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POLYNOMIAL DECAY TO THERMOELASTIC PLATES WITH MEMORY

By

PEDRO GAMBOA ROMERO

Institute of Mathematics, Universidade Federal do Rio de Janeiro, Av. Brigadeiro Trompowski s/n, Caixa Postal 68530 CEP:21945-970.RJ., Brazil

Abstract. We consider the linear model of thermoelastic plates with memory and we show that the solution decays polynomially with rates that depend on the regularity of the initial data.

1. Introduction. Let Ω be an open bounded set of $\mathbb{R}^n$ with smooth boundary Γ. Here we consider the transverse oscillations of thermoelastic plate configurates over Ω. Denoting by u and θ the vertical deflection and the temperature of the plate, the model that defines the oscillations of the plate is given by

$$\varrho u_{tt} + \gamma\Delta^2 u + m\Delta\theta = 0 \qquad \text{in} \quad \Omega\times]0,+\infty[, \tag{1.1}$$

$$c\theta_t - \int_0^\infty \kappa(s)\Delta\theta(t-s)ds - m\Delta u_t = 0 \qquad \text{in} \quad \Omega\times]0,+\infty[. \tag{1.2}$$

We consider Dirichlet or clamped boundary conditions for u and a Dirichlet boundary condition for θ that is

$$u = \Delta u = \theta = 0 \quad \text{in} \quad \partial\Omega\times]0,+\infty[\tag{1.3}$$

or

$$u = \frac{\partial u}{\partial \nu} = \theta = 0 \quad \text{in} \quad \partial\Omega\times]0,+\infty[\tag{1.4}$$

and has initial value

$$u(0) = u^0, \quad u'(0) = u^1, \quad \theta(0) = \theta^0 \qquad \text{in} \quad \Omega, \tag{1.5}$$

$$\theta(-s) = \psi(s) \qquad \text{in} \quad \Omega\times]0,+\infty[, \tag{1.6}$$

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E-mail address: pgamboa@im.ufrj.br

where u^0, u^1, θ^0 and ψ are given functions and $\kappa : [0, +\infty[\to \mathbb{R}$ is the memory kernel. Denoting by $\mu(s) = -\kappa'(s)$, the hypotheses we impose on μ are the following:

$$\mu \in \mathcal{C}^1(\mathbb{R}^+) \cap H^1(\mathbb{R}^+), \quad \mu(s) \geq 0, \quad \mu'(s) \leq 0 \qquad \forall s \in \mathbb{R}^+, \tag{1.7}$$

$$\mu'(s) + \sigma\mu(s) \leq 0. \tag{1.8}$$

Here we follow the same notations as in [5]. In that article the authors proved that the system is not exponentially stable. By using the La Salle principle, the authors showed that the solution in general goes to zero but they did not report any rate of decay. The main result of this paper is to show that there exists a polynomial rate of decay of the solution that can be improved by improving the regularity of the initial data. We consider both clamped and Dirichlet boundary conditions for the displacement u.

2. Functional spaces and notations. Let us denote by $\mathcal{H}$ the Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and let us denote by $\| \cdot \|_{\mathcal{H}}$ the induced norm of $\mathcal{H}$. Let us introduce the history space

$$\mathcal{M}_1 = L^2_\mu(0, +\infty; H^1_0(\Omega))$$

which is a Hilbert space. Let us denote by $\mathfrak{T} : \mathcal{M}_1 \to \mathcal{M}_1$ the operator given by

$$\mathfrak{T}\eta = -\eta_s \qquad \eta \in \mathcal{D}(\mathfrak{T}) = \{ \eta \in \mathcal{M}_1;\, \eta_s \in \mathcal{M}_1 \quad \text{and} \quad \eta(0) = 0 \}, \tag{2.1}$$

with

$$\langle \mathfrak{T}\eta, \eta \rangle_{\mathcal{M}_1} = \int_0^\infty \mu'(s) ||\nabla \eta(s)||^2 ds \leq 0 \quad \forall \eta \in \mathcal{D}(\mathfrak{T}). \tag{2.2}$$

Let us introduce the *summed past history* of θ as

$$\eta^t(s) = \int_0^s \theta(t - y) dy \qquad (t, s) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{2.3}$$

It is not difficult to see that η satisfies

$$\eta^t_t + \eta_s = \theta \qquad t \in \mathbb{R}^+. \tag{2.4}$$

Moreover

$$\eta^0(s) = \int_0^s \psi(y) dy \quad s \in \mathbb{R}^+ \text{ in } \Omega, \qquad \eta^t(0) = 0 \quad \forall t \geq 0.$$

