WELL-POSEDNESS AND REGULARITY
FOR NON-UNIFORM SCHRÖDINGER
AND EULER-BERNOULLI EQUATIONS
WITH BOUNDARY CONTROL AND OBSERVATION

By

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Abstract. The open-loop systems of a Schrödinger equation and an Euler-Bernoulli equation with variable coefficients and boundary controls and collocated observations are considered. It is shown, with the help of a multiplier method on a Riemannian manifold, that both systems are well-posed in the sense of D. Salamon and regular in the sense of G. Weiss. The feed-through operators are found to be zero. The results imply particularly that the exact controlability of each open-loop system is equivalent to the exponential stability of the associated closed-loop system under the output proportional feedback.

1. Introduction and main results. Well-posed and regular linear infinite dimensional systems first introduced by D. Salamon and G. Weiss in the 1980s ([25, 26, 29, 30]) are a quite general class of linear infinite-dimensional systems. They cover many control systems described by time delay equations and partial differential equations with actuators and sensors supported at isolated points, subdomains, or on a part of the boundary

of the spatial region [6]. This class of linear infinite-dimensional systems, although the input and output operators are usually unbounded, may possess many properties paralleling in many ways to finite-dimensional systems, for instance, representation, transfer function, internal model based tracking and disturbance rejection, stabilizing controller parametrization, and quadratic optimal control, to name just a few [7].

The abstract theory of well-posed and regular linear systems has been quite fruitful. However, only a few multidimensional PDEs have been verified to be well-posed and regular. For the systems with constant coefficients, the well-posedness and regularity of a multidimensional heat equation with both Dirichlet and Neumann type boundary controls were established in [5], and that for a wave equation with boundary Dirichlet input and collocated output was proved in [1] and [14]. The well-posedness and regularity for multi-dimensional Schrödinger and Euler-Bernoulli equations were reported in [11, 12, 19]. The well-posedness and regularity for a wave equation with variable coefficients and Dirichlet boundary control and observation was established in [15], and that for a plate equation with variable coefficients under Neumann boundary control and observation was obtained in [16]. In [13], the exact controllability, well-posedness and regularity for the transmission problem of plate equations with variable coefficients and Neumann boundary control and observation were considered. Recently, the well-posedness and regularity of linear elasticity with Dirichlet boundary control and collocated observation were established in [3, 17].

The objective of this paper is, on the one hand, to generalize the well-posedness for a Schrödinger equation with Dirichlet boundary control [19] and an Euler-Bernoulli plate equation with boundary moment control [22], all with collocated observation, to the variable coefficients cases, and on the other hand, to show that, in both cases, the systems are regular with zero feed-through operators.

The systems to be considered are described, respectively, by the Schrödinger equation with Dirichlet boundary control and collocated observation

$$
\begin{cases}
v_t(x,t) - iPv(x,t) = 0, \ x = (x_1, x_2, \ldots, x_n) \in \Omega, t > 0, \\
v(x,t) = u(x,t), \ x \in \Gamma, t \geq 0, \\
y(x,t) = -i\dfrac{\partial \mathcal{A}^{-1}v(x,t)}{\partial \nu_{\mathcal{A}}}, \ x \in \Gamma, t \geq 0,
\end{cases}
\tag{1.1}
$$

and the Euler-Bernoulli equation with boundary moment control and collocated observation

$$
\begin{cases}
w_{tt}(x,t) + P^2 w(x,t) = 0, \ x \in \Omega, t > 0, \\
w(x,t) = 0, \ x \in \Gamma, t \geq 0, \\
Pw(x,t) = u(x,t), \ x \in \Gamma, t \geq 0, \\
y(x,t) = \dfrac{\partial \mathcal{A}^{-1} w_t(x,t)}{\partial \nu_{\mathcal{A}}}, \ x \in \Gamma, t \geq 0,
\end{cases}
\tag{1.2}
$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is an open bounded region with C^2-boundary Γ. P is a second order partial differential operator

$$P = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right),$$

and for some constants $a, b > 0$,

$$a \sum_{i=1}^{n} |\xi_i|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\overline{\xi_j} \leq b \sum_{i=1}^{n} |\xi_i|^2, \ \forall \ x \in \overline{\Omega}, \ \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{C}^n, \tag{1.3}$$

$$a_{ij} = a_{ji} \in C^{\infty}(\mathbb{R}^n), \ \forall \ i, j = 1, 2, \ldots, n.$$

