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IMPULSIVE DISPLACEMENT OF A LIQUID IN A PIPE AT HIGH REYNOLDS NUMBERS

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Abstract. We consider the problem of an impulsive displacement of a liquid, originally at rest in a circular pipe, which is displaced by another liquid. The purpose of this analysis is to show that at a sufficiently high inertia the initial essentially inviscid motion can be extended to cover the entire displacement process, thus creating an inviscid window to which an inviscid analysis can be applied. We simplify the problem first, by considering a 1-liquid problem where the displacing liquid and displaced liquid are the same. We identify two characteristic times in this problem: the time it takes an inviscid liquid to be displaced, and the time it takes a viscous liquid to attain a steady state. Taking the ratio of the two defines the Reynolds number for the problem and we show that the motion becomes essentially inviscid once the Reynolds number is sufficiently high. We obtain the general solution of the 1-liquid problem which determines the nondimensional viscous displacement time as a function of the Reynolds number. We derive from the general solution: a critical Reynolds number above which the motion remains unsteady throughout the entire displacement process, and a formula which determines quantitatively whether applying an inviscid analysis to the 1-liquid viscous problem at a given Reynolds number is admissible within an acceptable error tolerance. We also show that at the limit the Reynolds number approaches infinity the viscous displacement time approaches the inviscid displacement time and that the velocity profile and the shape of the material surface separating the displacing from the displaced liquid approach their counterpart in the inviscid solution. Second, based on these results we propose that an inviscid solution is applicable to the 2-liquid viscous problem once the condition of a high Reynolds number is independently met by the two participating liquids. We obtain the solution to the inviscid 2-liquid displacement problem and calculate various examples. Finally, we present a stability analysis of the flat interface between the two inviscid

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liquids, which shows which of the examples is stable, neutrally stable, or unstable. The paucity of data for an impulsive displacement in the high Reynolds number range makes quantitative comparisons difficult. However, the excellent agreement obtained between the critical Reynolds number derived in this analysis and the result obtained in a numerical analysis of the viscous 2-liquid problem elsewhere constitutes at least a partial validation of the theory. Additional confirmation is obviously recommended.

1. Introduction. We consider a liquid at rest in an inclined circular pipe connecting two reservoirs A and B containing two immiscible liquids A and B, respectively. The liquid originally at rest in the pipe is a liquid B. When the constraints maintaining the liquid at rest are suddenly removed, the liquid is set impulsively into motion. The forces contributing to this motion are the pressure difference between the entrance pressure p_A and the exit pressure p_B, the gravitational forces, and the viscous forces resisting the motion. The pipe is inclined at an angle θ to the vertical direction so that when θ equals zero the pipe axis is in the direction of the gravitational acceleration. Liquid displacement under impulsive conditions has been studied extensively for some time; however most of the effort, whether experimental or theoretical, was directed at steady state and low inertia. The papers by Taylor and Saffman (1958) and Reinlet and Saffman (1985) are just two examples. More recently, a numerical solution of the time-dependent viscous displacement problem, including capillarity effects, was presented by Dimakopoulos and Tsamopoulos (2003). The effect of inertia was presented in their paper as a sequence of Reynolds numbers in the low to mid inertia range. It is our purpose to present a theoretical analysis of the impulsive displacement motion of a viscous liquid at the high inertia range and show that for a high Reynolds number the solution for the displacement time can be obtained by an inviscid analysis including a quantified estimate of its deviation from the viscous solution. It is well known that under impulsive conditions the initial motion is irrotational (see Batchelor (1967)). As shown by Kleinstein (1988), this irrotational motion cannot persist and must become rotational through the influence of the boundaries. In the present problem, similar impulsive conditions exist, and it is expected that an essentially inviscid motion exists initially. We seek to determine the necessary condition for the initial essentially inviscid motion to persist throughout the entire displacement process. We designate the time interval where the motion remains essentially inviscid during the entire displacement process as an inviscid window. First, we simplify the problem by considering a 1-liquid model where the displacing liquid and the displaced liquid are the same. This model is free of capillarity effects as well as the penetration and finger formation associated with the 2-liquid problem. By considering the dimensional viscous and inviscid solutions of the 1-liquid displacement problem we identify two characteristic times: an inviscid time, the time it takes to displace an inviscid liquid, and a viscous time, the time it takes a viscous liquid to attain a steady state. Taking the ratio of these two time scales results in the pipe displacement Reynolds number. Expressing the dimensional solution of the 1-liquid problem in terms of this Reynolds number we obtain a general solution which gives the nondimensional viscous displacement time as a function of the Reynolds number. From this general solution we derive two central results to this analysis: we determine a critical Reynolds number above which the motion remains unsteady throughout the displacement problem, and we derive a formula which gives the percent deviation of the inviscid displacement time from the viscous displacement time as a function of the Reynolds number. Hence we can test if at a specified finite Reynolds number an inviscid solution is admissible or inadmissible according to whether the percent deviation falls within an acceptable tolerance. We also show that at the limit as the Reynolds number approaches infinity, the viscous displacement time approaches the inviscid displacement time, and both the form of the velocity profile and the shape of the material surface separating the displacing liquid from the displaced liquid approach their counterparts in the inviscid motion. Based on the above results, we propose that an inviscid window exists for the 2-liquid viscous problem once the condition of a high Reynolds number, as compared with the critical Reynolds number, is independently met by the two participating liquids. In section 4, we present the solution to the 2-liquid inviscid problem. Various numerical examples are presented in section 5 where, in addition to the displacement times, the Reynolds numbers associated with the two participating liquids are also computed to check for admissibility. In section 6 we apply Taylor's analysis (Taylor (1950)) of the stability of an accelerating infinite interface to the study of the stability of the finite material interface in the 2-liquid inviscid problem. The results obtained are in general agreement with Taylor's results. In section 7 we make comparisons with the numerical calculations of Dimakopoulos and Tsamopoulos (2003). We show a very good agreement between our predictions of the critical Reynolds number and the results observed in their calculations. Since the high Reynolds numbers they present are only slightly above the critical Reynolds number, the comparison we can make is at best qualitative. For a quantitative assessment of the validity of the theory, additional data at a high Reynolds number, whether numerical or experimental, is highly desirable.

2. Hypotheses and definitions. We make the following hypotheses:

H1. The liquids A and B are Newtonian, incompressible and immiscible.

H2. The liquid pressure at the pipe entrance p_A is a constant.

H3. The liquid pressure at the pipe exit p_B is a constant.

H4. The velocity of the liquids in the pipe on either side of the 2-liquid interface is unidirectional, i.e., $V = (u, v, w) = (u, 0, 0)$.

H5. The liquids are inviscid (in the 2-liquid inviscid analysis).

H6. The liquids are initially at rest.

H7. The interface remains flat and its normal remains parallel to the pipe axis (in the 2-liquid inviscid analysis).

H8. The pressure on the interface is uniform.

We also make the following definitions:

t_{DI} An inviscid displacement time

t_{DV} A viscous displacement time

$\tilde{t} = \nu t/a^2$ Nondimensional viscous time

$\bar{t} = t/t_{DI}$ Nondimensional displacement time

$\hat{t} = \sqrt{g/l}\,t$

t_∞ The characteristic time it takes to attain a steady state (see Appendix A)

$\mathrm{R}_{ep} = \frac{d}{\nu}\sqrt{\frac{p_A - p_B + \rho g l \cos\theta}{2\rho}}\left(\frac{l}{d}\right)^{-1}$ The pipe displacement Reynolds number

ρ_A The displacing liquid density

ρ_B The displaced liquid density

$\lambda = 1 - \rho_A/\rho_B$

$P = (p_A - p_B)/(\rho_B g l)$ Nondimensionalised pressure difference

θ The pipe inclination to the vertical direction (when $\theta = 0$ the pipe axis is in the direction of the gravitational acceleration).

