{\overline{\beta}\le\beta_1} \Big|w^\theta \partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_\sigma \Big|\partial^{\alpha-\alpha_1} a(x)\Big| \Big|w^\theta [1+|v|]^{\frac{\gamma+2}{2}} \partial_j \partial_{\beta-\beta_1}\mu^{1/4}\Big|_2 \Big|w^\theta [1+|v|]^{\frac{\gamma+2}{2}} \partial^\alpha_\beta g_3\Big|_2 \\ \le\ & C\sum_{\overline{\beta}\le\beta_1} \Big|\partial^{\alpha-\alpha_1} a(x)\Big| \Big|w^\theta \partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_\sigma \Big|w^\theta \partial^\alpha_\beta g_3\Big|_\sigma. \end{aligned} \tag{3.17}$$

By (3.14) and (3.16), we deduce from (3.5) that

$$\begin{aligned} & \Big\{\phi^{ij} * \partial_{\beta_1}\Big[\mu^{1/2}\partial_j\partial^{\alpha_1} g_1\Big]\Big\} + \Big\{\phi^{ij} * \partial_{\beta_1}\Big[v_i\mu^{1/2}\partial_j\partial^{\alpha_1} g_1\Big]\Big\} \\ \le\ & \Big[|\phi^{ij}|^2 * \mu^{1/4}\Big]^{1/2} \sum_{\overline{\beta}\le\beta_1} \Big|\mu^{1/8}\partial_j\partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_2 \\ \le\ & C[1+|v|]^{\gamma+2} \sum_{\overline{\beta}\le\beta_1} \Big|w^\theta \partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_\sigma. \end{aligned} \tag{3.18}$$

By using (3.18) and the fact that $g_2(x,v) = a(x)\mu^{1/4}(v)$, similar arguments as those used in (3.15) and (3.17) imply that (3.10) and (3.11) can be controlled by the right-hand side of (3.7).

Since $|\partial_i[w^{2\theta}]| \le C[1+|v|]^{-1}w^{2\theta}$ and $g_2(x,v) = a(x)\mu^{1/4}(v)$, (3.12) is bounded by

$$\begin{aligned}
& C\sum_{\overline{\beta}\le\beta_1}\Big|w^\theta\partial^{\alpha_1}_{\overline{\beta}}g_1\Big|_\sigma\int w^{2\theta}[1+|v|]^{\gamma+1}\Big|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial^\alpha_\beta g_3\Big|dv \\
\le\ & C\sum_{\overline{\beta}\le\beta_1}\Big|w^\theta\partial^{\alpha_1}_{\overline{\beta}}g_1\Big|_\sigma\Big|\partial^{\alpha-\alpha_1}a(x)\Big|\Big|w^\theta[1+|v|]^{\frac{\gamma}{2}}\partial_j\partial_{\beta-\beta_1}\mu^{1/4}\Big|_2\Big|w^\theta[1+|v|]^{\frac{\gamma+2}{2}}\partial^\alpha_\beta g_3\Big|_2 \\
\le\ & C\sum_{\overline{\beta}\le\beta_1}\Big|\partial^{\alpha-\alpha_1}a(x)\Big|\Big|w^\theta\partial^{\alpha_1}_{\overline{\beta}}g_1\Big|_\sigma\Big|w^\theta\partial^\alpha_\beta g_3\Big|_\sigma, \qquad (3.19)
\end{aligned}$$

where we have used (3.5) and (3.14). By similar arguments as those used in (3.19), (3.13) is controlled by the last line of (3.19).

In the following we will establish (3.6). We first estimate (3.11) and (3.13).

Since $|\partial_i[w^{2\theta}]| \le C[1+|v|]^{-1}w^{2\theta}$ and $g_1(x,v) = a(x)\mu^{1/4}(v)$, (3.11) and (3.13) are both bounded by

$$\begin{aligned}
& C\sum_{\overline{\beta}\le\beta_1}\Big|w^\theta\partial^{\alpha_1}_{\overline{\beta}}g_1\Big|_\sigma\int w^{2\theta}[1+|v|]^{\gamma+2}\Big|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial^\alpha_\beta g_3\Big|dv \\
\le\ & C\sum_{\overline{\beta}\le\beta_1}\Big|\partial^{\alpha_1}a(x)\Big|\Big|w^\theta\partial_{\overline{\beta}}\mu^{1/4}\Big|_\sigma\Big|w^\theta[1+|v|]^{\frac{\gamma+2}{2}}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\Big|_2\Big|w^\theta[1+|v|]^{\frac{\gamma+2}{2}}\partial^\alpha_\beta g_3\Big|_2 \\
\le\ & C\sum_{\overline{\beta}\le\beta_1}\Big|\partial^{\alpha_1}a(x)\Big|\Big|w^\theta\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\Big|_\sigma\Big|w^\theta\partial^\alpha_\beta g_3\Big|_\sigma,
\end{aligned}$$

where we have used (3.5) and (3.18).

Since $|\partial_i[w^{2\theta}]| \le C[1+|v|]^{-1}w^{2\theta}$ and $g_1(x,v) = a(x)\mu^{1/4}(v)$, we easily deduce, from (3.5) and (3.18), that (3.12) is bounded by

$$\begin{aligned}
& C\sum_{\overline{\beta}\le\beta_1}\Big|w^\theta\partial^{\alpha_1}_{\overline{\beta}}g_1\Big|_\sigma\int w^{2\theta}[1+|v|]^{\gamma+1}\Big|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial^\alpha_\beta g_3\Big|dv \\
\le\ & C\sum_{\overline{\beta}\le\beta_1}\Big|\partial^{\alpha_1}a(x)\Big|\Big|w^\theta\partial_{\overline{\beta}}\mu^{1/4}\Big|_\sigma\Big|w^\theta[1+|v|]^{\frac{\gamma}{2}}\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\Big|_2\Big|w^\theta[1+|v|]^{\frac{\gamma+2}{2}}\partial^\alpha_\beta g_3\Big|_2 \\
\le\ & C\sum_{\overline{\beta}\le\beta_1}\Big|\partial^{\alpha_1}a(x)\Big|\Big|w^\theta\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\Big|_\sigma\Big|w^\theta\partial^\alpha_\beta g_3\Big|_\sigma. \qquad (3.20)
\end{aligned}$$

We now estimate (3.8) to (3.10). We decompose their double integration region $[v,v'] \in \mathbf{R}^3\times\mathbf{R}^3$ into three parts:

$$\{|v|\le 1\},\quad \{2|v'|\ge|v|, |v|\ge 1\}\ \text{ and }\ \{2|v'|\le|v|, |v|\ge 1\}.$$

For the first part, $\{|v| \le 1\}$, recall that $\phi^{ij}(v) = O(|v|^{\gamma+2}) \in L^2_{loc}$. By (3.14), (3.16) and (3.18), we have

$$\begin{aligned}
&\Big|\phi^{ij} * \partial_{\beta_1}\Big[\mu^{1/2}\partial_j\partial^{\alpha_1} g_1\Big]\Big| + \Big|\phi^{ij} * \partial_{\beta_1}\Big[v_i\mu^{1/2}\partial_j\partial^{\alpha_1} g_1\Big]\Big| \\
&\quad \le C[1+|v|]^{\gamma+2}\sum_{\overline{\beta}\le\beta_1}\Big|w^\theta\partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_\sigma, \\
&\Big|\phi^{ij} * \partial_{\beta_1}\Big[\mu^{1/2}\partial^{\alpha_1} g_1\Big]\Big| + \Big|\phi^{ij} * \partial_{\beta_1}\Big[v_i\mu^{1/2}\partial^{\alpha_1} g_1\Big]\Big| \\
&\quad \le C[1+|v|]^{\gamma+2}\sum_{\overline{\beta}\le\beta_1}\Big|w^\theta\partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_\sigma. \qquad (3.21)
\end{aligned}$$

Hence their corresponding integrands over the region $\{|v| \le 1\}$ are bounded by

$$C\sum\Big|w^\theta\partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_\sigma w^{2\theta}[1+|v|]^{\gamma+2}\Big[\Big|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2\Big| + \Big|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2\Big|\Big]\Big[\Big|\partial_i\partial^\alpha_\beta g_3\Big| + \Big|\partial^\alpha_\beta g_3\Big|\Big].$$

Its v-integral over $\{|v| \le 1\}$ is clearly bounded by the right-hand side of (3.6) because of (3.5) and the fact that $g_1(x,v) = a(x)\mu^{1/4}(v)$. We thus conclude the first part of $\{|v| \le 1\}$ for (3.8) to (3.10).

For the second part, $\{2|v'| \ge |v|, |v| \ge 1\}$, we have

$$|\partial_{\beta_1}\{\mu^{1/2}(v')\}| + |\partial_{\beta_1}\{v'_i\mu^{1/2}(v')\}| \le C\mu^{1/8}(v')\mu^{1/32}(v).$$

By the same types of estimates as in (3.21), the v-integrands in (3.8) to (3.10) are bounded by

$$\begin{aligned}
&\mu^{1/32}(v)w^{2\theta}\Big|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2\Big|\Big[\Big|\partial_i\partial^\alpha_\beta g_3\Big| + \Big|\partial^\alpha_\beta g_3\Big|\Big]\int\Big|\phi^{ij}(v-v')\mu^{1/8}(v')\partial^{\alpha_1}_{\overline{\beta}} g_1(v')\Big|dv' \\
&\quad +\mu^{1/32}(v)w^{2\theta}\Big|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2\Big|\Big[\Big|\partial_i\partial^\alpha_\beta g_3\Big| + \Big|\partial^\alpha_\beta g_3\Big|\Big]\int\Big|\phi^{ij}(v-v')\mu^{1/8}(v')\partial_j\partial^{\alpha_1}_{\overline{\beta}} g_1(v')\Big|dv' \\
\le\ & C\Big|w^{2\theta}\partial^{\alpha_1}_{\overline{\beta}} g_1\Big|_\sigma[1+|v|]^{\gamma+2}\mu^{1/32}(v)w^{2\theta}\Big[\Big|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2\Big| + \Big|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2\Big|\Big]\Big[\Big|\partial_i\partial^\alpha_\beta g_3\Big| + \Big|\partial^\alpha_\beta g_3\Big|\Big] \\
\le\ & C\Big|\partial^{\alpha_1} a(x)\Big|[1+|v|]^{\gamma+2}\mu^{1/32}(v)w^{2\theta}\Big[\Big|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2\Big| + \Big|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2\Big|\Big]\Big[\Big|\partial_i\partial^\alpha_\beta g_3\Big| + \Big|\partial^\alpha_\beta g_3\Big|\Big],
\end{aligned}$$

where we have used $g_1(x,v) = a(x)\mu^{1/4}(v)$. Its v-integral is bounded by the right-hand side of (3.6) because of the fast decay factor $\mu^{1/32}(v)$.

We finally consider the third part, $\{2|v'| \le |v|, |v| \ge 1\}$, for which we shall estimate each term from (3.8) to (3.10).