The operator $\mathcal{A}$ is defined by

$$\mathcal{A}f := Pf, \ \forall f \in D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega) \tag{1.4}$$

and

$$\frac{\partial}{\partial \nu_{\mathcal{A}}} \equiv \nu_{\mathcal{A}} = \sum_{i,j=1}^{n} a_{ij}\nu_j \frac{\partial}{\partial x_i}, \tag{1.5}$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is the unit normal of $\partial\Omega$ pointing towards the exterior of Ω. u and y are the input functions (or controls) and the output functions (or observations), respectively. It is seen that both operators $\mathcal{A}$ and P appear in the system (1.1) and (1.2) simultaneously. That is why we put them together and treat the two with a similar approach.

Let $H = H^{-1}(\Omega)$, $\mathcal{H} = H_0^1(\Omega) \times H^{-1}(\Omega)$ and $U = L^2(\Gamma)$. Theorem 1.1 is the generalization of Proposition 4.2 of [19] where the coefficients of the system (1.1) are constants.

THEOREM 1.1. Let $T > 0$, $v_0 \in H$ and $u \in L^2(0, T; U)$. Then there exists a unique solution $v \in C(0, T; H)$ to the Equation (1.1) satisfying the initial condition $v(\cdot, 0) = v_0$. Moreover, there exists a constant $C_T > 0$, independent of (v_0, u), such that

$$\|v(\cdot, T)\|_H^2 + \|y\|_{L^2(0,T;U)}^2 \leq C_T \left[\|v_0\|_H^2 + \|u\|_{L^2(0,T;U)}^2 \right].$$

Theorem 1.2 below generalizes the well-posedness result of Theorem 4.29 of [22] to the variable coefficients case.

THEOREM 1.2. Let $T > 0$, $(w_0, w_1) \in \mathcal{H}$, and $u \in L^2(0, T; U)$. Then there exists a unique solution $(w, w_t) \in C([0, T]; \mathcal{H})$ to the Equation (1.2) satisfying $w(\cdot, 0) = w_0$ and $w_t(\cdot, 0) = w_1$. Moreover, there exists a constant $C_T > 0$, independent of (w_0, w_1, u), such that

$$\|(w(\cdot, T), w_t(\cdot, T))\|_{\mathcal{H}}^2 + \|y\|_{L^2(0,T;U)}^2 \leq C_T \left[\|(w_0, w_1)\|_{\mathcal{H}}^2 + \|u\|_{L^2(0,T;U)}^2 \right].$$

Theorems 1.1 and 1.2 imply that the open-loop systems (1.1) and (1.2) are well-posed in the sense of D. Salamon with the state spaces H and $\mathcal{H}$, respectively, with input and output space U [27]. These results together with Theorem 2.2 of [2] and Proposition 3.1 of [22] (see also Theorem 3 of [10]) give immediately the following corollary.

COROLLARY 1.3. The systems (1.1) or (1.2) are exactly controllable in some time interval $[0, T]$ if and only if the corresponding closed-loop system under the proportional output feedback $u = -ky, k > 0$ are exponentially stable.

For the exponential stability of the system (1.1) under the feedback $u = -ky, k > 0$, we refer to [24]. For the exact controllability of the system (1.2), we refer the reader to [34] and [20].