3. Analysis of the 1-liquid viscous displacement problem.

3.1. *The viscous and inviscid solutions.* The initial boundary value problem describing the impulsively starting motion of a viscous fluid initially at rest in a horizontal circular pipe of a fixed diameter d, subject to a constant pressure gradient, is defined by the differential equation

$$\frac{\partial u}{\partial t} - \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{3.1}$$

with the initial conditions at $t = 0$, $u = 0$ for $r \in (0, a)$, and the following boundary conditions: at $r = 0$, $\partial \mathrm{u}/\partial \mathrm{r} = 0$, and at $r = a$, $u = 0$ for all t, where $a = d/2$ is the radius of the pipe. The solution for the velocity distribution, first derived by Szymanski (1932), can be found for example in Batchelor (1967) as

$$u = -\frac{1}{\rho\nu}\frac{dp}{dx}a^2\left(\frac{1-(r/a)^2}{4} - 2\sum_{n=1}^{\infty}\frac{J_0(\beta_n\bar{r})}{(\beta_n)^3 J_1(\beta_n)}e^{-\beta_n^2\nu t/a^2}\right), \tag{3.2}$$

where β_n are the positive roots of the Bessel function $J_0(\beta_n) = 0$.

The mass flow rate

$$\dot{m} = \int \rho u dA = \int_0^a \rho u 2\pi r dr,$$

as computed from (3.2), is given by

$$\dot{m} = -\frac{1}{\nu}\frac{dp}{dx}\frac{\pi}{8}a^4\left(1 - 32\sum_{n=1}^{\infty}\frac{1}{{\beta_n}^4}e^{-(\beta_n)^2\nu t/a^2}\right). \tag{3.3}$$

Integrating equation (3.3) over time results in the total mass $M(t)$ displaced out of the pipe at t as

$$M(t) = \int_0^t \dot{m}dt = -\frac{1}{\nu}\frac{dp}{dx}\frac{\pi}{8}a^4\left(t - 32\frac{a^2}{\nu}\sum_{n=1}^{\infty}\frac{1}{(\beta_n)^6}\left(1 - \exp\left(-(\beta_n)^2\frac{\nu}{a^2}t\right)\right)\right). \tag{3.4}$$

Now, the mass contained initially in the pipe is $\pi a^2 l\rho$; thus, the time it takes to displace this mass out of the pipe is obtained by solving for t from the equation

$$-\frac{1}{\nu}\frac{dp}{dx}\frac{\pi}{8}a^4\left(t - 32\frac{a^2}{\nu}\sum_{n=1}^{\infty}\frac{1}{(\beta_n)^6}\left(1 - \exp\left(-(\beta_n)^2\frac{\nu}{a^2}t\right)\right)\right) = \pi a^2 l\rho, \tag{3.5}$$

which determines t_{DV}, *the viscous displacement time.*

Following the same procedure for an inviscid fluid, starting with the differential equation

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{dp}{dx},$$

and the initial conditions $t = 0,\ u = 0$, gives the inviscid velocity as

$$u = -\frac{1}{\rho}\frac{dp}{dx}t. \tag{3.6}$$

Equating the corresponding inviscid fluid displaced mass $M(t)$ to the mass originally at rest in the pipe, $\pi a^2 l\rho$, we obtain an explicit solution for the *inviscid displacement time* as

$$t_{DI} = \frac{l}{\sqrt{\frac{p_A - p_B}{2\rho}}}, \tag{3.7}$$

where $\sqrt{(p_A - p_B)/2\rho}$ is the average inviscid velocity.

In equation (3.7) we set $-\frac{dp}{dx} = \frac{p_A - p_B}{l}$, where p_A is the pressure at the entrance, p_B the pressure at the exit, ρ is the density and l is the length of the pipe.

The state of the motion of the liquid in the pipe, following an impulsive start, changes with time from an initially unsteady, essentially inviscid motion, through an unsteady viscous motion, to finally a fully developed viscous steady state, a Poiseuille flow. In the displacement problem the displacement process can terminate in any of these states depending on a particular combination of the five dimensional parameters: the pipe geometry defined by its length l and radius a, the pressure difference $p_A - p_B$, the density ρ, and the kinematic viscosity ν. It is our primary interest to determine those solutions of equation (3.5) which at the end of the displacement process, when $t = t_{DV}$, the motion, while slightly more viscous, still remains essentially inviscid. We define an interval $0 \le t \le t_{DV}$ where the motion remains essentially inviscid during the entire displacement process, *an inviscid window*. It is expected, from its definition, that in an interval where the motion remains essentially inviscid the viscous displacement time approaches the inviscid displacement time:

$$t_{DV} \to t_{DI}. \tag{3.8}$$

In the next section we determine the nondimensional parameter for the displacement problem and the condition imposed on it so that the resulting motion is essentially inviscid.

3.2. *The pipe displacement Reynolds number and the inviscid window.* We identify two characteristic times in the problem: an inviscid characteristic time t_{DI}, defined as the time it takes to displace the entire inviscid liquid originally at rest in the pipe, given by equation (3.7) as

$$t_{DI} = \frac{l}{\sqrt{\frac{p_A - p_B}{2\rho}}},$$

and a viscous characteristic time t_∞, defined as the time it takes for a viscous liquid to attain a fully developed viscous steady state in the pipe, i.e., a Poiseuille flow. The viscous characteristic time was derived in Appendix A, and it is given as

$$t_\infty = 0.1907 a^2/\nu. \tag{3.9}$$

We note first from equations (3.7) and (3.9) that the two characteristic times are independent, namely $t_{DI} = f(l, p_A - p_B, \rho)$ and $t_\infty = g(a, \nu)$, and that all of the five dimensional parameters of the displacement problem are included in the definition of the two characteristic times. Hence by taking the ratio t_{DI}/t_∞ we obtain the nondimensional parameter for the displacement problem. Second, it follows from the definition of the numerator that the more inviscid the motion the smaller t_{DI} is, while it follows from the definition of the denominator that the more inviscid the motion the larger t_∞ is. Thus by taking the ratio t_{DI}/t_∞ we obtain a parameter which becomes smaller the more inviscid the motion is. Substituting from equations (3.7) and (3.9) into the ratio results in the nondimensional parameter,

$$\frac{t_{DI}}{t_\infty} = 21\left(\frac{l}{d}\right)\left(\frac{d\sqrt{\frac{p_A - p_B}{2\rho}}}{\nu}\right)^{-1} = 21{R_{ep}}^{-1}, \tag{3.10}$$

where in (3.10) we introduce

$$R_{ep} = \left(\frac{d\sqrt{\frac{p_A - p_B}{2\rho}}}{\nu}\right)\left(\frac{l}{d}\right)^{-1} \tag{3.11}$$

as the *pipe displacement Reynolds number.* Now, when the motion is essentially inviscid, by the arguments presented above, $t_{DI}/t_\infty \ll 1$ and, by equation (3.10), the Reynolds number satisfies the condition $R_{ep} \gg 1$. Assuming $R_{ep} \gg 1$, we obtain from (3.10) that $t_{DV}/t_\infty \ll 1$ and thus the motion is essentially inviscid. Hence, the condition

$$R_{ep} \gg 1 \tag{3.12}$$

is a necessary and sufficient condition for an essentially inviscid motion to exist. In the next section we show that the condition (3.12) is also the necessary condition for an inviscid window.