To estimate (3.8) over this region we expand $\phi^{ij}(v-v')$ to get

$$\phi^{ij}(v-v') = \phi^{ij}(v) - \sum_k \partial_k\phi^{ij}(v)v'_k + \frac{1}{2}\sum_{k,l}\partial_{kl}\phi^{ij}(\overline{v})v'_kv'_l, \qquad (3.22)$$

where $\overline{v}$ is between v and $v-v'$. We plug (3.22) into the integrand of (3.8). Notice that for either fixed i or j,

$$\sum_i \phi^{ij}v_i = \sum_j \phi^{ij}v_j = 0. \qquad (3.23)$$

From (3.4) and (3.23), we can decompose $\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2$ and $\partial_i \partial^\alpha_\beta g_3$ into $\mathbf{P}_v$ parts as well as $\mathbf{I}-\mathbf{P}_v$ parts. For the first term in expansion (3.22), we have

$$\sum_{ij} w^{2\theta}\phi^{ij}(v)\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 \partial_i \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_3$$

$$= \sum_{ij} w^{2\theta}\phi^{ij}(v)\{[\mathbf{I}-\mathbf{P}_v]\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 [\mathbf{I}-\mathbf{P}_v]\partial_i \partial^\alpha_\beta g_3\}.$$

Here we have used (3.23) so that the sum of terms with either $\mathbf{P}_v \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2$ or $\mathbf{P}_v \partial_i \partial^\alpha_\beta g_3$ vanishes. The absolute value of this is bounded by

$$Cw^{2\theta}[1+|v|]^{\gamma+2}|[\mathbf{I}-\mathbf{P}_v]\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2| \times |[\mathbf{I}-\mathbf{P}_v]\partial_i \partial^\alpha_\beta g_3|. \tag{3.24}$$

For the second term in expansion (3.22), by taking the k derivative of

$$\sum_{ij} \phi^{ij}(v) v_i v_j = 0,$$

we have that

$$\sum_{ij} \partial_k \phi^{ij}(v) v_i v_j = -2 \sum_j \phi^{kj}(v) v_j = 0.$$

Therefore, expanding $\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2$ and $\partial_i \partial^\alpha_\beta g_3$ into their $\mathbf{P}_v$ and $\mathbf{I}-\mathbf{P}_v$ parts yields

$$\sum_{ij} w^{2\theta}\partial_k \phi^{ij}(v)\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v) \partial_i \partial^\alpha_\beta g_3(v)$$

$$= \sum_{ij} w^{2\theta}\partial_k \phi^{ij}(v)\{[\mathbf{P}_v \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2][\mathbf{I}-\mathbf{P}_v]\partial_i \partial^\alpha_\beta g_3 + w^{2\theta}[\mathbf{I}-\mathbf{P}_v]\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 [\mathbf{P}_v \partial_i \partial^\alpha_\beta g_3]\}$$

$$+ \sum_{ij} w^{2\theta}\partial_k \phi^{ij}(v)[\mathbf{I}-\mathbf{P}_v]\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 [\mathbf{I}-\mathbf{P}_v]\partial_i \partial^\alpha_\beta g_3,$$

where

$$\sum_{ij} w^{2\theta}\partial_k \phi^{ij}(v)[\mathbf{P}_v \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2][\mathbf{P}_v \partial_i \partial^\alpha_\beta g_3] = 0.$$

Noting that $|\partial_k \phi^{ij}(v)| \le C[1+|v|]^{\gamma+1}$ for $|v| \ge 1$, we majorize the above by

$$\begin{aligned} &Cw^\theta[1+|v|]^{\gamma/2}\{|\mathbf{P}_v \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2| + |\mathbf{P}_v \partial_j \partial^\alpha_\beta g_3|\} \\ &\times Cw^\theta[1+|v|]^{(\gamma+2)/2}\{|[\mathbf{I}-\mathbf{P}_v]\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2| + |[\mathbf{I}-\mathbf{P}_v]\partial_i \partial^\alpha_\beta g_3|\} \\ &+ Cw^{2\theta}[1+|v|]^{\gamma+2}|[\mathbf{I}-\mathbf{P}_v]\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2||[\mathbf{I}-\mathbf{P}_v]\partial_i \partial^\alpha_\beta g_3|. \end{aligned} \tag{3.25}$$

Next, we estimate the third term in (3.22). Using the region, we have

$$\frac{1}{2}|v| \le |v| - |v'| \le |\overline{v}| \le |v| + |v'| \le \frac{3}{2}|v|. \tag{3.26}$$

Thus we have

$$|\partial_{kl}\phi^{ij}(\overline{v})| \le C[1+|v|]^\gamma,$$

and

$$\left| w^{2\theta}\partial_{kl}\phi^{ij}(\overline{v})\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v) \partial_i \partial^\alpha_\beta g_3(v) \right| \le Cw^{2\theta}[1+|v|]^\gamma |\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 \partial_i \partial^\alpha_\beta g_3|. \tag{3.27}$$

Combining (3.22), (3.24), (3.25) and (3.27), we have

$$\begin{aligned}
&\int \Big|\sum_{ij} w^{2\theta}\partial_k\phi^{ij}(v-v')\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^{\alpha}_{\beta}g_3(v)\Big|dv\\
\leq\ & C[1+|v'|]^2\int \Big|\sum_{ij} w^{2\theta}\phi^{ij}(v)\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^{\alpha}_{\beta}g_3(v)\Big|dv\\
&+C[1+|v'|]^2\int \Big|\sum_{ij} w^{2\theta}\partial_k\phi^{ij}(v)\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^{\alpha}_{\beta}g_3(v)\Big|dv\\
&+C[1+|v'|]^2\int \Big|\sum_{ij} w^{2\theta}\partial_{kl}\phi^{ij}(\overline{v})\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^{\alpha}_{\beta}g_3(v)\Big|dv\\
\leq\ & C[1+|v'|]^2|w^{\theta}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma}|w^{\theta}\partial^{\alpha}_{\beta}g_3|_{\sigma},
\end{aligned}$$

where we have used (3.5). Therefore, (3.8) over $\{2|v'|\leq|v|, |v|\geq 1\}$ is bounded by

$$\begin{aligned}
&Cw^{2\theta}\int[1+|v'|]^2|\partial_{\beta_1}[\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')]|dv'\times|w^{\theta}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma}|w^{\theta}\partial^{\alpha}_{\beta}g_3|_{\sigma}\\
\leq\ & C|\partial^{\alpha_1}a(x)|\int[1+|v'|]^2|\partial_{\beta_1}[\mu^{1/2}(v')\mu^{1/4}(v')]|dv'\times|w^{\theta}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma}|w^{\theta}\partial^{\alpha}_{\beta}g_3|_{\sigma}\\
\leq\ & C|\partial^{\alpha_1}a(x)||w^{\theta}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma}|w^{\theta}\partial^{\alpha}_{\beta}g_3|_{\sigma},
\end{aligned}$$

where we have used $g_1(x,v)=a(x)\mu^{1/4}(v)$.

Now we consider the second term in (3.9). We again expand $\phi^{ij}(v-v')$ as

$$\phi^{ij}(v-v')=\phi^{ij}(v)-\sum_k\partial_k\phi^{ij}(\overline{v})v'_k,$$

with $\overline{v}$ between v and $v-v'$. Since $\sum_j\phi^{ij}(v)v_j=0$, we obtain

$$\begin{aligned}
&\sum_j w^{2\theta}\phi^{ij}(v)\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial^{\alpha}_{\beta}g_3(v)\\
&=\sum_j w^{2\theta}\phi^{ij}(v)[\mathbf{I}-\mathbf{P}_v]\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial^{\alpha}_{\beta}g_3(v)\\
&\leq Cw^{2\theta}[1+|v|]^{\gamma+2}|[\mathbf{I}-\mathbf{P}_v]\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)||\partial^{\alpha}_{\beta}g_3(v)|\\
&\leq C|w^{\theta}[1+|v|]^{\frac{\gamma+2}{2}}[\mathbf{I}-\mathbf{P}_v]\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)||w^{\theta}[1+|v|]^{\frac{\gamma+2}{2}}\partial^{\alpha}_{\beta}g_3(v)|. \qquad (3.28)
\end{aligned}$$

From (3.26), $|\partial_k\phi^{ij}(\overline{v})|\leq C[1+|v|]^{\gamma+1}$. Hence, we have that

$$\begin{aligned}
&|w^{2\theta}\partial_k\phi^{ij}(\overline{v})\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial^{\alpha}_{\beta}g_3(v)|\\
&\leq Cw^{2\theta}[1+|v|]^{\gamma+1}|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)||\partial^{\alpha}_{\beta}g_3(v)|\\
&\leq C|w^{\theta}[1+|v|]^{\frac{\gamma}{2}}\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)||w^{\theta}[1+|v|]^{\frac{\gamma+2}{2}}\partial^{\alpha}_{\beta}g_3(v)|. \qquad (3.29)
\end{aligned}$$

From (3.28) and (3.29), we thus conclude

$$\int\Big|\sum_j w^{2\theta}\phi^{ij}(v-v')\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial^{\alpha}_{\beta}g_3(v)\Big|dv\leq C|w^{\theta}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma}|w^{\theta}\partial^{\alpha}_{\beta}g_3|_{\sigma}.$$

We conclude that (3.9) over $\{2|v'| \le |v|, |v| \ge 1\}$ can be majorized by

$$\begin{aligned}
&\int\int \Big| w^{2\theta}\phi^{ij}(v-v')\partial_{\beta_1}\Big[v'_i\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')\Big]\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial^\alpha_\beta g_3(v)\Big| dv'dv \\
= &\int\int \Big| w^{2\theta}\Big[\phi^{ij}(v)-\partial_k\phi^{ij}(\overline{v})v'_k\Big]\partial_{\beta_1}\Big[v'_i\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')\Big]\Big| \\
&\qquad\qquad \times\Big|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial^\alpha_\beta g_3(v)\Big| dv'dv \\
\le &\ C\int [1+|v'|]|\partial^{\alpha_1}a(x)|\partial_{\beta_1}\Big[v'_i\mu^{3/4}(v')\Big]\Big| dv' |w^\theta\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_\sigma |w^\theta\partial^\alpha_\beta g_3|_\sigma \\
\le &\ C|\partial^{\alpha_1}a(x)||w^\theta\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_\sigma|w^\theta\partial^\alpha_\beta g_3|_\sigma,
\end{aligned}$$

where we have used the Cauchy-Schwartz inequality and $g_1(x,v) = a(x)\mu^{1/4}(v)$.

We now consider the third term in (3.10) over $\{2|v'| \le |v|, |v| \ge 1\}$. We use integration by parts to split (3.10) into two parts:

$$\phi^{ij} * \partial_{\beta_1}\Big[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1\Big] = \partial_j\phi^{ij} * \partial_{\beta_1}\Big[\mu^{1/2}\partial^{\alpha_1}g_1\Big] - \phi^{ij} * \partial_{\beta_1}\Big[\partial_j\mu^{1/2}\partial^{\alpha_1}g_1\Big]. \tag{3.30}$$

We first decompose

$$\partial_i\partial^\alpha_\beta g_3 = \mathbf{P}_v\partial_i\partial^\alpha_\beta g_3 + [\mathbf{I}-\mathbf{P}_v]\partial_i\partial^\alpha_\beta g_3.$$

By similar estimates to (3.28) and (3.29), the second part of (3.10) over $\{2|v'| \le |v|, |v| \ge 1\}$ can be estimated as

$$\begin{aligned}
&\int\int \Big| w^{2\theta}\phi^{ij}(v-v')\partial_{\beta_1}\Big[\partial_j\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')\Big]\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^\alpha_\beta g_3(v)\Big| dv'dv \\
= &\int\int \Big| w^{2\theta}\Big[\phi^{ij}(v)-\partial_k\phi^{ij}(\overline{v})v'_k\Big]\partial_{\beta_1}\Big[\partial_j\mu^{1/2}(v')\partial^{\alpha_1}a(x)\mu^{1/4}(v')\Big]\Big| \\
&\qquad \times\Big|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^\alpha_\beta g_3(v)\Big| dv'dv \\
\le &\ C\Big|\partial^{\alpha_1}a(x)\Big|\Big|w^\theta[1+|v|]^{\frac{\gamma+2}{2}}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\Big|_2\Big|w^\theta[1+|v|]^{\frac{\gamma+2}{2}}[\mathbf{I}-\mathbf{P}_v]\partial_i\partial^\alpha_\beta g_3\Big|_2 \\
&\quad + C\Big|\partial^{\alpha_1}a(x)\Big|\Big|w^\theta[1+|v|]^{\frac{\gamma+2}{2}}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\Big|_2\Big|w^\theta[1+|v|]^{\frac{\gamma}{2}}\partial_i\partial^\alpha_\beta g_3\Big|_2.
\end{aligned}$$

By (3.5), this is bounded by the right-hand side of (3.6).