THEOREM 1.4. The system (1.1) is regular. More precisely, if $v(\cdot, 0) = 0$ and $u(\cdot, t) \equiv u(\cdot) \in U$ is a step input, then the corresponding step response y satisfies

$$\lim_{\sigma \to 0+} \int_\Gamma \left| \frac{1}{\sigma} \int_0^\sigma y(x, t)\, dt \right|^2 dx = 0. \tag{1.6}$$

THEOREM 1.5. The system (1.2) is regular. More precisely, if $w(\cdot, 0) = w_t(\cdot, 0) = 0$ and $u(\cdot, t) \equiv u(\cdot) \in U$ is a step input, then the corresponding output y satisfies

$$\lim_{\sigma \to 0+} \int_\Gamma \left| \frac{1}{\sigma} \int_0^\sigma y(x, t)\, dt \right|^2 dx = 0. \tag{1.7}$$

Theorems 1.1, 1.2, 1.4 and 1.5 together show that the systems (1.1) and (1.2) are well-posed regular linear systems with feedthrough operators zero. This makes the systems (1.1) and (1.2) parallel in many ways to linear finite-dimensional ones.

Our main approach in proving above mentioned results is the Riemannian geometry method introduced in [33] for the proof of exact controllability of wave equation with variable coefficients. The difficulty of our proofs also arises from the occurrence of the variable coefficients as that in [33]. By the geometry approach, similar to [34], the computations on the Riemannian manifold become much more simple, which it would be extremely tedious and even hardly carried out in Euclidean space. The key point is that by the introduction of the Riemannian metric, the corresponding differential operators with variable coefficients in Euclidean space become the differential operators with constant coefficients on the Riemannian manifold, except some low order perturbations. Therefore, the techniques in dealing with problems with constant coefficients in Euclidean space can then be moved to Riemannian manifold. From this point view, it seems that the Riemannian geometry method becomes natural in dealing with the non-uniform problems.

The remainder of the paper is organized as follows. In Section 2, the systems (1.1) and (1.2) are casted into the abstract settings separately, and some basic background on Riemannian geometry is introduced. The proofs of Theorems 1.1 and 1.2 are given in Section 3 and Section 4, respectively. Section 5 is devoted to the proofs of Theorems 1.4 and 1.5.

2. Abstract formulations and preliminaries. In order to formulate the system (1.1) and (1.2) into abstract forms, we need some notation and facts in Riemannian geometry.

By the ellipticity condition (1.3), we let $G(x)$ and $\rho(x)$ be the following positive matrix and its determinant, respectively,

$$G(x) := [g_{ij}(x)]_{n \times n} = [a_{ij}(x)]_{n \times n}^{-1}, \ \ \rho(x) := \det[g_{ij}(x)]_{n \times n}, \ \forall \, x \in \mathbb{R}^n. \tag{2.1}$$

For each $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define the inner product and norm over the tangent space $\mathbb{R}_x^n$ of $\mathbb{R}^n$ by

$$\langle X, Y \rangle_g := \sum_{i,j=1}^{n} g_{ij} \alpha_i \beta_j,$$
$$|X|_g := \langle X, X \rangle_g^{1/2}, \ \forall \, X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}, \ Y = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}_x^n. \tag{2.2}$$

Then $(\mathbb{R}^n, g)$ becomes a Riemannian manifold with Riemannian metric g [34]. Denote by D the Levi-Civita connection with respect to g. Let N be a smooth vector field on $(\mathbb{R}^n, g)$. Then for each $x \in \mathbb{R}^n$, the covariant differential DN of N determines a bilinear form on $\mathbb{R}_x^n$,

$$DN(X, Y) = \langle D_X N, Y \rangle_g, \ \forall \, X, Y \in \mathbb{R}_x^n, \tag{2.3}$$

where $D_X N$ stands for the covariant derivative of the vector field N with respect to X.