Denoting by $z(t) = (u(t), v(t), \theta(t), \eta^t)^T, z^0 = (u^0, v^0, \theta^0, \eta^0)^T$ in $\mathcal{H}$, the semigroup formulation of the system (1.1)–(1.6) is given by

$$\begin{cases} \dfrac{d}{dt} z(t) = \mathcal{L} z(t), \\ z(0) = z^0, \end{cases} \tag{2.5}$$

where $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$ is defined as

$$\mathcal{L} \begin{pmatrix} u \\ v \\ \theta \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ -\frac{1}{\varrho}\Delta(\gamma \Delta u + m\theta) \\ +\frac{m}{c}\Delta v + \frac{1}{c}\int_0^\infty \mu(s)\Delta\eta(s) ds \\ \theta + \mathfrak{T}\eta \end{pmatrix}$$

with domain

$$\mathcal{D}(\mathcal{L}) = \left\{ z \in \mathcal{H}_D;\ \gamma\Delta u + m\theta,\ v \in H_0^1 \cap H^2,\ \theta \in H_0^1,\ \mu * \Delta\eta \in L^2,\ \eta \in \mathcal{D}(\mathfrak{T}) \right\}$$

or

$$\mathcal{D}(\mathcal{L}) = \left\{ z \in \mathcal{H}_C;\ \gamma\Delta u + m\theta \in H^2,\ v \in H_0^2,\ \theta \in H_0^1,\ \mu * \Delta\eta \in L^2,\ \eta \in \mathcal{D}(\mathfrak{T}) \right\},$$

where $\mathcal{H}_C = H_0^2 \times L^2 \times L^2 \times \mathcal{M}_1$ and $\mathcal{H}_D = H_0^1 \cap H^2 \times L^2 \times L^2 \times \mathcal{M}_1$. From now on, we denote by $\mathcal{H}$ any of the spaces $\mathcal{H}_D$ or $\mathcal{H}_C$.

THEOREM 2.1. The operator $\mathcal{L}$ is the infinitesimal generator of a C_0 semigroup of contractions over $\mathcal{H}$ that we denote as $S(t)$.

Proof. See [5]. □

In particular we have that problem (2.5) is well posed in the corresponding semigroup spaces.

3. Decay of solutions. In this section we study the polynomial decay of energy to the system

$$\varrho u_{tt} + \gamma\Delta^2 u + m\Delta\theta = 0, \tag{3.1}$$

$$c\theta_t - \int_0^\infty \mu(s)\Delta\eta(s)ds - m\Delta u_t = 0, \tag{3.2}$$

$$\eta_t + \eta_s = \theta, \tag{3.3}$$

with boundary conditions (1.3) or (1.4) and initial condition (1.5)–(1.6). To facilitate our analysis we introduce the following notation:

$$(\mu * \eta)(t) = \int_0^\infty \mu(s)\eta^t(s)ds,$$

$$(\mu\Box\ \nabla\eta)(t) = \int_0^\infty \mu(s)|\nabla\eta^t(s)|^2 ds.$$

Let us define the first- and second-order energy

$$\mathbf{E}_1(t) = \frac{1}{2}\int_\Omega \left\{ \varrho|u_t(t)|^2 + \gamma|\Delta u(t)|^2 + c|\theta(t)|^2 + (\mu\Box\ \nabla\eta)(t) \right\} dx,$$

$$\mathbf{E}_2(t) = \frac{1}{2}\int_\Omega \left\{ \varrho|u_{tt}(t)|^2 + \gamma|\Delta u_t(t)|^2 + c|\theta_t(t)|^2 + (\mu\Box\ \nabla\eta_t)(t) \right\} dx.$$

Using the usual multiplicative techniques we can show that

$$\frac{d}{dt}\mathbf{E}_1(t) = \int_0^\infty \mu'(s)\int_\Omega |\nabla\eta(s)|^2 dxds, \tag{3.4}$$

$$\frac{d}{dt}\mathbf{E}_2(t) = \int_0^\infty \mu'(s)\int_\Omega |\nabla\eta_t(s)|^2 dxds. \tag{3.5}$$

Let us take $q_k \in C^2$ such that $q_k = \nu_k$ on Γ. Let us denote by J_0 the functional

$$J_0(t) = \int_\Omega \varrho u_t q_k \frac{\partial u}{\partial x_k} dx + \frac{1}{\mu_0}\int_\Omega [c\theta - m\Delta u] q_k \frac{\partial \mu * \eta}{\partial x_k}\ .$$

The next Lemma is only necessary when we use clamped boundary conditions.