We now turn to the proof of the first part of (3.10). By (3.30), noticing that our integration region implies

$$|\partial_j\phi^{ij}(v-v')| \le C[1+|v|]^{\gamma+1},$$

the first part of (3.10) over $\{2|v'| \le |v|, |v| \ge 1\}$ can be estimated as

$$\begin{aligned}
&\int\int w^{2\theta}\Big|\partial_j\phi^{ij}(v-v')\Big|\Big|\partial_{\beta_1}[\partial_j\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')]\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^\alpha_\beta g_3(v)\Big| dv'dv \\
&\qquad \le C\Big|\partial^{\alpha_1}a(x)\Big|\int w^{2\theta}[1+|v|]^{\gamma+1}\Big|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^\alpha_\beta g_3(v)\Big| dv \\
&\qquad \le C\Big|\partial^{\alpha_1}a(x)\Big|\Big|w^\theta[1+|v|]^{\frac{\gamma+2}{2}}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\Big|_2\Big|w^\theta[1+|v|]^{\frac{\gamma}{2}}\partial_i\partial^\alpha_\beta g_3\Big|_2,
\end{aligned}$$

where we have used $g_1(x,v) = a(x)\mu^{1/4}(v)$. By (3.5), this is bounded by the right-hand side of (3.6). We conclude our theorem.

COROLLARY 3.2. Let $|\alpha|+|\beta|\le N$ and $|\beta|\le\theta$. If $g_1(x,v)$ belongs to the null space N of L, namely, $g_1(x,v)=a(x)\sqrt{\mu}+\sum_{j=1}^3 b_j(x)v_j\sqrt{\mu}+c(x)|v|^2\sqrt{\mu}$, then

$$\Big|\langle w^{2\theta}\partial_\beta^\alpha\Gamma(g_1,g_2),\partial_\beta^\alpha g_3\rangle\Big|\le C\sum\Big[|\partial^{\alpha_1}a(x)|+|\partial^{\alpha_1}b_j(x)|+|\partial^{\alpha_1}c(x)|\Big]\times\Big|w^\theta\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(x)\Big|_\sigma\Big|w^\theta\partial_\beta^\alpha g_3(x)\Big|_\sigma. \quad (3.31)$$

If $g_2(x,v)=a(x)\sqrt{\mu}+\sum_{j=1}^3 b_j(x)v_j\sqrt{\mu}+c(x)|v|^2\sqrt{\mu}$, then

$$\Big|\langle w^{2\theta}\partial_\beta^\alpha\Gamma(g_1,g_2),\partial_\beta^\alpha g_3\rangle\Big|\le C\sum\Big[|\partial^{\alpha-\alpha_1}a(x)|+|\partial^{\alpha-\alpha_1}b_j(x)|+|\partial^{\alpha-\alpha_1}c(x)|\Big]\times\Big|w^\theta\partial_{\beta_1}^{\alpha_1}g_1(x)\Big|_\sigma\Big|w^\theta\partial_\beta^\alpha g_3(x)\Big|_\sigma, \quad (3.32)$$

where the summation is over j, $|\alpha_1|+|\beta_1|\le N$ and $\beta_1\le\beta$.

LEMMA 3.3. Let $0<|\alpha_1|+|\alpha_2|=|\alpha|\le N$. Let $g_1(x,v)$, $g_2(x,v)$ and $h(v)$ be smooth functions. We have

$$\Big|\langle\Gamma(g_1,g_2),h\rangle\Big|_2\le\begin{cases} C\sum_{|\beta|\le2}|\partial_\beta h|_\sigma\sum_{|\alpha|\le N}\|\partial^\alpha g_1\|\|g_2\|_\sigma,\\ C\sum_{|\beta|\le2}|\partial_\beta h|_2\sum_{|\alpha|\le N}\|\partial^\alpha g_2\|\|g_1\|_\sigma,\end{cases} \quad (3.33)$$

$$\Big|\langle\Gamma(\partial^{\alpha_1}g_1,\partial^{\alpha-\alpha_1}g_2),h\rangle\Big|_2 \le\begin{cases} C\sum_{|\beta|\le2}|\partial_\beta h|_\sigma\sum_{|\alpha|\le N}\|\partial^\alpha g_1\|\|\partial^{\alpha-\alpha_1}g_2\|_\sigma, & \text{if } |\alpha_1|\le\frac{N}{2};\\ C\sum_{|\beta|\le2}|\partial_\beta h|_2\sum_{|\alpha|\le N}\|\partial^\alpha g_2\|\|\partial^{\alpha_1}g_1\|_\sigma, & \text{if } |\alpha_1|>\frac{N}{2}.\end{cases} \quad (3.34)$$

Moreover,

$$|\langle L\partial^\alpha g,h\rangle|_2\le C\sum_{|\beta|\le2}|\partial_\beta h|_\sigma\|\partial^\alpha g\|_\sigma. \quad (3.35)$$

Proof. We begin with the linear term. By (3.1) and (3.2), we have that

$$Lg=-\partial_i\Big[\sigma^{ij}\partial_j g\Big]+\sigma^{ij}v_iv_jg-\partial_i\Big[v_j\sigma^{ij}\Big]g+\mu^{-1/2}\partial_i\Big\{\mu\Big[\phi^{ij}*\Big\{\mu^{1/2}[\partial_jg+v_jg]\Big\}\Big]\Big\}.$$

Using integrations by parts, $\langle L\partial^\alpha g,h\rangle$ is given by

$$\int\Big\{-\partial^\alpha g\cdot\partial_j\Big[\sigma^{ij}\partial_ih(v)\Big]+\sigma^{ij}v_iv_j\partial^\alpha g\cdot h(v)-\partial_i\Big[v_j\sigma^{ij}\Big]\partial^\alpha g\cdot h(v)\Big\}dv -\int\mu\Big[\phi^{ij}*\Big\{\mu^{1/2}[\partial_j\partial^\alpha g+v_j\partial^\alpha g]\Big\}\Big]\partial_i\Big[\mu^{-1/2}h(v)\Big]dv, \quad (3.36)$$

where we implicitly sum over $i,j\in\{1,2,3\}$. Using the Cauchy-Schwartz inequality and the following inequality

$$|\sigma^{ij}|+|\partial_j\sigma^{ij}|+|\partial_i[v_j\sigma^{ij}]|\le C[1+|v|]^{\gamma+2},$$

we obtain that

$$\begin{aligned}
\Big|\int\Big\{-\partial^\alpha g\cdot\partial_j\Big[\sigma^{ij}\partial_i h(v)\Big]\Big\}dv\Big|_2 &\le C\sum_{|\beta|\le 2}|[1+|v|]^{\frac{\gamma+2}{2}}\partial_\beta h|_2\|[1+|v|]^{\frac{\gamma+2}{2}}\partial^\alpha g\| \\
&\le C\sum_{|\beta|\le 2}|\partial_\beta h|_\sigma\|\partial^\alpha g\|_\sigma, \\
\Big|\int\Big\{\partial_i\Big[v_j\sigma^{ij}\Big]\partial^\alpha g\cdot h(v)\Big\}dv\Big|_2 &\le C|[1+|v|]^{\frac{\gamma+2}{2}}h|_2\|[1+|v|]^{\frac{\gamma+2}{2}}\partial^\alpha g\| \\
&\le C\sum_{|\beta|\le 2}|\partial_\beta h|_\sigma\|\partial^\alpha g\|_\sigma, \\
\Big|\int\sigma^{ij}v_iv_j\partial^\alpha g\cdot h(v)dv\Big|_2 &\le C\Big\{\int\sigma^{ij}v_iv_jh^2dv\Big\}^{1/2}\Big|\Big\{\int\sigma^{ij}v_iv_j|\partial^\alpha g|^2dv\Big\}^{1/2}\Big|_2 \\
&\le C\sum_{|\beta|\le 2}|\partial_\beta h|_\sigma\|\partial^\alpha g\|_\sigma,
\end{aligned}$$

where we have used the definition of the norm $\|\cdot\|_\sigma$.

Using the inequality (3.21), we have that the second line of (3.36) is bounded by

$$\begin{aligned}
& C\|\partial^\alpha g\|_\sigma\Big|\int\mu^{1/4}[1+|v|]^{\gamma+2}\Big[|h|+|\partial_i h|\Big]dv\Big|_2 \\
\le\ & C\|\partial^\alpha g\|_\sigma\Big\{\int\mu^{1/4}[1+|v|]^{2\gamma+4}dv\Big\}^{1/2} \\
& \qquad\times\Big\{\int\mu^{1/4}\Big[|h|^2+|\partial_i h|^2\Big]dv\Big\}^{1/2} \\
\le\ & C\sum_{|\beta|\le 2}|\partial_\beta h|_\sigma\|\partial^\alpha g\|_\sigma.
\end{aligned}$$

This completes the proof of estimate (3.35).