For any $\varphi \in C^2(\mathbb{R}^n)$ and $N = \sum_{i=1}^{n} h^i(x) \frac{\partial}{\partial x_i}$, denote

$$\text{div}_0(N) := \sum_{i=1}^{n} \frac{\partial h^i}{\partial x_i}, \ D\varphi := \nabla_g \varphi = \sum_{i,j=1}^{n} a_{ij} \frac{\partial \varphi}{\partial x_j} \frac{\partial}{\partial x_i},$$

$$\text{div}_g(N) := \sum_{i=1}^{n} \rho^{-1/2} \frac{\partial}{\partial x_i} (\rho^{1/2} h^i),$$

$$\Delta_g \varphi := \sum_{i,j=1}^{n} \rho^{-1/2} \frac{\partial}{\partial x_i} \left(\rho^{1/2} a_{ij} \frac{\partial \varphi}{\partial x_j} \right) = -P\varphi + (Dq)\varphi, \ q(x) = 1/2 \log(\rho(x)),$$

where div_0 is the divergence operator in Euclidean space $\mathbb{R}^n$, and ∇_g, div_g and Δ_g are the gradient operator, the divergence operator and the Beltrami-Laplace operator in $(\mathbb{R}^n, g)$, respectively.

Denote by $\mu = \frac{\nu_A}{|\nu_A|_g}$ the unit outward-pointing normal to $\partial \Omega$ in terms of the Riemannian metric g. The following Lemma 2.1, providing some useful identities [28, pp. 128, 138], will be used frequently in subsequent sections.

LEMMA 2.1. Let $\varphi, \psi \in C^1(\overline{\Omega})$ and N be a vector field on $(\mathbb{R}^n, g)$. Then we have the following.

(1) Divergence formulae and theorems:

$$\text{div}_0(\varphi N) = \varphi \text{div}_0(N) + N(\varphi), \quad \text{div}_g(\varphi N) = \varphi \text{div}_g(N) + N(\varphi),$$
$$\int_\Omega \text{div}_0(N) \, dx = \int_\Gamma N \cdot \nu \, d\Gamma, \quad \int_\Omega \text{div}_g(N) \, dx = \int_\Gamma \langle N, \mu \rangle_g \, d\Gamma.$$

(2) Green's first formulae:

$$\int_{\Omega} \psi P\varphi \, dx = -\int_{\Gamma} \psi \frac{\partial \varphi}{\partial \nu_{\mathcal{A}}} \, d\Gamma + \int_{\Omega} \langle \nabla_g \varphi, \nabla_g \psi \rangle_g \, dx,$$

$$\int_{\Omega} \psi \Delta_g \varphi \, d\Omega = \int_{\Gamma} \psi \frac{\partial \varphi}{\partial \mu} \, dS - \int_{\Omega} \langle \nabla_g \varphi, \nabla_g \psi \rangle_g \, d\Omega.$$

(3) Green's second formulae:

$$\int_{\Omega} \psi P\varphi \, dx - \int_{\Omega} P\psi \, \varphi \, dx = -\int_{\Gamma} \psi \frac{\partial \varphi}{\partial \nu_{\mathcal{A}}} \, d\Gamma + \int_{\Gamma} \frac{\partial \psi}{\partial \nu_{\mathcal{A}}} \varphi \, d\Gamma,$$

$$\int_{\Omega} \psi \Delta_g \varphi \, d\Omega - \int_{\Omega} \Delta_g \psi \, \varphi \, d\Omega = \int_{\Gamma} \psi \frac{\partial \varphi}{\partial \mu} \, dS - \int_{\Gamma} \frac{\partial \psi}{\partial \mu} \varphi \, d\Gamma,$$

where $d\Omega$ and dS are the canonical volume elements on $\overline{\Omega}$ and Γ, respectively.

Similar to (4) of Lemma 2.1 in [33], we have the following multiplier identity.

LEMMA 2.2. Let N be a smooth real vector field on $(\mathbb{R}^n, g)$. For $z \in C^1(\overline{\Omega})$, the following formula holds:

$$\langle \nabla_g z, \nabla_g(N(z)) \rangle_g = DN(\nabla_g z, \nabla_g z) + \frac{1}{2}\mathrm{div}_g(|\nabla_g z|_g^2 N) - \frac{1}{2}|\nabla_g z|_g^2 \mathrm{div}_g(N). \qquad (2.4)$$

The following Lemma 2.3 is straightforward.