3.3. *The general solution of the* 1*-liquid displacement problem.* Expressing the dimensional solution to the displacement problem, equation (3.5), in terms of the Reynolds number defined by equation (3.11) and the nondimensional displacement time as $\tilde{t}_{DV} = \nu t_{DV}/a^2$, we arrive at a general solution of the 1-liquid displacement problem in the form

$$\tilde{t}_{DV} - 32\sum_{n=1}^{\infty}\frac{1}{(\beta_n)^6}\left(1 - \exp\left(-(\beta_n)^2\tilde{t}_{DV}\right)\right) = 64{R_{ep}}^{-2}, \tag{3.13}$$

which gives $\tilde{t}_{DV}$as a function of the Reynolds numberR_{ep}. An analysis of equation (3.13) shows how the various states of motion correlate with specific ranges of the Reynolds number. The condition that a Poiseuille flow emerges just as the displacement process ends is defined by equating the viscous displacement time to the viscous characteristic times, i.e.,

$$\frac{t_{DV}}{t_\infty} = \frac{\tilde{t}_{DV}}{\tilde{t}_\infty} = 1. \tag{3.14}$$

This is the time the motion undergoes a transition from an unsteady to a steady state. From equations (3.14) and (3.9) we obtain that $\tilde{t}_{DV} = \tilde{t}_\infty = 0.1907$. Thus we calculate from equation (3.13) the critical Reynolds number corresponding to this time as

$$(R_{ep})_C = 28.47. \tag{3.15}$$

Accordingly, for all Reynolds numbers in the range $R_{ep} < (R_{ep})_C$, the motion undergoes a transition from its initial essentially inviscid motion through an unsteady viscous motion into a fully developed viscous steady state, *before* the displacement process is complete, while for all $R_{ep} > (R_{ep})_C$, the motion is unsteady throughout the entire displacement process. In this range, the velocity profile which is initially a flat uniform profile becomes curved but falls short of becoming a parabolic profile. With the critical Reynolds number defined, we can sharpen the inequality (3.12) to

$$R_{ep} \gg (R_{ep})_C = 28.47. \tag{3.16}$$

Next we consider the limit of equation (3.13) as $R_{ep} \to \infty$. By inspection, equation (3.13) shows that as $R_{ep} \to \infty$, $\tilde{t}_{DV} \to 0$. As shown in (3.8), it is anticipated that t_{DV} should approach t_{DI}, a finite positive number, or, nondimensionalised that $\tilde{t}_{DV} \to \tilde{t}_{DI}$. To show that indeed this is the case we stretch the coordinate $\tilde{t}$ by introducing a new time variable scaled by the inviscid characteristic time, namely, $\bar{t} = t/t_{DI}$. We have then

$$\bar{t} = (1/4)\,\tilde{t}R_{ep}. \tag{3.17}$$

Expressing $\tilde{t}_{DV}$ in equation (3.13) in terms of the new variable by taking $\tilde{t}_{DV} = 4\bar{t}_{DV}{R_{ep}}^{-1}$, then expanding (3.13) with respect to ${R_{ep}}^{-1}$ as $R_{ep} \to \infty$, gives, after retaining a few leading terms,

$$\bar{t}^2_{DV} - \frac{16}{3}{R_{ep}}^{-1}\bar{t}^3_{DV} - 1 + O\left({R_{ep}}^{-2}\bar{t}^4_{DV}\right) = 0. \tag{3.18}$$

Taking the limit of (3.18) as $R_{ep} \to \infty$ gives

$$\bar{t}_{DV} \to 1. \tag{3.19}$$

Expressing (3.19) dimensionally we obtain

$$t_{DV} \to t_{DI}, \tag{3.20}$$

which is what is expected at an inviscid window. Thus we showed that the condition $R_{ep} \to \infty$ is the necessary and sufficient condition for both an essentially inviscid motion and an inviscid window.

The application of an inviscid analysis to the viscous displacement problem involves an approximation which can be calculated with the aid of equation (3.13). We define the percent deviation of the inviscid displacement time from the viscous displacement time as

$$pdiv\,(R_{ep}) = 100\left(1 - \frac{t_{DI}}{t_{DV}}\right) = 100\left(1 - \frac{4{R_{ep}}^{-1}}{\tilde{t}_{DV}(R_{ep})}\right). \tag{3.21}$$

Equation (3.21) assigns a quantitative measure to the quality of the inviscid window. The percent deviation at Reynolds numbers higher than 1.45×10^4 is less than 1 percent, at a Reynolds number equal to 612 the percent deviation increases to 5 percent. It increases to 10 percent at a Reynolds number equal to 162 and to 20 percent at a Reynolds number equal to 45.

The above analysis can be extended to an inclined pipe by replacing the constant pressure gradient $\frac{\partial p}{\partial x}$ by the constant modified pressure gradient $\frac{\partial \Pi}{\partial x}$, where the modified pressure Π is defined by $\Pi = p - \rho g_i x_i$ (Batchelor 1967, p.176). Accordingly the pressure difference $p_A - p_B \geq 0$ can be replaced with $p_A - p_B + \rho gl \cos\theta \geq 0$, where the radial component, which is of order $O(\rho gd \sin\theta)$, has been neglected.

In the following subsections we investigate the form of the velocity profile, the displacement thickness, and the shape of the material interface in an inviscid window and show that as the Reynolds number increases they approach their inviscid counterpart.

3.4. *The effect of high Reynolds numbers on the velocity profile, the displacement thickness, and on the initial material surface.*

3.4.1. *The velocity distribution at high Reynolds numbers.* Writing the velocity, equation (3.2), in terms of the time variable $\bar{t} = (1/4)\, R_{ep}\tilde{t}$ (equation (3.17)), then expanding the exponential term for small ${R_{ep}}^{-1}$ we obtain to second order

$$u = -\frac{1}{\rho\nu}\frac{dp}{dx}a^2$$
$$\times \left(\frac{1}{4}\left(1-(\bar{r})^2\right) - 2\sum_{n=1}^{\infty}\frac{J_0\left(\beta_n\bar{r}\right)}{\left(\beta_n\right)^3 J_1\left(\beta_n\right)}\left(1-4(\beta_n)^2\bar{t}{R_{ep}}^{-1}+8(\beta_n)^4\left(\bar{t}{R_{ep}}^{-1}\right)^2+..\right)\right). \tag{3.22}$$

In the above expansion the first term in the summation cancels the parabolic velocity profile, and the remaining two terms become

$$u = -\frac{1}{\rho}\frac{dp}{dx}t\left(2\sum_{n=1}^{\infty}\frac{J_0\left(\beta_n\bar{r}\right)}{\left(\beta_n\right)J_1\left(\beta_n\right)} - 4\bar{t}{R_{ep}}^{-1}\left(\sum_{n=1}^{\infty}\frac{J_0\left(\beta_n\bar{r}\right)\left(\beta_n\right)}{J_1\left(\beta_n\right)}\right)+..\right). \tag{3.23}$$

We define an axisymmetric Heaviside function as

$$H(1-\bar{r}) = 1 \text{ when } 0 \leq \bar{r} < 1 \text{ and } H(1-\bar{r}) = 0 \text{ when } \bar{r} = 1. \tag{3.24}$$

Then, expanding the function $H(1-\bar{r})$ in terms of the Bessel functions $J_0\left(\beta_n\bar{r}\right)$, we obtain

$$H(1-\bar{r}) = 2\sum_{n=1}^{\infty}\frac{1}{\beta_n}\frac{J_0(\beta_n\bar{r})}{J_1(\beta_n)}, \tag{3.25}$$

which equals the first summation in (3.23). We designate the second summation in (3.23) as the function

$$f\left(\bar{r}\right) = \sum_{n=1}^{\infty}\frac{J_0\left(\beta_n\bar{r}\right)\left(\beta_n\right)}{J_1\left(\beta_n\right)}. \tag{3.26}$$

The integral of the function $f(\bar{r})$ over the cross section of the pipe is given by

$$\int_0^1 f\left(\bar{r}\right)\bar{r}d\bar{r} = 1.$$

Substituting equations (3.24) to (3.26) into (3.23) gives the viscous velocity profile at high Reynolds numbers as

$$u(\bar{r},t) = -\frac{1}{\rho}\frac{dp}{dx}t\left(H(1-\bar{r}) - 4f\left(\bar{r}\right)\bar{t}{R_{ep}}^{-1}...\right) \tag{3.27}$$

and at the limit as $R_{ep} \to \infty$, we obtain the viscous velocity profile as

$$u(\bar{r},t) = -\frac{1}{\rho}\frac{dp}{dx}H(1-\bar{r})t. \tag{3.28}$$

Comparing equation (3.28) with the inviscid solution equation (3.6) shows that the two solutions coincide for all $\bar{r}$ in $0 \leq \bar{r} < 1$ and deviate at $\bar{r} = 1$, where the velocity of the viscous solution vanishes at the wall. We have thus demonstrated that within $O\left({R_{ep}}^{-1}\right)$ the viscous motion is inviscid everywhere in the pipe, except next to the wall, where it satisfies the no-slip condition.