Recalling (3.3), $\Gamma(\partial^{\alpha_1}g_1,\partial^{\alpha-\alpha_1}g_2)h$ takes the form

$$\begin{aligned}
&\partial_i\Big[\Big\{\phi^{ij}*\Big[\mu^{1/2}\partial^{\alpha_1}g_1\Big]\Big\}\partial_j\partial^{\alpha-\alpha_1}g_2\Big]h-\Big\{\phi^{ij}*\Big[v_i\mu^{1/2}\partial^{\alpha_1}g_1\Big]\Big\}\partial_j\partial^{\alpha-\alpha_1}g_2h \\
&-\partial_i\Big[\Big\{\phi^{ij}*\Big[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1\Big]\Big\}\partial^{\alpha-\alpha_1}g_2\Big]h+\Big\{\phi^{ij}*\Big[v_i\mu^{1/2}\partial_j\partial^{\alpha_1}g_1\Big]\Big\}\partial^{\alpha-\alpha_1}g_2h \\
&\equiv I_1+I_2+I_3+I_4.
\end{aligned}$$

We first consider the term I_1. We rewrite I_1 as

$$\begin{aligned}
I_1=\ & \partial_i\Big[\Big\{\phi^{ij}*\Big[\mu^{1/2}\partial^{\alpha_1}g_1\Big]\Big\}\partial_j\partial^{\alpha-\alpha_1}g_2h\Big]-\partial_j\Big[\Big\{\phi^{ij}*\Big[\mu^{1/2}\partial^{\alpha_1}g_1\Big]\Big\}\partial^{\alpha-\alpha_1}g_2\partial_ih\Big] \\
&+\Big\{\partial_j\phi^{ij}*\Big[\mu^{1/2}\partial^{\alpha_1}g_1\Big]\Big\}\partial^{\alpha-\alpha_1}g_2\partial_ih+\Big\{\phi^{ij}*\Big[\mu^{1/2}\partial^{\alpha_1}g_1\Big]\Big\}\partial^{\alpha-\alpha_1}g_2\partial_{ij}h.
\end{aligned}$$

The Cauchy-Schwartz inequality implies that

$$\begin{aligned}
\Big\{\partial_j\phi^{ij}*\Big[\mu^{1/2}\partial^{\alpha_1}g_1\Big]\Big\}\ &\le\Big\{|\partial_j\phi^{ij}|^2*\mu^{1/2}\Big\}^{1/2}\Big\{\int\mu^{1/2}(v')|\partial^{\alpha_1}g_1(v')|^2dv'\Big\}^{1/2} \\
&\le C[1+|v|]^{\gamma+1}\min\{|\partial^{\alpha_1}g_1|_2,|\partial^{\alpha_1}g_1|_\sigma\}.
\end{aligned}$$

If $|\alpha_1| \le N/2$, by $H^2(\mathbf{R}^3) \subset L^\infty(\mathbf{R}^3)$, we have that

$$\sup_x |\partial^{\alpha_1} g(x)|_2^2 \le C \sum_{|\alpha|\le N} |\partial^\alpha g(x)|_2^2.$$

If $|\alpha_1| \le N/2$, we have, from (3.5) and the Cauchy-Schwartz inequality, that

$$\begin{aligned}
&\Big| \int \Big\{ \partial_j \phi^{ij} * \Big[\mu^{1/2} \partial^{\alpha_1} g_1 \Big] \Big\} \partial^{\alpha-\alpha_1} g_2 \partial_i h dv \Big|_2 \\
\le\ & C \Big| |\partial^{\alpha_1} g_1|_2 \int [1+|v|]^{\gamma+2} |\partial^{\alpha-\alpha_1} g_2(x,v) \partial_i h(v)| dv \Big|_2 \\
\le\ & C \Big| |\partial^{\alpha_1} g_1|_2 |\partial^{\alpha-\alpha_1} g_2(x)|_\sigma |\partial_i h|_\sigma \Big|_2 \\
\le\ & C \sum_{|\beta|\le 2} |\partial_\beta h|_\sigma \sum_{|\alpha|\le N} \|\partial^\alpha g_1\| \|\partial^{\alpha-\alpha_1} g_2\|_\sigma.
\end{aligned}$$

If $|\alpha - \alpha_1| \le N/2$, we also have, from $\gamma + 1 \le -1$, that

$$\begin{aligned}
&\Big| \int \Big\{ \partial_j \phi^{ij} * \Big[\mu^{1/2} \partial^{\alpha_1} g_1 \Big] \Big\} \partial^{\alpha-\alpha_1} g_2 \partial_i h dv \Big|_2 \\
\le\ & C \Big| |\partial^{\alpha_1} g_1(x)|_\sigma \int [1+|v|]^{\gamma+1} |\partial^{\alpha-\alpha_1} g_2(x,v) \partial_i h(v)| dv \Big|_2 \\
\le\ & C \Big| |\partial^{\alpha_1} g_1(x)|_\sigma |\partial^{\alpha-\alpha_1} g_2(x)|_2 |\partial_i h|_2 \Big|_2 \\
\le\ & C \sum_{|\beta|\le 2} |\partial_\beta h|_2 \sum_{|\alpha|\le N} \|\partial^\alpha g_2\| \|\partial^{\alpha_1} g_1\|_\sigma.
\end{aligned}$$

The fourth term of I_1 has the same upper bound by the same arguments as above. This completes the proof of (3.34) about the term I_1.

For the term I_2, we split I_2 as

$$\begin{aligned}
I_2 = -\partial_j \Big[\phi^{ij} * \Big[v_i \mu^{1/2} \partial^{\alpha_1} g_1 \Big] \partial_j \partial^{\alpha-\alpha_1} g_2 h \Big] \quad &+ \Big\{ \partial_j \phi^{ij} * \Big[v_i \mu^{1/2} \partial^{\alpha_1} g_1 \Big] \Big\} \partial^{\alpha-\alpha_1} g_2 h \\
&+ \Big\{ \phi^{ij} * \Big[v_i \mu^{1/2} \partial^{\alpha_1} g_1 \Big] \Big\} \partial^{\alpha-\alpha_1} g_2 \partial_j h.
\end{aligned}$$

By (3.16) and similar arguments as we used for I_1, we can obtain the estimates of I_2.

Similarly, we can rewrite I_3 as

$$\begin{aligned}
I_3 &= -\partial_i \Big[\Big\{ \phi^{ij} * \Big[\mu^{1/2} \partial_j \partial^{\alpha_1} g_1 \Big] \Big\} \partial^{\alpha-\alpha_1} g_2 h \Big] + \Big\{ \phi^{ij} * \Big[\mu^{1/2} \partial_j \partial^{\alpha_1} g_1 \Big] \Big\} \partial^{\alpha-\alpha_1} g_2 \partial_i h \\
&= -\partial_i \Big[\Big\{ \phi^{ij} * \Big[\mu^{1/2} \partial_j \partial^{\alpha_1} g_1 \Big] \Big\} \partial^{\alpha-\alpha_1} g_2 h \Big] + \Big\{ \partial_j \phi^{ij} * \Big[\mu^{1/2} \partial^{\alpha_1} g_1 \Big] \Big\} \partial^{\alpha-\alpha_1} g_2 \partial_i h \\
&\qquad - \Big\{ \phi^{ij} * \Big[\partial_j \mu^{1/2} \partial^{\alpha_1} g_1 \Big] \Big\} \partial^{\alpha-\alpha_1} g_2 \partial_i h.
\end{aligned}$$

Applying similar estimates as we used for I_1, we also can obtain (3.34) about the term I_3. We now split I_4 as follows:

$$\Big\{ \partial_j \phi^{ij} * \Big[v_i \mu^{1/2} \partial^{\alpha_1} g_1 \Big] \Big\} \partial^{\alpha-\alpha_1} g_2 h - \Big\{ \phi^{ij} * \Big[\partial_j [v_i \mu^{1/2}] \partial^{\alpha_1} g_1 \Big] \Big\} \partial^{\alpha-\alpha_1} g_2 h.$$

Similar arguments as the last two terms of I_3 imply that (3.34) about the term I_4 holds. This completes the proof of (3.34). The similar argument as we used for (3.34) implies that (3.33) holds.

In the following, we recall the basic estimates in [12].

LEMMA 3.4. Let $|\beta|+|\alpha|\le N$ and $\theta\ge 0$. For any $\eta>0$, there exists $C_\eta>0$ such that

$$\langle w^{2\theta}\partial_\beta[Lg],\partial_\beta g\rangle \ge |w^\theta\partial_\beta g|_\sigma^2-\eta\sum_{|\beta_1|\le|\beta|}|w^\theta\partial_{\beta_1}g|_\sigma^2-C_\eta|g|_\sigma^2\quad(|\beta|>0), \tag{3.37}$$

$$\begin{aligned}&|\langle w^{2\theta}\partial_\beta^\alpha\Gamma(f,g),\partial_\beta^\alpha h\rangle|\\ &\le C\sum\Big[|\partial_{\overline{\beta}}^{\alpha_1}f|_{2,\theta}|\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g|_{\sigma,\theta}+|\partial_{\overline{\beta}}^{\alpha_1}f|_{\sigma,\theta}|\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g|_{2,\theta}\Big]|\partial_\beta^\alpha h|_{\sigma,\theta},\end{aligned} \tag{3.38}$$

where the summation is over $|\alpha_1|+|\beta_1|\le N$, $\alpha_1\le\alpha$, $\overline{\beta}\le\beta_1\le\beta$.

4. Positivity of L. In this section, we shall establish the positivity of the linearized operator L for any classical solution $f(t,x,v)$ to (1.3), which plays an important role in obtaining the global existence result of (1.3).

LEMMA 4.1. It holds that $\partial^\alpha\mathbf{P}f=\mathbf{P}\partial^\alpha f$. Moreover, there exists $C>1$ such that

$$\frac{1}{C}\|w^{|\beta|}\partial_\beta^\alpha\mathbf{P}f\|_w^2\le\|\partial^\alpha a\|^2+\|\partial^\alpha b\|^2+\|\partial^\alpha c\|^2\le C\|w^{|\beta|}\partial_\beta^\alpha\mathbf{P}f\|_w^2 \tag{4.1}$$

for any smooth function $f(t,x,v)$ and any multi-indices β.

Proof. A direct computation implies that $\partial^\alpha\mathbf{P}f=\mathbf{P}\partial^\alpha f$. We plug the expression $w^{|\beta|}\partial_\beta^\alpha\mathbf{P}f=w^{|\beta|}\partial^\alpha a(t,x)\partial_\beta[\mu^{1/2}]+w^{|\beta|}\partial^\alpha b(t,x)\cdot\partial_\beta[v\mu^{1/2}]+w^{|\beta|}\partial^\alpha c(t,x)\partial_\beta[|v|^2\mu^{1/2}]$ into the norms $\|\cdot\|_\nu$ and $\|\cdot\|_\sigma$. Using $\nu(v)\le C[1+|v|]^\gamma$, $|\sigma^{ij}|\le C[1+|v|]^{\gamma+2}$ and the exponential decay of μ, we can obtain the first half of (4.1) by a direct computation. For the second half of (4.1), since a, b and c are the coefficients of a basis to the finite dimensional space N, $|\partial^\alpha a|^2+|\partial^\alpha b|^2+|\partial^\alpha c|^2$ is bounded by $C\int_{\mathbf{R}^3}w^{2|\beta|}\nu(v)|\partial_\beta^\alpha\mathbf{P}f|^2dv$, $C\sum_{i,j=1}^3\int_{\mathbf{R}^3}w^{2|\beta|}\sigma^{ij}\partial_i\partial_\beta^\alpha\mathbf{P}f\partial_j\partial_\beta^\alpha\mathbf{P}fdv$ and $C\sum_{i,j=1}^3\int_{\mathbf{R}^3}w^{2|\beta|}\sigma^{ij}v_iv_j|\partial_\beta^\alpha\mathbf{P}f|^2dv$ for any (t,x). We then deduce (4.1) by a further integration over x.

We know that $\mathbf{P}$ is a projection from $L^2(\mathbf{R}^3)$ to the null space N of the linearized operator L. Thus for any fixed (t,x), we can decompose any function $f(t,x,v)$ uniquely as

$$f(t,x,v)=\{\mathbf{P}f\}(t,x,v)+\{\mathbf{I}-\mathbf{P}\}f(t,x,v),$$

where $\mathbf{P}f$ is called the hydrodynamic part, and $\{\mathbf{I}-\mathbf{P}\}f$ is called the microscopic part [14, 21, 22]. We plug $f=\{\mathbf{P}f\}+\{\mathbf{I}-\mathbf{P}\}f$ into equation (1.3). By separating its linear and nonlinear parts, and using $L\{\mathbf{P}f\}=0$, we can express the hydrodynamic part through the microscopic part $\{\mathbf{I}-\mathbf{P}\}f$:

$$[\partial_t+v\cdot\nabla_x]\mathbf{P}f=l(\{\mathbf{I}-\mathbf{P}\}f)+h(f), \tag{4.2}$$

where $l(\{\mathbf{I}-\mathbf{P}\}f)=-[\partial_t+v\cdot\nabla_x+L]\{\mathbf{I}-\mathbf{P}\}f$, and $h(f)=\Gamma(f,f)$.