LEMMA 2.3. Let φ be a complex function defined on $\overline{\Omega}$ with suitable regularity. Then there exist some constants C's possibly depending on g, N and Ω, such that

(1)

$$\sup_{x \in \overline{\Omega}} |N|_g \leq C, \quad \sup_{x \in \overline{\Omega}} |DN|_g \leq C, \quad \sup_{x \in \overline{\Omega}} |\mathrm{div}_g(N)| \leq C, \quad \sup_{x \in \overline{\Omega}} |Dq|_g \leq C,$$

$$\sup_{x \in \overline{\Omega}} |\nabla_g(\mathrm{div}_g N)|_g \leq C;$$

(2)

$$|N(\varphi)| \leq C|\nabla_g \varphi|_g, \quad |Dq(\varphi)| \leq C|\nabla_g \varphi|_g, \quad |DN(\nabla_g \varphi, \nabla_g \overline{\varphi})| \leq C \, |\nabla_g \varphi|_g^2,$$

$$|\langle \nabla_g \varphi, \nabla_g(\mathrm{div}_g N) \rangle_g| \leq C|\nabla_g \varphi|_g,$$

where $q(x) = 1/2 \log(\rho(x))$ and ρ is given by (2.1);

(3)

$$\int_{\Omega} |\varphi|^2 \, dx \leq C\|\varphi\|_{H^1(\Omega)}^2, \int_{\Omega} |\nabla_g \varphi|_g^2 \, dx \leq C\|\varphi\|_{H^1(\Omega)}^2, \int_{\Omega} |\nabla_g(P\varphi)|_g^2 \, dx \leq C\|\varphi\|_{H^3(\Omega)}^2.$$

Now we are in a position to cast the system (1.1) into an abstract form of a first-order system in the Hilbert state space $H = H^{-1}(\Omega)$.

By Green's first formula, an extension $\mathscr{A}_1 \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ of $\mathcal{A}$ can be defined by

$$\langle \mathscr{A}_1 \varphi, \psi \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = \int_{\Omega} \langle \nabla_g \varphi, \nabla_g \phi \rangle_g \, dx, \ \forall \ \varphi, \psi \in H_0^1(\Omega).$$

Then $\mathscr{A}_1$ is a positive self-adjoint operator in H, and by means of the Lax-Milgram theorem, $\mathscr{A}_1$ is a canonical isomorphism from $D(\mathscr{A}_1) = H_0^1(\Omega)$ onto H. It is easily shown (see e.g., [18]) that $D(\mathscr{A}_1^{1/2}) = L^2(\Omega)$ and $\mathscr{A}_1^{1/2}$ is a canonical isomorphism from $L^2(\Omega)$ onto H.

Define the Dirichlet map $\Upsilon \in \mathcal{L}(L^2(\Gamma), H^{1/2}(\Omega))$ ([21] p. 189) as follows:

$$\Upsilon u = \varphi \Longleftrightarrow \begin{cases} P\varphi = 0 \text{ in } \Omega, \\ \varphi|_\Gamma = u. \end{cases} \tag{2.5}$$

Then (1.1) can be written as

$$\dot{v} - i\mathscr{A}_1(v - \Upsilon u) = 0. \tag{2.6}$$

Identifying H with its dual H', we have the following relations:

$$D(\mathscr{A}_1) \subset D(\mathscr{A}_1^{1/2}) \hookrightarrow H = H' \hookrightarrow [D(\mathscr{A}_1^{1/2})]' \subset [D(\mathscr{A}_1)]'.$$

Define an extension $\tilde{A}_1 \in \mathcal{L}(D(\mathscr{A}_1^{1/2}), [D(\mathscr{A}_1^{1/2})]')$ of $\mathscr{A}_1$ by

$$\langle \tilde{A}_1\varphi, \psi \rangle_{[D(\mathscr{A}_1^{1/2})]' \times D(\mathscr{A}_1^{1/2})} = \langle \mathscr{A}_1^{1/2}\varphi, \mathscr{A}_1^{1/2}\psi \rangle_H, \ \forall \ \varphi, \psi \in D(\mathscr{A}_1^{1/2}), \tag{2.7}$$