It is instructive to investigate the viscous layer next to the wall and to determine the shear stress τ_w at the wall. We stretch the coordinate $(1 - \bar{r})$ by defining a stretched coordinate η as

$$\eta = R_{ep}\left(1 - \bar{r}\right),$$

which vanishes at the wall ar $\bar{r} = 1$. Expressing the Bessel function $J_0\left(\beta_n \bar{r}\right)$ in terms of η we obtain:

$$J_0(\beta_n \bar{r}) = J_0\left(\beta_n\left(1 - \eta {R_{ep}}^{-1}\right)\right).$$

Expanding with respect to small η gives:

$$J_0\left(\beta_n\left(1 - \eta {R_{ep}}^{-1}\right)\right) = J_0\left(\beta_n\right) + J_1(\beta_n){R_{ep}}^{-1}\beta_n \eta + \cdots.$$

Substituting this result into the Heaviside function (3.25), and then substituting the result into equation (3.28), gives the velocity in the viscous layer in the form

$$u = -\frac{2}{\rho}\frac{dp}{dx}t\mathrm{R}_{ep}{}^{-1}\eta. \tag{3.29}$$

The shear, computed from (3.29) and nondimensionalised with respect to the pressure difference, gives

$$\frac{\tau_w}{p_A - p_B} = 4{R_{ep}}^{-1}\frac{d}{l}\bar{t}.$$

At $t = t_{DV}$, $\bar{t} = t_{DV}/t_{DI} = 1 + O({R_{ep}}^{-1})$; hence this result gives the highest value of the shear at the wall, to first order in ${R_{ep}}^{-1}$, as

$$\frac{(\tau_w)_{\max}}{p_A - p_B} = 4{R_{ep}}^{-1}\frac{d}{l}.$$

Multiplying the numerator of the above equation by the pipe surface area $2\pi a l$ and the denominator by the cross-sectional area πa^2, then taking the inverse, we obtain the ratio of inertia forces to viscous forces as

$$\frac{\text{Inertia Forces}}{\text{Viscous Forces}} = \frac{1}{16}R_{ep}, \tag{3.30}$$

which shows that the Reynolds number R_{ep} is proportional to the ratio of inertia forces to the viscous forces acting on the liquid volume in the pipe. At the critical Reynolds number the ratio of inertia to viscous forces is about 1.8, which is far from high inertia.

3.4.2. *The displacement thickness.* The displacement thickness, δ^*, in an axisymmetric internal flow is defined as the thickness of a fluid ring next to the wall that can accommodate the difference between the inviscid mass flow rate and the viscous mass flow rate. Accordingly we have

$$\frac{\delta^*}{a} = \int_0^1\left(1 - \frac{u_v}{u_I}\right)\frac{r}{a}d\frac{r}{a},$$

where u_v and u_I are the viscous and inviscid velocities, respectively. Writing this equation directly in terms of the mass flow rates we obtain

$$2\pi\rho u_I a\delta^* = \dot{m}_I - \dot{m}_v,$$

where $\dot{m}_v$ is given by equation (3.3) and $\dot{m}_I = -\frac{dp}{dx}\pi a^2 t$ as obtained from equation (3.6). Expanding equation (3.3) in powers of the small parameter${R_{ep}}^{-1}$ to second order results in

$$\dot{m}_v = -\frac{1}{\nu}\frac{dp}{dx}\frac{\pi}{8}a^4\left(1 - 32\sum_{n=1}^{\infty}\frac{1}{(\beta_n)^4}\left(1 - 4\bar{t}{R_{ep}}^{-1}{\beta_n}^2 + \frac{1}{2}\left({\beta_n}^2 4\bar{t}{R_{ep}}^{-1}\right)^2 +\right)\right)$$

and after replacing the infinite sums of the Bessel roots in the above by the appropriate constants (see Watson (1952)), we obtain

$$\dot{m}_v = -\frac{dp}{dx}\pi a^2 t\left(1 - 8{R_{ep}}^{-1}\bar{t}\right).$$

Thus, the difference in the mass flow rates becomes

$$\dot{m}_I - \dot{m}_v = -8\frac{dp}{dx}\pi a^2{R_{ep}}^{-1}\bar{t}t.$$

Dividing this result by $\rho u_I 2\pi a^2$ gives

$$\frac{\delta^*}{a} = 4{R_{ep}}^{-1}\bar{t}.$$

Thus δ^*grows linearly with time and at the end of the displacement process at $\bar{t} = \bar{t}_{DV}$ we have, since $\bar{t}_{DV} = t_{DV}/t_{DI} \to 1$, to first order in${R_{ep}}^{-1}$,

$$\frac{\delta^*}{a} = \frac{4}{\mathrm{R}_{ep}}.$$

This shows that the viscous layer at the wall thickens as the inverse of the Reynolds number. This result also supports the notion that as the Reynolds number increases the velocity profile remains essentially flat.

3.4.3. *The motion of the material surface.* The geometry of a material surface as it evolves in time is determined by the integral

$$x(r,t) = \int_0^t u\,(r,t)\,dt + x(r,0),\;\; r \in [0,a],$$

where $x(r,0)$ is the initial distribution of the fluid at $t = 0$. In the displacement problem we select the initial position of the material surface at $t = 0$ as $x\,(r,0) = 0$, for all $r \in (0,a)$, which describes a disk perpendicular to the pipe axis. Carrying out the integration using the velocity from equation (3.27) we obtain the description of the initial material surface as it changes with time as

$$x(r,t) = -\frac{1}{\rho}\frac{dp}{dx}\frac{t^2}{2}\left((H(1-\bar{r}) - \frac{8}{3}{R_{ep}}^{-1}f\,(\bar{r})\,\bar{t}\right). \tag{3.31}$$

As seen from the above equation, at the limit as $R_{ep} \to \infty$, the geometry of the material surface is described as an expanding cylindrical shell, anchored at $x(a,0) = 0$, which is capped at $x(r,t)$ by a disk defined over the interval $0 \leq \bar{r} < 1$. The disk moves at the velocity

$$w = \frac{dx}{dt} = -\frac{1}{\rho}\frac{dp}{dx}H\,(1-\bar{r})\,t,\ \ \bar{r} \in [0,1],$$

which is zero at the wall.

From the inviscid velocity, equation (3.6), we obtain the description of the initial material surface as it changes in time as

$$x(\bar{r},t) = -\frac{1}{\rho}\frac{dp}{dx}\frac{t^2}{2} \text{ for all } \bar{r} \text{ in } 0 \leq \bar{r} \leq 1\ , \tag{3.32}$$

as compared with equation (3.31).