By further expanding $\mathbf{P}f$ as a linear combination of the basis in (1.4),

$$\Big[a(t,x)+\sum_{j=1}^3b_j(t,x)v_j+c(t,x)|v|^2\Big]\sqrt{\mu},$$

we can derive the macroscopic equations for $\mathbf{P}f$'s coefficients a, b and c. In fact, the left-hand side of (4.2) now becomes

$$\sum_i \Big[v_i\partial^i c|v|^2 + [\partial^0 c + \partial^i b_i]v_i^2 + \sum_{j>i}[\partial^i b_j + \partial^j b_i]v_i v_j + [\partial^0 b_i + \partial^i a]v_i + \partial^0 a\Big]\sqrt{\mu}, \quad (4.3)$$

where $\partial^0 = \partial_t$, $\partial^j = \partial_{x_j}$ and $\partial^i = \partial_{x_i}$. For fixed (t,x), this is an expansion to the left-hand side of (4.2) with respect to the basis of $(1 \le i \ne j \le 3)$:

$$\sqrt{\mu}, \quad v_i\sqrt{\mu}, \quad v_i^2\sqrt{\mu}, \quad v_i v_j\sqrt{\mu}, \quad |v|^2 v_i\sqrt{\mu}.$$

We denote an orthogonal basis for this 13-dimensional space by ϵ_j, $1 \le j \le 13$ as in [9]. We expand the right-hand side of (4.2) with respect to the same basis, and compare with their coefficients on both sides. Then we have

(1) $\nabla_x c = l_c + h_c$, (2) $\partial^0 c + \partial^i b_i = l_i + h_i$,

(3) $\partial^i b_j + \partial^j b_i = l_{ij} + h_{ij},\ i \ne j$ (4) $\partial^0 b_i + \partial^i a = l_{bi} + h_{bi}$, (5) $\partial^0 a = l_a + h_a$.

Here $l_c(t,x)$, $l_i(t,x)$, $l_{ij}(t,x)$, $l_{bi}(t,x)$ and $l_a(t,x)$ are the corresponding coefficients of such an expansion to the linear term $-[\partial_t + v\cdot\nabla_x + L]\{\mathbf{I}-\mathbf{P}\}f$, while $h_c(t,x)$, $h_i(t,x)$, $h_{ij}(t,x)$, $h_{bi}(t,x)$ and $h_a(t,x)$ are the corresponding coefficients of the same expansion of the higher-order term $\Gamma(f,f)$. □

Now we estimate the L^2 norm of $l(\{\mathbf{I}-\mathbf{P}\}f)$ through the macroscopic equations (1-5).

LEMMA 4.2. Let $\alpha = [\alpha_0, \alpha_1, \alpha_1, \alpha_2]$; then for any $1 \le i, j \le 3$,

$$\sum_{|\alpha|\le N-1} [\|\partial^\alpha l_c\| + \|\partial^\alpha l_i\| + \|\partial^\alpha l_{ij}\| + \|\partial^\alpha l_{bi}\| + \|\partial^\alpha l_a\|] \le C \sum_{|\alpha|\le N} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_w.$$

Proof. We first normalize the basis $\{\epsilon_j\}$ $(1 \le j \le 13)$. Let

$$[\mu^{1/2}, v_i\mu^{1/2}, v_i^2\mu^{1/2}, v_i v_j\mu^{1/2}, |v|^2 v_i\mu^{1/2}]A_{13\times 13} = [\epsilon_j^*],$$

with $detA \ne 0$ [9]. Then for any fixed (t,x), $l_c(t,x)$, $l_i(t,x)$, $l_{ij}(t,x)$, $l_{bi}(t,x)$ and $l_a(t,x)$ take the form

$$\sum_{i,n=1}^{13} \lambda^{ij}\lambda^{in} \int_{\mathbf{R}^3} l(\{\mathbf{I}-\mathbf{P}\}f)\epsilon_n(v)dv,$$

where λ^{ij} and λ^{in} are the entries of the matrix A.

The same is true after we take ∂^α. Let $|\alpha| \le N-1$. Notice that

$$\int_{\mathbf{R}^3} \partial^\alpha l(\{\mathbf{I}-\mathbf{P}\}f)\epsilon_n(v)dv = -\int_{\mathbf{R}^3} \{\partial_t + v\cdot\nabla_x + L\}\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f(v)\cdot\epsilon_n(v)dv.$$

We now estimate the first two terms:

$$\begin{aligned} & \Big\|\langle\{\partial_t + v\cdot\nabla_x\}\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f(v), \epsilon_n(v)\rangle\Big\|^2 \\ \le\ & 2\int_{\mathbf{R}^3} |\epsilon_n(v)|dv \int_{\mathbf{R}^3\times\mathbf{R}^3} |\epsilon_n(v)|(|\{\mathbf{I}-\mathbf{P}\}\partial^0\partial^\alpha f|^2 + |v|^2|\{\mathbf{I}-\mathbf{P}\}\nabla_x\partial^\alpha f|^2)dxdv \\ \le\ & C\{\|\{\mathbf{I}-\mathbf{P}\}\partial^0\partial^\alpha f\|_w + \|\{\mathbf{I}-\mathbf{P}\}\nabla_x\partial^\alpha f\|_w\}^2, \end{aligned}$$

where $\partial^0 = \partial_t$, and we have used (3.5) and the exponential decay of $\epsilon_n(v)$.

By (2.11) in Lemma 2.4 and (3.35) in Lemma 3.3, we have that

$$\|\langle L\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f,\epsilon_n\rangle\| \le C\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_w.$$

We next estimate the coefficients of the higher-order term $h(f)$. For convenience, we define the temporary energy norm:

$$[[f]]_w^2 = \|\{\mathbf{I}-\mathbf{P}\}f(t)\|_w^2 + \sum_{|\alpha|\le N,\alpha\ne 0}\|\partial^\alpha f(t)\|_w^2.$$

Noticing that the weaker dissipation norm $[[f]]_w^2$ without the L^2 norms of a, b and c, we carefully estimate coefficients of the higher-order terms in macroscopic equations.

LEMMA 4.3. Let $\alpha=[\alpha_0,\alpha_1,\alpha_1,\alpha_2]$; then for any $1\le i,j\le 3$,

$$\sum_{|\alpha|\le N}[\|\partial^\alpha h_c\|+\|\partial^\alpha h_i\|+\|\partial^\alpha h_{ij}\|+\|\partial^\alpha h_{bi}\|+\|\partial^\alpha h_a\|]\le C|||f|||_{|\beta|}\cdot[[f]]_w.$$

Proof. Let $|\alpha|\le N$. Notice that $\partial^\alpha h_c$, $\partial^\alpha h_i$, $\partial^\alpha h_{ij}$, $\partial^\alpha h_{bi}$ and $\partial^\alpha h_a$ are again of the form

$$\sum_{i,n=1}^{13}\lambda^{ij}\lambda^{in}\int_{\mathbf{R}^3}\partial^\alpha\Gamma(f,f)\cdot\epsilon_n(v)dv,$$

where λ^{ij} and λ^{in} are the entries of the matrix A. It again suffices to estimate $\langle\partial^\alpha\Gamma(f,f),\epsilon_n\rangle$. If $0<|\alpha|\le N$, then we are able to apply (2.8) for the Boltzmann collision operator to get

$$\begin{aligned}\|\langle\Gamma(\partial^{\alpha_1}f,\partial^{\alpha-\alpha_1}f),\epsilon_n\rangle\| &\le C\Big[\sum_{|\alpha_i|+|\beta_i|\le N}\|w^{|\beta_i|}\partial^{\alpha_i}_{\beta_i}f\|\Big]\|\partial^{\alpha-\alpha_1}f\|_\nu\\ &\le |||f|||_{|\beta|}\cdot[[f]]_\nu,\end{aligned}$$

and we are able to apply (3.34) for the Landau collision operator to get

$$\begin{aligned}\|\langle\Gamma(\partial^{\alpha_1}f,\partial^{\alpha-\alpha_1}f),\epsilon_n\rangle\| &\le C\Big[\sum_{|\alpha_1|\le N}\|\partial^\alpha f\|\Big]\|\partial^{\alpha-\alpha_1}f\|_\sigma\\ &\le |||f|||_{|\beta|}\cdot[[f]]_\sigma,\end{aligned}$$

where, without loss of generality, we have assumed that $|\alpha-\alpha_1|>0$.

In the following, we consider the case $\alpha=0$. We split $f=\mathbf{P}f+\{\mathbf{I}-\mathbf{P}\}f$ so that $\|\langle\Gamma(f,f),\epsilon_n\rangle\|$ is decomposed into

$$\begin{aligned}&\|\langle\Gamma(\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f),\epsilon_n\rangle\|+\|\langle\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\mathbf{P}f),\epsilon_n\rangle\|\\ &\quad+\|\langle\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f),\epsilon_n\rangle\|+\|\langle\Gamma(\mathbf{P}f,\mathbf{P}f),\epsilon_n\rangle\|.\end{aligned}$$

For the Boltzmann collision operator, by applying (2.7) in Lemma 2.3, we easily obtain that the first three terms are bounded by

$$C\Big[\sum_{|\alpha_i|+|\beta_i|\le N}\|w^{|\beta_i|}\partial^{\alpha_i}_{\beta_i}f\|\Big]\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu\le C|||f|||_{|\beta|}\cdot[[f]]_\nu.$$

For the Landau collision operator, by applying (3.33) in Lemma 3.3, we easily obtain that the first three terms are bounded by

$$C\Big[\sum_{|\alpha|\le N}\|\partial^\alpha f\|\Big]\|\{\mathbf{I}-\mathbf{P}\}f\|_\sigma \le C|||f|||_{|\beta|}\cdot[[f]]_\sigma.$$

On the other hand, we plug $\mathbf{P}f=[a(t,x)+b(t,x)\cdot v+c(t,x)|v|^2]\sqrt{\mu}$ into the last term to get

$$\|\langle\Gamma(\mathbf{P}f,\mathbf{P}f),\epsilon_n\rangle\|\le C\|a^2+|b|^2+c^2\|.$$

For the estimate of the terms a^2, $|b|^2$ and c^2, we shall use the similar method as used in [16]. By the generalized Hölder inequality, we obtain

$$\begin{aligned}\|a^2+|b|^2+c^2\| \le C\{&\|a(t,\cdot)\|_{L^6}+\|b(t,\cdot)\|_{L^6}+\|c(t,\cdot)\|_{L^6}\}\\ &\times\{\|a(t,\cdot)\|_{L^3}+\|b(t,\cdot)\|_{L^3}+\|c(t,\cdot)\|_{L^3}\}.\end{aligned}\tag{4.4}$$

This first factor is bounded by Sobolev's inequality in $\mathbf{R}^3$ and Lemma 4.1 as

$$\Big[\|\nabla_x a(t,\cdot)\|+\|\nabla_x b(t,\cdot)\|+\|\nabla_x c(t,\cdot)\|\Big]\le C[[f]]_w,$$

while the second factor is bounded by an interpolation as

$$\Big[\|a(t,\cdot)\|_{H^1}+\|b(t,\cdot)\|_{H^1}+\|c(t,\cdot)\|_{H^1}\Big]\le C|||f|||_{|\beta|}.$$

Thus, we have the following $L^2(\mathbf{R}^3)$ estimate for $\Gamma(f,f)$:

$$\sum_{|\alpha|\le N}\Big[\|\partial^\alpha h_c\|+\|\partial^\alpha h_i\|+\|\partial^\alpha h_{ij}\|+\|\partial^\alpha h_{ai}\|+\|\partial^\alpha h_a\|\Big]\le C|||f|||_{|\beta|}\cdot[[f]]_w.\tag{4.5}$$

We thus conclude the proof of Lemma 4.3.