where $\langle \cdot, \cdot \rangle_H$ is the inner product on $H^{-1}(\Omega)$. In this way, $i\tilde{A}_1$ generates a C_0-group on $[D(\mathscr{A}_1^{1/2})]'$. Hence (2.6) can be written in $[D(\mathscr{A}_1)]'$ as

$$\dot{v} = i\tilde{A}_1 v + B_1 u, \tag{2.8}$$

where $B_1 \in \mathcal{L}(U, [D(\mathscr{A}_1^{1/2})]')$ is given by

$$B_1 u = -i\tilde{A}_1 \Upsilon u, \ \forall \ u \in U. \tag{2.9}$$

Define $B_1^* \in \mathcal{L}(D(\mathscr{A}_1^{1/2}), U)$, the adjoint of B_1, by

$$\langle B_1 u, f \rangle_{[D(\mathscr{A}_1^{1/2})]' \times D(\mathscr{A}_1^{1/2})} = \langle u, B_1^* f \rangle_U, \ \forall \ f \in D(\mathscr{A}_1^{1/2}) = L^2(\Omega), u \in U.$$

Then for any $f \in L^2(\Omega)$ and $u \in C_0^\infty(\Gamma)$, we have

$$\langle B_1 u, f \rangle_{[D(\mathscr{A}_1^{1/2})]' \times D(\mathscr{A}_1^{1/2})}$$

$$= \langle -i\tilde{A}_1 \Upsilon u, f \rangle_{[D(\mathscr{A}_1^{1/2})]' \times D(\mathscr{A}_1^{1/2})} = -i\langle \mathscr{A}_1^{1/2}\Upsilon u, \mathscr{A}_1^{1/2} f \rangle_H$$

$$= -i\langle \Upsilon u, \mathcal{A}(\mathcal{A}^{-1}f \rangle_{L^2(\Omega)} = \int_\Omega \Upsilon u P(\overline{i\mathcal{A}^{-1}f}) \, dx$$

$$= \int_\Omega P(\Upsilon u) \overline{i\mathcal{A}^{-1}f} \, dx - \int_\Gamma \Upsilon u \frac{\partial(\overline{i\mathcal{A}^{-1}f})}{\partial \nu_{\mathcal{A}}} \, d\Gamma + \int_\Gamma \frac{\partial(\Upsilon u)}{\partial \nu_{\mathcal{A}}} (\overline{i\mathcal{A}^{-1}f}) \, d\Gamma$$

$$= \left\langle u, -i\frac{\partial \mathcal{A}^{-1}f}{\partial \nu_{\mathcal{A}}} \right\rangle_U,$$

where in the last two steps, we used Green's second formula and the definitions of Υ and $\mathcal{A}$. Since $C_0^\infty(\Gamma)$ is dense in $L^2(\Gamma)$, we obtain

$$B_1^* f = -i\frac{\partial \mathcal{A}^{-1}f}{\partial \nu_{\mathcal{A}}}, \ \forall \ f \in L^2(\Omega). \tag{2.10}$$

System (1.1) is then formulated as an abstract form of a first-order system in the state Hilbert space H:

$$\begin{cases} \dot{v} = i\tilde{A}_1 v + B_1 u, \\ y = B_1^* v, \end{cases} \tag{2.11}$$

where $\tilde{A}_1$, B_1, and B_1^* are defined by (2.7), (2.9), and (2.10), respectively. The abstract study for system (2.11) can be found in [22].

Next, we formulate the system (1.2) as an abstract second-order system in $\mathcal{H} = H_0^1(\Omega) \times H$. Let A be the positive self-adjoint operator in $L^2(\Omega)$ defined by

$$\begin{cases} A\varphi = P^2\varphi, \\ \quad D(A) \; = \{\varphi \in L^2(\Omega)\,;\; P^2\varphi \in L^2(\Omega), \varphi|_\Gamma = P\varphi|_\Gamma = 0\} \\ \qquad\quad = \{\varphi \in H^4(\Omega)\,;\; \varphi|_\Gamma = P\varphi|_\Gamma = 0\}. \end{cases}$$