In sum, we have demonstrated that when the pipe Reynolds numbers $R_{ep} \gg 1$, the deviation between the viscous and the inviscid solutions for the displacement time, the velocity profile, and the deformation of the material surface separating the displacing liquid from the displaced liquid, vanishes asymptotically. Thus, in an inviscid window of a viscous liquid the application of an inviscid analysis is justified. We propose therefore that in the 2-liquid viscous displacement problem, if both liquids satisfy the condition $R_{ep} \gg 1$, an inviscid window exists, and thus an inviscid analysis is justified. However, when the material surface of the 1-liquid problem is replaced with a material interface in the 2-liquid problem, this interface is susceptible to the Rayleigh-Taylor instability, which we address in section 6. In the following section we solve the 2-liquid inviscid problem.

4. Control volume analysis of the 2-liquid inviscid displacement problem.

4.1. *Formulation of the* 2*-liquid inviscid displacement problem.* We derive the equations of motion for the two liquids in the pipe during the displacement process subject to the hypothesis that the liquids are incompressible (H1), and the motion is unidirectional (H4). Let the variables $\rho, u, g, \theta, p, x, l$ designate the density, the velocity of the liquid along the axis, the gravitational acceleration, the inclination of the pipe, the pressure, the position of the interface in the pipe, and the pipe length, respectively. As shown by Batchelor (1970, p.179), subject to these assumptions the velocity component u becomes independent of x. If in addition the fluid is inviscid (H5) and at time $t = 0$, $u = constant$ (H6), then $u = u(t)$ for all $t \geq 0$. Applying a control volume analysis to the liquids in the pipe subject to these assumptions we obtain the following pair of differential equations:

For the displacing liquid, liquid A:

$$\rho_A x\frac{du_A}{dt} = \rho_A x g\cos\theta + (p_A - p_{IA}) \tag{4.1}$$

and for the displaced liquid, liquid B:

$$\rho_B(l-x)\frac{du_B}{dt} = \rho_B g(l-x)\cos\theta + (p_{IB} - p_B), \tag{4.2}$$

where $p_{IA}(x(t))$ is the time-dependent pressure at the A side of the interface and $p_{IB}(x(t))$ is the time-dependent pressure at the B side of the interface. Applying the jump conditions at the material interface we obtain

$$p_{IA} = p_{IB} = p_I,$$

where p_I is the pressure at the interface, and

$$u_A = u_B = \frac{dx}{dt},$$

where dx/dt is the velocity of the interface.

Substituting the jump conditions into equations (4.1) and (4.2) gives the two differential equations for the position of the interface and the pressure at the interface as

$$\rho_A x \frac{d^2x}{dt^2} = \rho_A x g \cos\theta + (p_A - p_I) \tag{4.3}$$

and

$$\rho_B (l - x) \frac{d^2x}{dt^2} = \rho_B g (l - x) \cos\theta + (p_I - p_B). \tag{4.4}$$

Eliminating the unknown p_I between the two equations we obtain a single ordinary, second order, nonlinear differential equation for the position of the interface $x(t)$ as

$$\rho_B (l - \lambda x) \frac{d^2x}{dt^2} = \rho_B (l - \lambda x) g \cos\theta + (p_A - p_B), \tag{4.5}$$

which is subject to the initial conditions at $t = 0$: $x = 0$, and $dx/dt = 0$, and where $\lambda = 1 - \rho_A/\rho_B$.

Rewriting equation (4.3) as

$$\rho_A x \left(\frac{d^2x}{dt^2} - g \cos\theta \right) = (p_A - p_I)$$

and equation (4.5) as

$$\rho_B (l - \lambda x) \left(\frac{d^2x}{dt^2} - g \cos\theta \right) = (p_A - p_B),$$

then taking the ratio we obtain

$$\frac{p_A - p_I\,(x(t))}{p_A - p_B} = \frac{(1 - \lambda) x(t)}{l - \lambda x(t)}, \tag{4.6}$$

which gives the pressure at the interface p_I, as a function of the position of the interface $x(t)$.

We define the time it takes the interface to reach the pipe exit l the displacement time t_{DI}; thus $x(t_{DI}) = l$. As can be seen from equation (4.6), when the interface is at $x = 0$ the interface pressure is equal to p_A, and when the interface reaches the end of the pipe the interface pressure becomes equal to p_B. We note also that when $\rho_A/\rho_B \to 0$ and hence $\lambda \to 1$, the pressure in the pipe between the entrance and the interface is a constant equal to the pressure at the entrance p_A. Consequently, the same effect on the displacement time would be obtained if the displacing fluid is a gas which is maintained at constant pressure. This completes the formulation of the 2-liquid inviscid displacement problem.

4.2. *The general solution of the* 2*-liquid inviscid displacement problem.* Substituting the nondimensional pressure parameter $P = (p_A - p_B)/\rho_B g l$, the nondimensional time variable $\hat{t} = \sqrt{g/l}t$, and the nondimensional space variable $\bar{x} = x/l$ into (4.5) we arrive at the nondimensional differential equation

$$\frac{d^2\bar{x}}{d\hat{t}^2} = \cos\theta + \frac{P}{1-\lambda\bar{x}}, \tag{4.7}$$

which holds in $0 \le \bar{x} \le 1$, subject to the initial conditions at $\hat{t} = 0$, $\bar{x} = 0$ and $d\bar{x}/d\hat{t} = 0$.

Multiplying (4.7) by $d\bar{x}/d\hat{t}$ and integrating with respect to $\hat{t}$ gives

$$\left(\frac{d\bar{x}}{d\hat{t}}\right)^2 = 2\left(\bar{x}\cos\theta + P\ln\left(\frac{1}{1-\lambda\bar{x}}\right)^{1/\lambda}\right) + c.$$

Applying the initial conditions determines the constant c to be zero, and the velocity of the interface becomes

$$\bar{w} = \frac{d\bar{x}}{d\hat{t}} = \sqrt{2\left(\bar{x}\cos\theta + P\ln\left(\frac{1}{1-\lambda\bar{x}}\right)^{1/\lambda}\right)}, \tag{4.8}$$

where the dimensional interface velocity is related to $\bar{w}$ by $w = \bar{w}\sqrt{gl}$. The position of the interface in the pipe is determined by the integral

$$\hat{t} = \int_0^{\bar{x}} \frac{d\xi}{\sqrt{2\xi(\cos\theta + \frac{P}{\xi\lambda}\ln(\frac{1}{1-\lambda\xi}))}}, \tag{4.9}$$

and the nondimensional inviscid displacement time becomes

$$\hat{t}_{DI} = \int_0^{1} \frac{d\xi}{\sqrt{2\xi(\cos\theta + \frac{P}{\xi\lambda}\ln(\frac{1}{1-\lambda\xi}))}}. \tag{4.10}$$

Given the parameters $P = (p_A - p_B)/\rho_B g l$, $\lambda = 1 - \rho_A/\rho_B$, and the inclination of the pipe θ, the position of the interface in the pipe as a function of time is determined from equation (4.9) and subsequently the velocity from equation (4.8).

5. Examples. In this section we consider various examples for high inertia displacement. In all of the examples considered the pipe length is taken as $l = .30m$ and its diameter as $d = 0.025m$. The liquids selected are water with a density $\rho = 997.5kg/m^3$ and a kinematic viscosity $\nu = 8.64 \times 10^{-7}m^2/s$, and oil with a density $\rho = 840kg/m^3$ and a kinematic viscosity $\nu = 2.03 \times 10^{-5}m^2/s$. Any additional data required for the solution are presented in the examples. The first three examples are closed form solutions; the others are computed numerically. Verification of the accuracy of the numerical integration is obtained by comparing results with the closed form solutions. The percentage error computed was found to be better than 10^{-4}. In addition to the displacement times, the results presented include the Reynolds numbers, which are required to satisfy the condition $R_{ep} \gg 28$ in order for the inviscid theory to apply and the percent deviation to be as computed from equation (3.21).