LEMMA 3.1. If u and θ satisfy the boundary conditions (1.4), then there exists a positive constant C for which we have

$$\frac{d}{dt}J_0(t) \leq -\frac{\gamma}{2}\int_\Gamma |\Delta u|^2\, d\Gamma + CE(t).$$

Proof. Multiplying equation (3.1) by $q_k\frac{\partial u}{\partial x_k}$ and equation (3.2) by $\frac{1}{\mu_0}q_k\frac{\partial \mu*\eta}{\partial x_k}$, summing the product result, and using similar arguments as in [2], our conclusion follows. □

Let us define the functional J as

$$J(t) = -N_1\int_\Omega \theta(\mu*\eta) + \nabla u(\mu*\nabla\eta)dx - \int_\Omega \theta p_t\, dx + \frac{1}{4}\int_\Omega uu_t dx + \frac{2\delta}{\gamma}J_0(t).$$

LEMMA 3.2. Let us suppose that u and θ satisfy (1.4). Then there exist positive constants for which we have

$$\frac{d}{dt}J(t) \leq -\frac{N\mu_0}{2}\int_\Omega |\theta|^2\, dx - \frac{1}{2}\int_\Omega |u_t|^2 dx.$$

Proof. Multiplying equation (3.2) by $\mu*\eta$ we get

$$\int_\Omega \theta_t(\mu*\eta)(t)dx + \int_\Omega |\mu*\nabla\eta|^2 dx + \int_\Omega u_t\mu*\Delta\eta dx = 0. \tag{3.6}$$

Using

$$\frac{d}{dt}\int_\Omega \theta\mu*\eta dx = \int_\Omega \theta_t\mu*\eta dx + \int_\Omega \theta(\mu*\eta_t)dx, \tag{3.7}$$

we get from (3.6) and (3.7) that

$$\frac{d}{dt}\int_\Omega \theta\mu*\eta = -\int_\Omega |\mu*\nabla\eta|^2 dx - \int_\Omega u_t\mu*\Delta\eta dx + \int_\Omega \theta(\mu*\eta_t)dx.$$

Note that

$$-\int_\Omega u_t\mu*\Delta\eta dx = \frac{d}{dt}\int_\Omega \nabla u(\mu*\nabla\eta)dx - \int_\Omega \nabla u(\mu*\nabla\eta_t)dx.$$

From here we have

$$\frac{d}{dt}\underbrace{\left\{\int_\Omega \theta\mu*\eta + \nabla u(\mu*\nabla\eta)dx\right\}}_{:=-F(t)} \tag{3.8}$$

$$= -\int_\Omega |\mu*\nabla\eta|^2 dx - \int_\Omega \nabla u(\mu*\nabla\eta_t)\, dx + \int_\Omega \theta(\mu*\eta_t)dx.$$

Substituting equations (3.3) into (3.8) and using

$$\int_\Omega |\mu*\nabla\eta|^2\, dx \leq \mu_0\int_\Omega \mu\Box\, \nabla\eta dx, \tag{3.9}$$

we get

$$\frac{d}{dt}F(t) \leq -\frac{\mu_0}{2}\int_\Omega |\theta|^2 dx + C_\epsilon\int_\Omega \mu\Box\, \nabla\eta + \mu\Box\, \nabla\eta_t dx + \epsilon\int_\Omega |\nabla u|^2. \tag{3.10}$$

Let us introduce the following multipliers:

$$\Delta\omega = \theta, \quad \Delta p = u, \quad \text{in} \quad \Omega, \qquad p = \theta = 0, \quad \text{on} \quad \Gamma. \tag{3.11}$$

Multiplying equation (3.2) by p_t defined above, we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega \theta p_t &= \int_\Omega \theta_t p_t + \int_\Omega \theta p_{tt}\\ &= \int_\Omega (\mu * \Delta\eta + \Delta u_t)p_t + \int_\Omega \Delta\omega p_{tt}\\ &= \int_\Omega |u_t|^2 + \int_\Omega (\mu*\eta)u_t + \int_\Omega \omega(-\Delta^2 u - \Delta\theta)\\ &= \int_\Omega |u_t|^2 + \int_\Omega (\mu*\eta)u_t + \int_\Gamma \frac{\partial\omega}{\partial\nu}\Delta u\, d\Gamma - \int_\Omega \theta\Delta u - \int_\Omega |\theta|^2. \end{aligned} \tag{3.12}$$

Note that for any $\epsilon > 0$ there exists $C_\epsilon > 0$ for which we have

$$\left|\int_\Omega (\mu*\eta)u_t\right| \le \epsilon\int_\Omega |u_t(t)|^2 + C_\epsilon \int_\Omega \mu\Box\,\nabla\eta dx, \tag{3.13}$$

and

$$\left|\int_\Omega \theta\Delta u\right| \le C_\epsilon\int_\Omega |\theta(t)|^2 dx + \epsilon\int_\Omega |\Delta u(t)|^2 dx. \tag{3.14}$$