THEOREM 4.4. Let $f(t,x,v)$ be the unique classical solution to (1.3). There exist positive constants M_0, $\delta_0=\delta_0(M_0)$, C_1 and C_2 such that if

$$\sum_{|\alpha|+|\beta|\le N}\|w^{|\beta|}\partial^\alpha_\beta f(t)\|^2\le M_0/2,\tag{4.6}$$

then

$$\begin{aligned}\sum_{0<|\alpha|\le N}(L\partial^\alpha f(s),\partial^\alpha f(s)) \ &\ge \delta_0\sum_{0<|\alpha|\le N}\|\partial^\alpha f(s)\|_w^2\\ &\quad -C_1\frac{d}{dt}\int_{\mathbf{R}^3}\nabla\cdot bcdx-C_2\|\{\mathbf{I}-\mathbf{P}\}f\|_w.\end{aligned}$$

Proof. Since $(L\partial^\alpha f(s),\partial^\alpha f(s))\ge\delta\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_w^2$, it thus suffices to show that if (4.6) is valid for some small $M_0>0$, then there is a constant $C>0$ such that

$$\begin{aligned}\sum_{0<|\alpha|\le N}\|\mathbf{P}\partial^\alpha f\|_w^2\le C\sum_{0<|\alpha|\le N}&\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f(s)\|_w^2+C_2\|\{\mathbf{I}-\mathbf{P}\}f\|_w^2\\ &+C_1\frac{d}{dt}\int_{\mathbf{R}^3}\nabla\cdot bcdx.\end{aligned}\tag{4.7}$$

By Lemma 4.1, equivalently, we therefore need only to establish that

$$\sum_{0<|\alpha|\le N}\{\|\partial^\alpha a\|+\|\partial^\alpha b\|+\|\partial^\alpha c\|\}^2 \le C\sum_{|\alpha|\le N}\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f(t)\|_w^2 + C|||f|||^2_{|\beta|}\cdot[[f]]^2_w + C_1\frac{d}{dt}\int_{\mathbf{R}^3}\nabla\cdot bcdx, \tag{4.8}$$

that is,

$$\begin{aligned}&\sum_{0<|\alpha|\le N}\{\|\partial^\alpha a\|+\|\partial^\alpha b\|+\|\partial^\alpha c\|\}^2\\ \le\ & C\sum_{|\alpha|\le N}\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f(t)\|_w^2 + C_1\frac{d}{dt}\int_{\mathbf{R}^3}\nabla\cdot bcdx\\ &+CM_0\{\sum_{0<|\alpha|\le N}\{\|\partial^\alpha a\|+\|\partial^\alpha b\|+\|\partial^\alpha c\|\}+\sum_{|\alpha|\le N}\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f(t)\|_w\}^2,\end{aligned}$$

which implies (4.7) when M_0 is sufficiently small. By Lemma 4.2 and Lemma 4.3, similar arguments as those used in [14, 16] imply that (4.7) holds.

5. Global Solutions. In this section we shall derive a refined energy estimate, which is a crucial step in constructing global solutions. We first review the local existence results. For periodic initial data, the local existence result of (1.3) was given in [12, 15]. By a straightforward modification of the arguments there, we have the following local existence result in the whole space.

LEMMA 5.1. There exist $M_0>0$ and $T^*>0$ such that if $T^*\le M_0/2$ and $\mathcal{E}(f_0)\le M_0/2$, then there is a unique solution $f(t,x,v)$ to (1.3) in $[0,T^*)\times\mathbf{R}^3\times\mathbf{R}^3$ such that

$$\frac{1}{2}|||f(t)|||^2_{|\beta|}+\sum_{|\alpha|+|\beta|\le N}\int_0^t\|\partial^\alpha_\beta f(s)\|^2_{w,|\beta|}ds\le M_0. \tag{5.1}$$

Moreover, $|||f(t)|||^2_{|\beta|}$ is continuous over $[0,T^*)$. If $F_0(x,v)=\mu+\sqrt{\mu}f_0(x,v)\ge 0$, then $F(t,x,v)=\mu+\sqrt{\mu}f(t,x,v)\ge 0$.

Proof of Theorem 1.1. Taking ∂^α on (1.3)$(0<|\alpha|\le N)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\ne 0}\|\partial^\alpha f(t)\|^2+\sum_{|\alpha|\ne 0}(L\partial^\alpha f,\partial^\alpha f)=\sum_{|\alpha|\ne 0}(\partial^\alpha\Gamma(f,f),\partial^\alpha f).$$

The goal is to show that

$$(\partial^\alpha\Gamma(f,f),\partial^\alpha f)\le C|||f|||_{|\beta|}\cdot|||f|||^2_{w,|\beta|}.$$

We split $f=\{\mathbf{P}f\}+\{\mathbf{I}-\mathbf{P}\}f$ to further decompose $(\partial^\alpha\Gamma(f,f),\partial^\alpha f)$ into

$$\begin{aligned}&(\partial^\alpha\Gamma(\mathbf{P}f,\mathbf{P}f),\partial^\alpha f)+(\partial^\alpha\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\mathbf{P}f),\partial^\alpha f)\\ &+(\partial^\alpha\Gamma(\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f),\partial^\alpha f)+(\partial^\alpha\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f),\partial^\alpha f).\end{aligned} \tag{5.2}$$

Now we consider the first part. We plug $\mathbf{P}f = [a(t,x) + b(t,x)\cdot v + c(t,x)|v|^2]\sqrt{\mu}$ into the expression and apply Corollary 2.2, Corollary 3.2 and Lemma 4.1 to get

$$\begin{aligned}
&(\partial^\alpha\Gamma(\mathbf{P}f,\mathbf{P}f),\partial^\alpha f)\\
\le\ & C\sum_{\alpha_1\le\alpha}\int_{\mathbf{R}^3}(|\partial^{\alpha_1}a|+|\partial^{\alpha_1}b|+|\partial^{\alpha_1}c|)(|\partial^{\alpha-\alpha_1}a|+|\partial^{\alpha-\alpha_1}b|+|\partial^{\alpha-\alpha_1}c|)|\partial^\alpha f|_w dx\\
\le\ & C\sum_{\alpha_1\le\alpha}\sup_x(|\partial^{\alpha_1}a|+|\partial^{\alpha_1}b|+|\partial^{\alpha_1}c|)\int_{\mathbf{R}^3}(|\partial^{\alpha-\alpha_1}a|+|\partial^{\alpha-\alpha_1}b|+|\partial^{\alpha-\alpha_1}c|)|\partial^\alpha f|_w dx\\
\le\ & C\sum_{\alpha_1\le\alpha}\sum_{|\ell|\le 2}\Big[\|\partial^{\alpha_1+\ell}a\|+\|\partial^{\alpha_1+\ell}b\|+\|\partial^{\alpha_1+\ell}c\|\Big]\\
&\times\Big[\|\partial^{\alpha-\alpha_1}a\|+\|\partial^{\alpha-\alpha_1}b\|+\|\partial^{\alpha-\alpha_1}c\|\Big]\cdot\|\partial^\alpha f\|_w\\
\le\ & C\sum_{\alpha_1\le\alpha}\|\partial^{\alpha_1}f\|\|\partial^{\alpha-\alpha_1}f\|_w\cdot\|\partial^\alpha f\|_w\\
\le\ & C|||f|||_{|\beta|}\cdot|||f|||^2_{w,|\beta|},
\end{aligned}$$

where we have assumed that $|\alpha_1|\le N/2$ and used $H^2(\mathbf{R}^3)\subset L^\infty(\mathbf{R}^3)$.

We apply Corollary 2.2 and Corollary 3.2 to get

$$\begin{aligned}
&(\partial^\alpha\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\mathbf{P}f),\partial^\alpha f)+(\partial^\alpha\Gamma(\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f),\partial^\alpha f)\\
\le\ & \sum_{\alpha_1\le\alpha}\int_{\mathbf{R}^3}(|\partial^{\alpha_1}a|+|\partial^{\alpha_1}b|+|\partial^{\alpha_1}c|)|\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha-\alpha_1}f|_w|\partial^\alpha f|_w dx\\
&+\sum_{\alpha_1\le\alpha}\int_{\mathbf{R}^3}(|\partial^{\alpha-\alpha_1}a|+|\partial^{\alpha-\alpha_1}b|+|\partial^{\alpha-\alpha_1}c|)|\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha_1}f|_w|\partial^\alpha f|_w dx.
\end{aligned}$$

The first summation of the above inequality is bounded by

$$\begin{aligned}
&\sum_{|\alpha_1|\le|\alpha|/2}\int_{\mathbf{R}^3}(|\partial^{\alpha_1}a|+|\partial^{\alpha_1}b|+|\partial^{\alpha_1}c|)|\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha-\alpha_1}f|_w|\partial^\alpha f|_w dx\\
&\quad+\sum_{|\alpha_1|\ge|\alpha|/2}\int_{\mathbf{R}^3}(|\partial^{\alpha_1}a|+|\partial^{\alpha_1}b|+|\partial^{\alpha_1}c|)|\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha-\alpha_1}f|_w|\partial^\alpha f|_w dx\\
&\le\sum_{|\alpha_1|\le|\alpha|/2}\sup_x(|\partial^{\alpha_1}a|+|\partial^{\alpha_1}b|+|\partial^{\alpha_1}c|)\int_{\mathbf{R}^3}|\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha-\alpha_1}f|_w|\partial^\alpha f|_w dx\\
&\quad+\sum_{|\alpha_1|\ge|\alpha|/2}\sup_x|\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha-\alpha_1}f|_w\int_{\mathbf{R}^3}(|\partial^{\alpha_1}a|+|\partial^{\alpha_1}b|+|\partial^{\alpha_1}c|)|\partial^\alpha f|_w dx\\
&\qquad\le\sum_{|\alpha_1|\le|\alpha|/2}\Big[\sum_{|\ell|\le N}\|\partial^\ell f\|\Big]\|\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha-\alpha_1}f\|_w\|\partial^\alpha f\|_w\\
&\qquad\quad+\sum_{|\alpha_1|\ge|\alpha|/2}\Big[\sum_{|\ell|\le N}\|\{\mathbf{I}-\mathbf{P}\}\partial^\ell f\|_w\Big]\|\partial^{\alpha_1}f\|\|\partial^\alpha f\|_w\\
&\qquad\le C|||f|||_{|\beta|}\cdot|||f|||^2_{w,|\beta|},
\end{aligned}$$

where we have used Lemma 4.1 and Sobolev's inequality.

For the last term of (5.2), we can apply Lemma 2.5 and Lemma 3.4 with $\theta = 0$ and it can be bounded by $C|||f|||_{|\beta|} \cdot |||f|||^2_{w,|\beta|}$.