EXAMPLE 5.1. Zero pressure difference. $p_A - p_B = 0,$ i.e., $P = 0.$

This example represents a free fall solution. We obtain from (4.10),

$$\hat{t}(\bar{x}) = \sqrt{\frac{2}{\cos\theta}}\bar{x}^{1/2},$$

which, in dimensional form gives the displacement time as

$$t_{DI} = \sqrt{\frac{2l}{g\cos\theta}}.$$

The result is independent of λ and hence valid for any pair of liquids. For a vertical pipe where $\theta = 0$, using water as the displacing liquid (liquid A) and oil as the displaced liquid (liquid B) the displacement time $t_{DI} = 0.247s$ and the pipe Reynolds numbers are $R_{epA} = 2.924 \times 10^3$ and $R_{epB} = 124$. The percent deviation for the oil as computed by equation (3.21) is 11.5 percent, which makes the inviscid solution unacceptable. Using instead a pipe with a diameter of $0.05m$, the displacement time remains unchanged while the Reynolds numbers become$R_{epA} = 1.17 \times 10^4$ and $R_{epB} = 498$, which reduces the percent deviation for the oil to about 5.5 percent.

EXAMPLE 5.2. Same density displacement. $\rho_A/\rho_B = 1$, $(\lambda = 0)$.

By taking the limit as $\lambda \to 0$ in (4.10) or directly from the differential equation (4.7), we obtain the nondimensional displacement time as

$$\hat{t}_{DI} = \sqrt{\frac{2}{\cos\theta + P}}. \tag{5.1}$$

which for a horizontal pipe, reduces to $\hat{t}_{DI} = \sqrt{2/P}$; or, dimensionally to $t_{DI} = l/\sqrt{(p_A - p_B)/2\rho}$.

Using water as the working substance and a pressure difference of one atmosphere gives the displacement time $t_{DI} = 0.0421s$ and $R_{ep} = 1.718 \times 10^4$. If we compute the viscous displacement time t_{DV} for this displacement problem using equation (13) with $R_{ep} = 1.718 \times 10^4$, we obtain $t_{DV} = 0.0425\text{s}$. Comparing this result with the inviscid solution t_{DI} shows a percent deviation better than one percent.

EXAMPLE 5.3. A horizontal pipe ($\theta{=}\pi/2$).

For a horizontal pipe, setting $\theta = \pi/2$, then introducing the transformation $y = \sqrt{\frac{1}{\lambda}\ln(\frac{1}{1-\lambda\xi})}$, we obtain the displacement time as a function of the position of the interface $\bar{x}$ and λ as

$$\hat{t}\,(\bar{x}, \lambda) = \sqrt{\frac{2}{P}}\int_0^{\sqrt{\frac{1}{\lambda}\ln\frac{1}{1-\lambda\bar{x}}}} e^{-\lambda y^2}dy, \tag{5.2}$$

which, by setting $\bar{x} = 1$, gives the nondimensional displacement time as

$$\hat{t}(1, \lambda)_{DI} = \sqrt{\frac{2}{P}}\int_0^{\sqrt{\frac{1}{\lambda}\ln\frac{1}{1-\lambda}}} e^{-\lambda y^2}dy. \tag{5.3}$$

Taking the limit of (5.3) as $\lambda \to 1$ corresponding to $\rho_A/\rho_B \to 0$, we arrive at the closed form solution

$$\hat{t}_{DI}\,(1, 1) = \sqrt{\frac{2}{P}}\int_0^{\infty} e^{-y^2}dy = \sqrt{\frac{\pi}{2P}}, \tag{5.4}$$

which in dimensional variables becomes

$$t_{DI} = l\sqrt{\frac{\pi}{2}\frac{\rho_B}{p_A - p_B}}. \tag{5.5}$$

Taking the limit of (5.3) as $\lambda \to 0$, corresponding to the case $\rho_A = \rho_B$, gives

$$\hat{t}_{DI}(1,0) = \sqrt{\frac{2}{P}}\int_0^1 dy = \sqrt{\frac{2}{P}};$$

hence, the ratio $t_{DI}(1,1)/t_{DI}(1,0) = \sqrt{\pi/4} = .886$, which makes the $\rho_A/\rho_B \to 0$ case almost 13.0 percent faster than the $\rho_A = \rho_B$ case.

As a numerical example, let the pressure difference be $p_A - p_B = 1atm$, the density ratio $\rho_A/\rho_B \to 0$, and water is the displaced liquid. Using the exact solution, equation (5.5), we find $t_{DI} = .037306s$ and the Reynolds number $R_{ep} = 1.718 \times 10^4$. Using air as the displacing fluid, which has a density $\rho = 1.2kg/m^3$, and hence $\lambda = 0.99879$, the displacement time as calculated by equation (5.3) becomes $t_{DI} = .037319s$ with the Reynolds number for water $R_{ep} = 1.718 \times 10^4$ and for air $R_{ep} = 2.854 \times 10^4$.

Next we show that for an inclined pipe at a prescribed $P = (p_A - p_B)/(\rho_B gl)$, the shortest displacement time for any density ratio $\rho_A/\rho_B \geq 0$ is obtained when$\rho_A/\rho_B \to 0$.

Proof. For an inclined pipe the forcing function for the acceleration of the interface as shown in (4.2) is given by $F(\bar{x},\theta,\lambda) = \cos\theta + P/(1-\lambda\bar{x})$.

Let $F_1 = F(\bar{x},\theta,1) = \cos\theta + P/(1-\bar{x})$ and $F_\lambda = F(\bar{x},\theta,\lambda) = \cos\theta + P/(1-\lambda\bar{x})$ for $\lambda \neq 1$. Now, for any $\rho_A/\rho_B \geq 0$, $\lambda = 1 - \rho_A/\rho_B \leq 1$; hence $1-\bar{x} \leq 1-\lambda\bar{x}$ and therefore $F_1 \geq F_\lambda$ for $\bar{x} \in [0,1]$, where the equality sign is for $\bar{x} = 0$, where $F_1 = F_\lambda = P + \cos\theta$. Since at $t = 0$, we have $\bar{x} = 0$, $d\bar{x}/d\hat{t} = 0$, and $F_1 = F_\lambda = P + \cos\theta$, and we have $F_1 > F_\lambda$ for $\bar{x} \in (0,1]$, the shortest displacement time is obtained when $\rho_A/\rho_B \to 0$. For a horizontal pipe the result is given by equation (5.5).

Example 5.4. Oil-water displacement.

We consider a pressure difference of one atmosphere with oil as the displacing liquid and water the displaced liquid with $\theta = (0,\ \pi/2,\ \pi)$. Using equation (4.10) the numerical integration gives $t_{DI} = (0.04095s,\ 0.04153s,\ 0.04212s)$, respectively. The corresponding pipes Reynolds numbers are: for the oil $R_{ep} = (807, 797, 787)$ and for the water $R_{ep} = \left(1.743 \times 10^4, 1.718 \times 10^4, 1.693 \times 10^4\right)$. The oil results indicate a percent deviation with the viscous solution (equation (3.21)) of between 4 to 5 percent.

Example 5.5. Water-oil displacement.

This is the same as Example 5.4, but the role of displacing and displaced liquids is interchanged. The numerical results for $\theta = (0, \pi/2, \pi)$ are $t_{DI} = (0.03873s, 0.03921s, 0.03972s)$, respectively, with corresponding pairs of Reynolds numbers as in Example 5.4.

Examples 5.4 and 5.5 show that an increase in the gravitational force shortens the displacement time, and when the heavier and lighter liquids exchange positions, when the heavier liquid is displacing the lighter liquid at a fixed angle, the displacing time is shorter. A general proof to that effect is presented in Appendix B. The solutions presented above are mechanically admissible; i.e., they satisfy the global equations of mass conservation and momentum conservation. In the following section we consider the stability of these solutions.