□

Since the normal derivative of ω is bounded by the L^2-norm of θ, from inequalities (3.12), (3.13), and (3.14), we get

$$\begin{aligned}-\frac{d}{dt}\int_\Omega \theta p_t &\le -\frac{1}{2}\int_\Omega |u_t(t)|^2 dx + c_{\epsilon\delta}\int_\Omega |\theta(t)|^2 dx\\ &\quad +\epsilon\int_\Omega |\Delta u(t)|^2 + \delta\int_\Gamma |\Delta u(t)|^2 + C_\epsilon\int_\Omega \mu\Box\,\nabla\eta dx. \end{aligned} \tag{3.15}$$

Multiplying equation (3.1) by u we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega u u_t dx &= \int_\Omega |u_t(t)|^2 dx + \int_\Omega u.u_{tt} dx\\ &= \int_\Omega |u_t(t)|^2 dx + \int_\Omega u(-\Delta^2 u - \Delta\theta)dx\\ &\le \int_\Omega |u_t(t)|^2 dx - \frac{1}{2}\int_\Omega |\Delta u|^2 + \frac{1}{2}\int_\Omega |\theta|^2 dx. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16) we get

$$\begin{aligned}-\frac{d}{dt}\left\{\int_\Omega \theta p_t - \frac{1}{4}\int_\Omega u u_t dx\right\} &\le -\frac{1}{8}\int_\Omega |u_t(t)|^2 + |\Delta u(t)|^2 dx\\ &\quad +c_\epsilon\int_\Omega |\theta(t)|^2 dx + C_\epsilon\int_\Omega \mu\Box\,\nabla\eta dx + \delta\int_\Gamma |\Delta u(t)|^2. \end{aligned} \tag{3.17}$$

Recalling the definition of J and taking ϵ and δ small enough, we conclude that there exists $\gamma_0 > 0$ for which we have

$$\frac{d}{dt}J(t) \le -\gamma_0\mathbf{E}(t) + c_\epsilon\int_\Omega \mu\Box\,\nabla\eta + \mu\Box\,\nabla\eta_t dx.$$

In the case where the boundary condition (1.3) holds, Lemma 3.2 also holds easily because in this case we have no boundary terms to estimate. We are now ready to prove the main result of this paper.

THEOREM 3.1. Let us take initial data $z_0 \in D(\mathcal{L})$. Then there exists a positive constant C for which we have

$$\|S(t)z_0\|_{\mathcal{H}} \leq \frac{C}{\sqrt{t}}\|z_0\|_{D(\mathcal{L})}. \tag{3.18}$$

Moreover for any $m \in \mathbb{N}$, we get

$$\|S(t)z_0\|_{\mathcal{H}} \leq \frac{C}{t^{m/2}}\|z_0\|_{D(\mathcal{L}^m)}. \tag{3.19}$$

Proof. Taking $\mathfrak{F}(t) = N\mathbf{E}(t) + N\mathbf{E}_2(t) + J(t)$, for N large enough, we get $\mathfrak{F}(t) \geq 0$ and also,

$$\frac{d}{dt}\mathfrak{F}(t) \leq -\gamma_0\mathbf{E}(t) \quad \Rightarrow \quad \gamma_0\int_0^\infty \mathbf{E}(t)\,dt \leq \mathfrak{F}(0) \leq C(\mathbf{E}(0) + \mathbf{E}_2(0)).$$

From here we get

$$\mathbf{E}(t) \leq C\frac{\{\mathbf{E}(0) + \mathbf{E}_2(0)\}}{t}. \tag{3.20}$$

So we get (3.18) (not optimal). Note that (3.18) implies

$$\|S(t)\mathcal{L}^{-1}z_0\|_{\mathcal{H}} \leq \frac{C}{\sqrt{t}}\|z_0\|_{\mathcal{H}} \;\Rightarrow\; ||s(t)\mathcal{L}^{-1}|| \leq \frac{C}{\sqrt{t}}.$$

Therefore to get (3.19) we consider

$$\|S(t)\mathcal{L}^{-m}\| = \|S(\frac{t}{m})^m\mathcal{L}^{-m}\| \leq \|S(\frac{t}{m})\mathcal{L}^{-1}\|^m \leq C^m\left[\left(\frac{t}{m}\right)^{-1/2}\right]^m.$$

From here our conclusion follows. □

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