We therefore conclude by Theorem 4.4 that

$$\frac{d}{dt}\Big[\sum_{|\alpha|\neq 0}\|\partial^\alpha f(t)\|^2 - 2C_1\int_{\mathbf{R}^3}\nabla\cdot bcdx\Big] + \delta_0\sum_{|\alpha|\neq 0}\|\partial^\alpha f(s)\|^2_w$$
$$\leq C|||f|||_{|\beta|}\cdot|||f|||^2_{w,|\beta|} + C\|\{\mathbf{I}-\mathbf{P}\}f\|_w. \tag{5.3}$$

The second step is to treat pure v-derivatives of $\{\mathbf{I}-\mathbf{P}\}f$. We first turn (1.3) into a microscopic-type equation:

$$[\partial_t + v\cdot\nabla_x + L]\{\mathbf{I}-\mathbf{P}\}f = -[\partial_t + v\cdot\nabla_x]\mathbf{P}f + \Gamma(f,f). \tag{5.4}$$

Taking the pure ∂_β ($0 < |\beta| \leq N$) derivatives of (5.4), we obtain

$$[\partial_t + v\cdot\nabla_x]\partial_\beta\{\mathbf{I}-\mathbf{P}\}f + \sum_{|\beta_1|=1} C^{\beta_1}_\beta \partial_{\beta_1} v\cdot\nabla_x\partial_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f + \partial_\beta L\{\mathbf{I}-\mathbf{P}\}f$$
$$= \partial_\beta\Gamma(f,f) - [\partial_t + v\cdot\nabla_x]\partial_\beta\mathbf{P}f - \sum_{|\beta_1|=1} C^{\beta_1}_\beta\partial_{\beta_1}v\cdot\nabla_x\partial_{\beta-\beta_1}\mathbf{P}f. \tag{5.5}$$

By multiplying $w^{2|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f$ to (5.5) and then integrating over $\mathbf{R}^3\times\mathbf{R}^3$, we get

$$\frac{1}{2}\frac{d}{dt}\|w^{|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + (w^{2|\beta|}\partial_\beta L\{\mathbf{I}-\mathbf{P}\}f, \partial_\beta\{\mathbf{I}-\mathbf{P}\}f)$$
$$+ \sum_{|\beta_1|=1} C^{\beta_1}_\beta (w^{2|\beta|}\partial_{\beta_1}v_j\partial^j\partial_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f, \partial_\beta\{\mathbf{I}-\mathbf{P}\}f)$$
$$+ \sum_{|\beta_1|=1} C^{\beta_1}_\beta (w^{2|\beta|}\partial_{\beta_1}v_j\partial^j\partial_{\beta-\beta_1}\mathbf{P}f, \partial_\beta\{\mathbf{I}-\mathbf{P}\}f)$$
$$= (w^{2|\beta|}\partial_\beta\Gamma(f,f), \partial_\beta\{\mathbf{I}-\mathbf{P}\}f) - (w^{2|\beta|}\partial_\beta[\partial_t + v\cdot\nabla_x]\mathbf{P}f, \partial_\beta\{\mathbf{I}-\mathbf{P}\}f). \tag{5.6}$$

We now estimate term by term. Applying (2.12) in Lemma 2.4 and (3.37) in Lemma 3.4 and integrating over $\mathbf{R}^3$, we deduce, for any $\eta > 0$, that

$$(w^{2|\beta|}\partial_\beta L\{\mathbf{I}-\mathbf{P}\}f, \partial_\beta\{\mathbf{I}-\mathbf{P}\}f) \geq \|w^{|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2_w$$
$$-\eta\sum_{|\beta_1|\leq|\beta|}\|w^{|\beta_1|}\partial_{\beta_1}\{\mathbf{I}-\mathbf{P}\}f\|^2_w - C_\eta\|\{\mathbf{I}-\mathbf{P}\}f\|^2_w.$$

We estimate the third term on the left-hand side of (5.6). For any $\eta > 0$, we have

$$(w^{2|\beta|}\partial_{\beta_1}v_j\partial^j\partial_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f, \partial_\beta\{\mathbf{I}-\mathbf{P}\}f)$$
$$\leq \int\int w^{2|\beta|}|\partial^j\partial_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f\partial_\beta\{\mathbf{I}-\mathbf{P}\}f|dxdv$$
$$\leq \|w^{1/2+|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|\|w^{1/2+\{|\beta|-1\}}\partial^j_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f\|$$
$$\leq \eta\|w^{|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2_w + C_\eta\|w^{|\beta-\beta_1|}\partial^j_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f\|^2_w.$$

Here $|\beta-\beta_1| = |\beta|-1$. For the Boltzmann equation, we have used the fact that $\|g\|_\nu$ is equivalent to $\|w^{1/2}g\|$ for $w = [1+|v|]^\gamma$, and for the Landau equation, we have used $\|w^{1/2}g\| \leq C\|g\|_\sigma$ by (3.5) for $w = [1+|v|]^{\gamma+2}$.

By similar arguments as used in the third term and by Lemma 4.1, for any $\eta > 0$, the fourth term on the left-hand side of (5.6) is bounded by

$$\begin{aligned}
&(w^{2|\beta|}\partial_{\beta_1} v_j \partial^j \partial_{\beta-\beta_1}\mathbf{P}f, \partial_\beta\{\mathbf{I}-\mathbf{P}\}f)\\
&\le \eta\|w^{|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|_w^2 + C_\eta\|w^{|\beta-\beta_1|}\partial^j_{\beta-\beta_1}\mathbf{P}f\|_w^2\\
&\le \eta\|w^{|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|_w^2 + C_\eta\sum_{|\alpha_1|=1}\|\partial^{\alpha_1}f\|_w^2.
\end{aligned}$$

By similar arguments as used above, we have that, for any $\eta > 0$, the second term on the right-hand side of (5.6) is bounded by

$$\eta\|w^{|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|_w^2 + C_\eta\sum_{|\alpha_1|=1}\|\partial^{\alpha_1}f\|_w^2.$$

Finally, we estimate the nonlinear collision term. Our goal is still to show

$$(w^{2|\beta|}\partial_\beta\Gamma(f,f), \partial_\beta\{\mathbf{I}-\mathbf{P}\}f) \le C|||f|||_{|\beta|}\cdot|||f|||^2_{w,|\beta|}. \tag{5.7}$$

We treat this term similarly as (5.2) and first decompose $(w^{2|\beta|}\partial_\beta\Gamma(f,f), \partial_\beta\{\mathbf{I}-\mathbf{P}\}f)$ as follows:

$$\begin{aligned}
&(w^{2|\beta|}\partial_\beta\Gamma(\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f), \partial_\beta\{\mathbf{I}-\mathbf{P}\}f) + (w^{2|\beta|}\partial_\beta\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\mathbf{P}f), \partial_\beta\{\mathbf{I}-\mathbf{P}\}f)\\
&+(w^{2|\beta|}\partial_\beta\Gamma(\mathbf{P}f,\mathbf{P}f), \partial_\beta\{\mathbf{I}-\mathbf{P}\}f) + (w^{2|\beta|}\partial_\beta\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f), \partial_\beta\{\mathbf{I}-\mathbf{P}\}f).
\end{aligned} \tag{5.8}$$

For the first term of (5.8), by (2.5) and (3.31) with $\theta = |\beta|$, we have

$$\begin{aligned}
&(w^{2|\beta|}\partial_\beta\Gamma(\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f), \partial_\beta\{\mathbf{I}-\mathbf{P}\}f)\\
&\le C\sum_{|\beta_1|\le|\beta|}\int_{\mathbf{R}^3}(|a|+|b|+|c|)|w^{|\beta_1|}\partial_{\beta_1}\{\mathbf{I}-\mathbf{P}\}f|_w|w^{|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f|_w dx\\
&\le C\sup_x(|a|+|b|+|c|)\sum_{|\beta_1|\le|\beta|}\int_{\mathbf{R}^3}|w^{|\beta_1|}\partial_{\beta_1}\{\mathbf{I}-\mathbf{P}\}f|_w|w^{|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f|_w dx\\
&\le C\sum_{|\ell|\le 2}\|\partial^\ell f\|\sum_{|\beta_1|\le|\beta|}\|w^{|\beta_1|}\partial_{\beta_1}\{\mathbf{I}-\mathbf{P}\}f\|_w\|w^{|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|_w,
\end{aligned}$$

where we have used Lemma 4.1, $w \le 1$ and $w^{|\beta|} \le w^{|\beta_1|}$ for $\beta_1 \le \beta$. Obviously, the last line of the above inequality is controlled by $C|||f|||_{|\beta|}|||f|||^2_{w,|\beta|}$.

The second term of (5.8) has the same upper bound as the first term by (2.6), (3.32) with $\theta = |\beta|$ and Lemma 4.1 as well as by Sobolev's inequality.

For the third term $\partial_\beta\Gamma(\mathbf{P}f,\mathbf{P}f)$, we apply (2.5) and (3.31) with $\theta = |\beta|$ to get

$$\begin{aligned}
&(w^{2|\beta|}\partial_\beta\Gamma(\mathbf{P}f,\mathbf{P}f), \partial_\beta\{\mathbf{I}-\mathbf{P}\}f)\\
&\le C\sum_{|\beta_1|\le|\beta|}\int_{\mathbf{R}^3}(|a|+|b|+|c|)|w^{|\beta_1|}\partial_{\beta_1}\mathbf{P}f|_w|w^{|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f|_w dx\\
&\le \sum_{|\beta_1|\le|\beta|}\int_{\mathbf{R}^3}(|a|^2+|b|^2+|c|^2)|w^{|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f|_w dx\\
&\le C\|a^2+|b|^2+c^2\|\|w^{|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|_w.
\end{aligned}$$

The similar argument as used in (4.4) implies that the first factor of the above is bounded by $C|||f|||_{|\beta|}|||f|||_{w,|\beta|}$. Thus this term is bounded by $C|||f|||_{|\beta|}|||f|||^2_{w,|\beta|}$.

For the last term of (5.8), we directly apply Lemma 2.5 with $\theta = |\beta|$ for the Boltzmann equation to get

$$\begin{aligned}
&(w^{2|\beta|}\partial_\beta\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f),\partial_\beta\{\mathbf{I}-\mathbf{P}\}f)\\
&\le C\Big[\sum_{|\alpha_i|+|\beta_i|\le N}\|w^{|\beta_i|}\partial^{\alpha_i}_{\beta_i}\{\mathbf{I}-\mathbf{P}\}f\|\Big]\|w^{|\beta_2|}\partial_{\beta_2}\{\mathbf{I}-\mathbf{P}\}f\|_\nu\|w^{|\beta|}\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|_\nu\\
&\le C|||f|||_{|\beta|}|||f|||^2_{\nu,|\beta|}.
\end{aligned}$$

For the last term of (5.8) in the Landau equation, we apply (3.38) in Lemma 3.4 with $\theta = |\beta|$ and the Sobolev embedding theorem to get

$$\begin{aligned}
&(w^{2|\beta|}\partial_\beta\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f),\partial_\beta\{\mathbf{I}-\mathbf{P}\}f)\\
&\le C\Big[\Big\{\sum\|\partial^{\alpha_1}_{\beta_1}\{\mathbf{I}-\mathbf{P}\}f\|_{|\beta_1|}\Big\}\Big\{\sum\|\partial^{\alpha_2}_{\beta_2}\{\mathbf{I}-\mathbf{P}\}f\|_{\sigma,|\beta_2|}\Big\}\\
&\qquad+\Big\{\sum\|\partial^{\alpha_1}_{\beta_1}\{\mathbf{I}-\mathbf{P}\}f\|_{|\beta_1|}\Big\}\Big\{\sum\|\partial^{\alpha_2}_{\beta_2}\{\mathbf{I}-\mathbf{P}\}f\|_{\sigma,|\beta_2|}\Big\}\Big]\|\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|_{\sigma,|\beta|}\\
&\le C|||f|||_{|\beta|}|||f|||^2_{\sigma,|\beta|},
\end{aligned}$$

where the summations are over $|\alpha_1|+|\beta_1|\le N$ and $|\alpha_2|+|\beta_2|\le N$ respectively.