6. Stability of the 2-liquid inviscid displacement problem. Taylor (1950) analyzes the stability of an infinite material interface between two inviscid immiscible fluids that is accelerating in the direction perpendicular to the interface. His major conclusions are that the surface is stable or unstable according to whether the acceleration is in the direction from the heavy fluid to the lighter fluid or vice versa, and that the growth rate of the instability is proportional to $\sqrt{(\rho_B - \rho_A)\,/\,(\rho_B + \rho_A)}$, where $\rho_A < \rho_B$. In applying Taylor's theory to the 2-liquid inviscid displacement problem, we are extending it from consideration of an infinite interface to a finite interface. For clarity in applying Taylor's analysis, we reproduce some of its basic elements here. Taylor considers the stability of the material interface between a liquid A, located in the half space $x < 0$ and a liquid B, located in the half space $x > 0$. The liquids are assumed inviscid and immiscible. The direction of x is taken as *positive* in the direction of the gravitational acceleration g, and the coordinate system is fixed on the moving interface which is accelerating at g_1 in the *negative* x direction. By applying potential theory, Taylor obtains the equation of the perturbed interface as

$$\eta = A_1 n^{-1} k e^{nt} \cos kz, \tag{6.1}$$

where A_1 is the amplitude of the perturbed surface, n the complex frequency, k the wave number, and the coordinate z is tangential to the unperturbed surface. By imposing the continuity of the normal velocity component of the interface and the pressure across the interface, he derives the square of the complex frequency as

$$n^2 = -k(g + g_1)\frac{\rho_B - \rho_A}{\rho_B + \rho_A}. \tag{6.2}$$

It follows directly from equation (6.2) that when $g_1 = 0$, the interface is stable if $\rho_A < \rho_B$, a familiar result for the Air-Ocean interface. Setting $\theta = 0$ in equation (4.5) so that the pipe axis is oriented in the direction of the gravitational acceleration, we replace the acceleration g_1 in equation (6.2) by the acceleration given in (4.5) with due account for the direction of g_1 to obtain

$$-g_1 = \frac{d^2x}{dt^2} = g + \frac{p_A - p_B}{\rho_B(l - \lambda x)}. \tag{6.3}$$

Substituting from equation (6.3) into equation (6.2) we obtain the square of the frequency as

$$n^2 = \frac{k}{l}\frac{p_A - p_B}{\rho_A + \rho_B}\frac{\rho_B - \rho_A}{\rho_B - (\rho_B - \rho_A)\bar{x}}. \tag{6.4}$$

In equation (6.4), the function $\rho_B - (\rho_B - \rho_A)\bar{x}$ is a linear interpolation of the density between ρ_B and ρ_A, and hence positive for $\bar{x} \in [0, 1]$. Since by definition the pressure difference $p_A - p_B \geq 0$, then, if $p_A - p_B > 0$, the interface is unstable when $\rho_A < \rho_B$ and stable when $\rho_A > \rho_B$, and the interface is neutrally stable when $p_A - p_B = 0$, which is in agreement with Taylor's result.

According to these results, the solutions in Examples 5.1 and 5.2 are neutrally stable, those in Examples 5.3 and 5.4 are unstable, and those in Example 5.5 are stable.

We consider next the growth rate of the unstable solutions, since if the interface can arrive at the pipe exit at a shorter time than the characteristic time required for the instability to grow, then the unstable solution still holds. The growth rate of the instability is its frequency n. We define its inverse τ, which is the e-folding time for the growth of the instability, as the characteristic time for the instability to grow. Hence we have

$$\tau^{-1} = n = \sqrt{\left(\frac{k}{l}\right) \frac{p_A - p_B}{\rho_B - (\rho_A - \rho_B)\bar{x}} \frac{\rho_B - \rho_A}{\rho_B + \rho_A}}, \tag{6.5}$$

where in (6.5), $\rho_A < \rho_B$. Replacing the wave number k with $k = 2\pi m/d$, where m is the sequence of positive integers $m = 1, 2, 3, ...$, we obtain

$$\tau_m{}^{-1} = \frac{1}{l}\sqrt{2\pi m \left(\frac{l}{d}\right) \frac{p_A - p_B}{\rho_B - (\rho_A - \rho_B)\bar{x}} \frac{\rho_B - \rho_A}{\rho_B + \rho_A}}. \tag{6.6}$$

As can be verified from (6.6) the growth rate is proportional to $\sqrt{(\rho_B - \rho_A) / (\rho_B + \rho_A)}$ and to $\sqrt{l/d}$; it is linearly proportional to the pipe Reynolds number R_{ep} and inversely proportional to the pipe cross-sectional area. As an example we consider two limiting cases, both in a horizontal pipe: the first is $\rho_A/\rho_B \to 0$ and the second is $\rho_A/\rho_B \to 1$. In the first case the rate of the displacement is the fastest; in the second case the growth of the instability is the slowest.

Case 1. $\rho_A/\rho_B \to 0$. As shown in Example 5.3, this density ratio provides the fastest displacement time for a horizontal pipe which is given by equation (5.5) as

$$t_{DI} = l\sqrt{\frac{\pi}{2}\frac{\rho_B}{p_A - p_B}}.$$

Applying the condition $\rho_A/\rho_B \to 0$ to (6.6) we derive

$$\tau_m{}^{-1} = \frac{1}{l}\sqrt{2\pi m \left(\frac{l}{d}\right) \frac{p_A - p_B}{\rho_B(1 + \bar{x})}}.$$

Hence the ratio is

$$\frac{t_{DI}}{\tau_m} = \pi\sqrt{\frac{l}{d}\frac{m}{1 + \bar{x}}} > 1,$$

which shows that for all positive integers, the instability grows faster than it takes the interface to arrive at the exit.

Case 2. $\rho_A/\rho_B \to 1$. The instability growth rate is the slowest when $\varepsilon = 1 - \rho_A/\rho_B < 1$ approaches zero. For sufficiently small ε, equation (5.4) takes the form

$$\tau_m{}^{-1} = n\tilde{}\frac{1}{l}\sqrt{\pi m \left(\frac{l}{d}\right) \frac{p_A - p_B}{\rho_B}\varepsilon}.$$

The corresponding t_{DI} as $\varepsilon \to 0$, as shown in Example 5.3, is

$$t_{DI} = \frac{l}{\sqrt{\frac{p_A - p_B}{2\rho_B}}},$$

and hence

$$\frac{t_{DI}}{\tau_m}\tilde{}\sqrt{2\pi m\varepsilon\frac{l}{d}}.$$

Consequently, although unstable, we find that the time it takes the interface to arrive at the exit is shorter or longer than the corresponding characteristic time τ_m, according to whether the positive integer m is smaller or larger than the real number $q(\varepsilon)$, where

$q(\varepsilon) = (d/l)/(2\pi\varepsilon)$. Hence at small $\varepsilon > 0$ the interface escapes the low frequencies, but it is vulnerable to the high frequencies.

7. Comparison of the theory with a numerical analysis. We compare the results of our analysis with the numerical data obtained by Dimakopoulos and Tsamopoulos (2003) where they analyze the displacement of a viscous liquid, including the effects of surface tension, with a highly pressurized air. In comparing our results we are mostly concerned with the effect of inertia, which is presented in their paper by a sequence of Reynolds numbers. In their analysis the Reynolds number is defined by

$$R_{dt} = \frac{\rho a^3 (p_A - p_B)}{l\mu^2},$$

where we use the subscript dt to distinguish between their Reynolds number and the one we use in our analysis. The relation between the two Reynolds numbers is given by

$$R_{ep} = 2\left(\frac{l}{d}\right)^{-1/2} {R_{dt}}^{1/2}.$$

The sequence of Reynolds numbers presented in their paper is

$$R_{dt} = 8.33, 83.33, 1250, 1666, \text{ and } 4165,$$

which, with a fixed aspect ratio of $l/d = 6$, translates into

$$R_{ep} = 2.36, 7.45, 28.87, 33.33, \text{ and } 52.69 \tag{7.1}$$

in our notation. By the criteria for a high Reynolds number, $R_{ep} \gg (R_{ep})_C = 28.47$, derived in section 3.3, equation (3.16), none of the above may be considered a high Reynolds number. According to equation (3.21) the percent deviation for the highest Reynolds number is about 18 percent, and according to equation (3.30) the ratio of the viscous forces to the inertia forces at the end of the displacement process is approximately 30 percent. As discussed in section 3.3, for Reynolds numbers smaller than $(R_{ep})_C$, the velocity attains a parabolic profile some time during the displacement process. The two lowest Reynolds numbers in equation (7.1) fall into this group. The observation made by Dimakopoulos and Tsamopoulos of a faster formation of the parabolic profile with the lower Reynolds number is in agreement with the general solution equation (3.13). As can be seen from the data presented in their paper for $R_{dt} = 1250$, the interface does not retain its original flat interface, neither does it become parabolic; it is a transitional case corresponding to $R_{ep} = 28.87$ in our notation. We find the close agreement between the transition point in the state of motion as predicted by our 1-liquid viscous model and the numerical results for a 2-liquid model, including surface tension, quite remarkable. Additional data to validate this result is obviously necessary.

For the Reynolds numbers higher than the critical number any comparison must be at best qualitative. We demonstrated a steepening of the velocity profile as well as the interface with increasing Reynolds numbers in the 1-liquid analysis which is in agreement with Dimakopoulos and Tsamopoulos observations for $R_{dt} = 1666$ and $R_{dt} = 4165$, who show that the interface becomes flat faster as the Reynolds number increases. In their paper the displacing fluid is air and the displaced fluid is a liquid. Hence, once the interface becomes flat, or at portions of the interface which become flat, the surface becomes susceptible to the Rayleigh-Taylor instability. For comparison with their results for the growth rate of the instability we consider our results for the density ratio $\rho_A/\rho_B \to 0$ determined in Case 1 of section 6. This growth rate can be written as

$$\tau_m{}^{-1} = \frac{\nu}{2s} R_{ep} \left(\frac{l}{d}\right)^{1/2} \sqrt{\frac{\pi^3 m}{1+\bar{x}}},$$

where s is the cross-sectional area of the pipe. As can be seen from this equation the growth rate at any frequency m is proportional to the pipe Reynolds number R_{ep} calculated for the displaced liquid, the square root of the aspect ratio l/d and inversely proportional to the cross-sectional area of the pipe. The data in the numerical results of Dimakopoulos and Tsamopoulos also indicate a faster instability growth rate as the Reynolds number increases. No qualitative or quantitative comparison could be made beyond these points.

For a quantitative comparison between the results of this analysis and either numerical or experimental results, additional data at high Reynolds numbers at either stable or unstable displacement problems are required.

Appendix A. The characteristic time to attain a fully developed viscous steady state. Taking the limit of equation (3.2) as $t \to \infty$ gives the steady state Poiseuille flow

$$U = -\frac{1}{\mu}\frac{dp}{dx}\frac{a^2}{4}\left(1 - \left(\frac{r}{a}\right)^2\right),$$

which has a maximum velocity at the center given by

$$U_{\max} = -\frac{1}{\mu}\frac{dp}{dx}\frac{a^2}{4}.$$

The characteristic time to approach steady state is determined as we let the nondimensionalised deviation from steady state defined by the expression

$$\Delta\bar{u} = \frac{U-u}{U_{\max}} = 8\sum_{n=1}^{\infty}\frac{J_0(\beta_n\bar{r})}{\beta_n{}^3 J_1(\beta_n)}\exp\left(-\beta_n\tilde{t}\right)$$

go to zero. As $\tilde{t} \to \infty$, the largest contribution to $\Delta\bar{u}$ comes from the smallest eigenvalue; hence

$$\Delta\bar{u} \;\approx 8\frac{J_0(\beta_1\bar{r})}{\beta_1{}^3 J_1(\beta_1)}\, e^{-\beta_1{}^2\tilde{t}}$$

with its maximum value on the axis as

$$\Delta\bar{u}_{\max} \approx 8\frac{J_0(0)}{\beta_1{}^3 J_1(\beta_1)}\, e^{-\beta_1{}^2\tilde{t}}.$$

Setting the nondimensional deviation equal to e^{-1} defines this e-folding time as the characteristic time for the flow to attain a steady state. Introducing the entries $\beta_1 = 2.4048,\ J_0(0) = 1,$ and $J_1(\beta_1) = .519$ (Watson (1952)), we obtain $\tilde{t}_\infty = .1907$ and thus the characteristic time for the velocity to attain a steady state is

$$t_\infty = 0.1907a^2/\nu.$$

Appendix B. A preferential choice for the denser liquid. We prove that given two liquids with densities ρ_1 and ρ_2 such that $\rho_1 > \rho_2$, a specified pressure difference $p_A - p_B$ such that $p_A > p_B$, and an inclination angle θ, the displacement time is shorter if the denser liquid ρ_1 is displacing the lighter liquid ρ_2.

Proof. A first integral of Equation (4.1) determines the velocity u of the interface as a function of $\bar{x}$, the displacing liquid ρ_A, and the displaced liquid ρ_B with $\rho_A \neq \rho_B$ in the form

$$\frac{1}{2}u^2 = xg\cos\theta + \frac{p_A - p_B}{\rho_A - \rho_B}\ln\left(1 + \left(\frac{\rho_A}{\rho_B} - 1\right)\bar{x}\right). \tag{B.1}$$

Consider two liquids ρ_1 and ρ_2 such that $\rho_1 > \rho_2$. We select first $\rho_1 = \rho_A$ as the displacing liquid and $\rho_2 = \rho_B$ as the displaced liquid to obtain the velocity u_1 as

$$\frac{1}{2}{u_1}^2 = xg\cos\theta + \frac{p_A - p_B}{\rho_1 - \rho_2}\ln\left(1 + \left(\frac{\rho_1}{\rho_2} - 1\right)\bar{x}\right). \tag{B.2}$$

Second, we select $\rho_2 = \rho_A$ as the displacing liquid and $\rho_1 = \rho_B$ as the displaced liquid to obtain the velocity u_2 as

$$\frac{1}{2}{u_2}^2 = xg\cos\theta + \frac{p_A - p_B}{\rho_2 - \rho_1}\ln\left(1 + \left(\frac{\rho_2}{\rho_1} - 1\right)\bar{x}\right). \tag{B.3}$$

Subtracting Equation (B.3) from Equation (B.2) gives

$$\frac{1}{2}{u_1}^2 - \frac{1}{2}{u_2}^2 = \frac{p_A - p_B}{\rho_1 - \rho_2}\ln\left(1 + \left(\sqrt{\frac{\rho_1}{\rho_2}} - \sqrt{\frac{\rho_2}{\rho_1}}\right)^2\left(\bar{x} - \bar{x}^2\right)\right). \tag{B.4}$$

Since $\bar{x} > \bar{x}^2$ in the interior of the closed interval $[0,1]$, and $\bar{x} = \bar{x}^2$ at its boundaries $\bar{x} = 0$ and $\bar{x} = 1$, the difference $\frac{1}{2}{u_1}^2 - \frac{1}{2}{u_2}^2 = \frac{1}{2}(u_1 - u_2)(u_1 + u_2)$ is positive, and hence $u_1 - u_2$ is also positive in $\bar{x} \in (0,1)$ and it is equal to zero at the boundaries. It follows that the displacement time is shorter when the denser liquid is the displacing liquid than when the lighter liquid is the displacing liquid. This completes the proof. (This proof was suggested by private communication with Prof. Lu Ting of the Courant Institute.)

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