If we collect the above inequalities and choose $\eta > 0$ small enough, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|\beta|\ne 0}\|\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2_{|\beta|}+\sum_{|\beta|\ne 0}\|\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2_{w,|\beta|}\\
&\le C|||f|||_{|\beta|}|||f|||^2_{w,|\beta|}+C\sum_{|\alpha_1|=1}\|\partial^{\alpha_1}f\|^2_w+C\|\{\mathbf{I}-\mathbf{P}\}f\|^2_w. \qquad (5.9)
\end{aligned}$$

The standard energy estimate for $\beta = 0$ is

$$\frac{1}{2}\frac{d}{dt}\|f\|^2+(Lf,f)=(\Gamma(f,f),f).$$

It is easily known that

$$\langle\Gamma(f,f),\mathbf{P}f\rangle = 0.$$

A similar argument to (5.2) implies that

$$(\Gamma(f,f),\{\mathbf{I}-\mathbf{P}\}f)\le C|||f|||_{|\beta|}|||f|||^2_{w,|\beta|}.$$

Therefore, we get

$$\frac{1}{2}\frac{d}{dt}\|f\|^2+\delta_1\|\{\mathbf{I}-\mathbf{P}\}f\|^2_w\le C|||f|||_{|\beta|}|||f|||^2_{w,|\beta|}. \qquad (5.10)$$

In the following we consider ∂^α_β derivatives of (1.3) where $|\alpha|\ne 0$, and $|\beta|\ne 0$. Assuming $|\alpha|+|\beta|\le N$ and taking ∂^α_β derivatives of (1.3), we obtain

$$[\partial_t+v\cdot\nabla_x]\partial^\alpha_\beta f+\sum_{|\beta_1|=1}C^{\beta_1}_\beta\partial_{\beta_1}v\cdot\nabla_x\partial^\alpha_{\beta-\beta_1}f+\partial_\beta[L\partial^\alpha f]=\partial^\alpha_\beta\Gamma(f,f). \qquad (5.11)$$

We take the inner product of (5.11) with $w^{2|\beta|}\partial^\alpha_\beta f$ over $\mathbf{R}^3\times\mathbf{R}^3$ and estimate this inner product term by term.

Now we consider the nonlinear collision term $\partial^\alpha_\beta\Gamma(f,f)$. We first decompose the inner product of $\partial^\alpha_\beta f$ with the collision term, $(w^{2|\beta|}\partial^\alpha_\beta\Gamma(f,f),\partial^\alpha_\beta f)$, as follows:

$$\begin{aligned}&(w^{2|\beta|}\partial^\alpha_\beta\Gamma(\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f),\partial^\alpha_\beta f)+(w^{2|\beta|}\partial^\alpha_\beta\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\mathbf{P}f),\partial^\alpha_\beta f)\\&+(w^{2|\beta|}\partial^\alpha_\beta\Gamma(\mathbf{P}f,\mathbf{P}f),\partial^\alpha_\beta f)+(w^{2|\beta|}\partial^\alpha_\beta\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f),\partial^\alpha_\beta f).\end{aligned}\tag{5.12}$$

For the first term of (5.12), we directly apply Corollary 2.1 and Corollary 3.1 to get

$$\begin{aligned}&(w^{2|\beta|}\partial^\alpha_\beta\Gamma(\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f),\partial^\alpha_\beta f)\\ \le\ & C\sum\int_{\mathbf{R}^3}(|\partial^{\alpha_1}a|+|\partial^{\alpha_1}b|+|\partial^{\alpha_1}c|)|w^{|\beta|}\partial^{\alpha_2}_{\beta_2}\{\mathbf{I}-\mathbf{P}\}f|_w|w^{|\beta|}\partial^\alpha_\beta f|_w dx\\ \le\ & C\sum\sup_x(|\partial^{\alpha_1}a|+|\partial^{\alpha_1}b|+|\partial^{\alpha_1}c|)\int_{\mathbf{R}^3}|w^{|\beta|}\partial^{\alpha_2}_{\beta_2}\{\mathbf{I}-\mathbf{P}\}f|_w|w^{|\beta|}\partial^\alpha_\beta f|_w dx\\ \le\ & C|||f|||_{|\beta|}\|w^{|\beta_2|}\partial^{\alpha_2}_{\beta_2}\{\mathbf{I}-\mathbf{P}\}f\|_w\|w^{|\beta|}\partial^\alpha_\beta f\|_w\le C|||f|||_{|\beta|}\cdot|||f|||^2_{w,|\beta|},\end{aligned}$$

where without loss of generality we have assumed that $|\alpha_2|+|\beta_2|\ge N/2$. For the second and third terms of (5.12), they have the same upper bound as the above by similar arguments.

For the last term of (5.12), we directly apply Lemma 2.3 and (3.38) in Lemma 3.4 with $\theta=|\beta|$ to get

$$\begin{aligned}&(w^{2|\beta|}\partial^\alpha_\beta\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f),\partial^\alpha_\beta f)\\ \le\ & C|||f|||_{|\beta|}\|w^{|\beta_2|}\partial^{\alpha_2}_{\beta_2}\{\mathbf{I}-\mathbf{P}\}f\|_w\|w^{|\beta|}\partial^\alpha_\beta f\|_w\le C|||f|||_{|\beta|}\cdot|||f|||^2_{w,|\beta|}.\end{aligned}$$

Using a similar method as above, the inner product of the second term on the left-hand side of (5.11) is bounded by

$$\sum_{|\beta_1|=1}(w^{2|\beta|}\partial_{\beta_1}v_j\cdot\partial^j\partial^\alpha_{\beta-\beta_1}f,\partial^\alpha_\beta f)\le C_\eta\sum_{|\beta_1|=1}\|w^{|\beta_1|}\partial^j\partial^\alpha_{\beta_1}f\|^2_w+\eta\|w^{|\beta|}\partial^\alpha_\beta f\|^2_w.$$

From Lemma 2.4 and Lemma 3.4, we deduce that, for $\eta>0$ small enough, the inner product of the third term on the left-hand side of (5.11) is controlled by

$$(w^{2|\beta|}\partial_\beta L\partial^\alpha f,\partial^\alpha_\beta f)\ge\|w^{|\beta|}\partial^\alpha_\beta f\|^2_w-\eta\sum_{\beta_1\le\beta}\|w^{|\beta_1|}\partial^\alpha_{\beta_1}f\|^2_w-C_\eta\|\partial^\alpha f\|^2_w.$$

Thus, for any $0<|\beta|\le N$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w^{|\beta|}\partial^\alpha_\beta f\|^2+\delta_N\|w^{|\beta|}\partial^\alpha_\beta f\|^2_w&\le C|||f|||_{|\beta|}\cdot|||f|||^2_{w,|\beta|}\\&+C\sum_{|\alpha|\ne0}\|\partial^\alpha f\|^2_w+C_\eta\sum_{|\beta_1|=1}\|w^{|\beta_1|}\partial^j\partial^\alpha_{\beta_1}f\|^2_w+\eta\sum_{\beta_1<\beta}\|w^{|\beta_1|}\partial^\alpha_{\beta_1}f\|^2_w.\end{aligned}\tag{5.13}$$

Thus, for any $0 < |\beta| \le N$ and suitable linear combination of (5.13) depending on $|\beta|$, we have that

$$\sum_{\alpha,\beta\in\Lambda}\Big[\frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta f\|^2_{|\beta|}+\delta_N\|\partial^\alpha_\beta f\|^2_{w,|\beta|}\Big] \le C|||f|||_{|\beta|}\cdot|||f|||^2_{w,|\beta|}+C\sum_{|\alpha|\neq 0}\|\partial^\alpha f\|^2_w, \quad (5.14)$$

where the indices set $\Lambda=\{|\alpha|+|\beta|\le N, |\alpha|\neq 0, |\beta|\neq 0\}$.

A suitable linear combination of (5.3), (5.9), (5.10) and (5.14) yields the following estimates:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big[C_1\sum_{\alpha,\beta\in\Lambda}\|\partial^\alpha_\beta f\|^2_{|\beta|}+C_2\sum_{0<|\beta|\le N}\|\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2_{|\beta|}\\ +C_3\sum_{0<|\alpha|\le N}\|\partial^\alpha f\|^2+C_4\|f\|^2-C_5\int_{\mathbf{R}^3}\nabla\cdot bcdx\Big]\\ +\delta\Big[\sum_{|\beta|\le N}\|\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2_{w,|\beta|}+\sum_{|\alpha|+|\beta|\le N,|\alpha|\neq 0}\|\partial^\alpha_\beta f(t)\|^2_{w,|\beta|}\Big]\\ \le C|||f|||_{|\beta|}\cdot|||f|||^2_{w,|\beta|}. \end{aligned} \quad (5.15)$$

Here C_1, C_2, C_3, C_4, C_5 and δ are positive constants where C_4 is sufficiently large.

We can choose C_4 large enough such that, for some constant $C_6>0$, we have

$$\begin{aligned} y(t)\equiv\ & C_1\sum_{\alpha,\beta\in\Lambda}\|\partial^\alpha_\beta f\|^2_{|\beta|}+C_2\sum_{0<|\beta|\le N}\|\partial_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2_{|\beta|}\\ &+C_3\sum_{0<|\alpha|\le N}\|\partial^\alpha f\|^2+C_4\|f\|^2-C_5\int_{\mathbf{R}^3}\nabla\cdot bcdx\ge C_6|||f|||^2_{|\beta|}.\end{aligned}$$

We easily obtain that $y(t)\le C_7|||f|||^2_{|\beta|}$ for some constant $C_7>0$. Therefore, (5.15) is rewritten as

$$\frac{1}{2}y'(t)+\delta|||f|||^2_{w,|\beta|}\le C|||f|||_{|\beta|}\cdot|||f|||^2_{w,|\beta|}. \quad (5.16)$$

To proceed, we define

$$M\equiv\min\Big\{\frac{\delta}{2C},M_0\Big\}>0,$$

and choose initial data so that $y(0)\le\frac{M}{2}<M_0$. From Lemma 5.1 we may choose $T>0$ so that

$$T=\sup\{t: y(t)\le M\}>0$$

since $y(t)$ is continuous. Notice that for $0\le t\le T$, $M\le M_0$. Thus, the small amplitude assumption (4.6) is valid. We apply (5.16) to get, for $0\le t\le T$, that

$$\frac{1}{2}y'(t)+\delta|||f|||^2_{w,|\beta|}\le C|||f|||\cdot|||f|||^2_{w,|\beta|}\le\frac{\delta}{2}|||f|||^2_{w,|\beta|}.$$

Therefore, an integration over $0\le t\le T$ yields

$$y(t)+\delta\int_0^t|||f(s)|||^2_{w,|\beta|}ds\le y(0)\le\frac{M}{2}<M. \quad (5.17)$$

Since $y(t)$ is continuous in t, this implies that $y(T) \leq M$ if $T < \infty$. This is a contradiction to the definition of T. Hence $T = \infty$. It is straightforward to verify that, for any $s > 0$, $\mathcal{E}(f(s)) \leq C_0 \mathcal{E}(f_0)$ by (5.17).

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