Just as in [15], it can be shown that

$$A^{1/2} = \mathcal{A},$$

where $\mathcal{A}$ is given by (1.4), and furthermore, we have the following space identifications with equivalent norms [9, 21]:

$$\begin{cases} D(A^\theta) = \{\varphi \in H^{4\theta}(\Omega)\,;\varphi|_\Gamma = 0\} = H^{4\theta}(\Omega) \cap H_0^1(\Omega), 1/8 < \theta < 5/8, \\ D(A^\theta) = \{\varphi \in H^{4\theta}(\Omega)\,;\; \varphi|_\Gamma = P\varphi|_\Gamma = 0\}, 5/8 < \theta \le 1. \end{cases} \tag{2.12}$$

By (2.12), we have

$$V \triangleq D(A^{3/4}) = \{\varphi \in H^3(\Omega)\,;\; \varphi|_\Gamma = P\varphi|_\Gamma = 0\}.$$

Define an extension operator $\mathscr{A}$ of A to the domain V as

$$\langle \mathscr{A}\varphi, \psi \rangle_H = \langle A^{1/2}\varphi, A^{1/2}\psi \rangle_H, \; \forall\, \varphi, \psi \in V. \tag{2.13}$$

Then $\mathscr{A}$ is positive self-adjoint in H. In fact, for some constants $C, C' > 0$, we have

$$\begin{aligned} \langle \mathscr{A}\varphi, \varphi \rangle_H \; &= \langle A^{1/2}\varphi, A^{1/2}\varphi \rangle_H = \langle A^{1/4}\varphi, A^{1/4}\varphi \rangle_{L^2(\Omega)} \\ &\ge C\|\varphi\|_{L^2(\Omega)}^2 \ge C'\|A^{-1/4}\varphi\|_{L^2(\Omega)}^2 = C'\|\varphi\|_H^2, \; \forall\, \varphi \in V. \end{aligned}$$

We identify H with its dual H'. Then the following relations hold:

$$D(\mathscr{A}^{1/2}) \hookrightarrow H = H' \hookrightarrow [D(\mathscr{A}^{1/2})]'.$$

An extension $\tilde{A} \in \mathcal{L}(D(\mathscr{A}^{1/2}), [D(\mathscr{A}^{1/2})]')$ of $\mathscr{A}$ is defined by

$$\langle \tilde{A}f, g \rangle_{[D(\mathscr{A}^{1/2})]' \times D(\mathscr{A}^{1/2})} = \langle \mathscr{A}^{1/2}f, \mathscr{A}^{1/2}g \rangle_H, \; \forall\, f, g \in D(\mathscr{A}^{1/2}). \tag{2.14}$$

Define the map $G \in \mathcal{L}(L^2(\Gamma), H^{5/2}(\Omega))$ [21, p. 189] so that $Gu = \phi$ if and only if

$$\begin{cases} P^2\phi(x) = 0, \; x \in \Omega, \\ \phi(x)|_\Gamma = 0, \; P\phi(x)|_\Gamma = u(x). \end{cases} \tag{2.15}$$

By virtue of the operators $\tilde{A}$ and G, (1.2) can be written in $[D(\mathscr{A}^{1/2})]'$ as

$$\ddot{w} + \tilde{A}w = Bu, \tag{2.16}$$

where $B \in \mathcal{L}(U, [D(\mathscr{A}^{1/2})]')$ is given by

$$Bu = \tilde{A}Gu, \ \forall \, u \in U. \tag{2.17}$$

Define $B^* \in \mathcal{L}(D(\mathscr{A}^{1/2}), U)$ by

$$\langle Bu, f \rangle_{[D(A^{1/2})]' \times D(A^{1/2})} = \langle u, B^*f \rangle_U, \ \forall \, f \in D(\mathscr{A}^{1/2}) = H_0^1(\Omega), u \in U.$$

Then for any $f \in D(\mathscr{A}^{1/2})$ and $u \in C_0^\infty(\Gamma)$, we have

$$\begin{aligned}
&\langle Bu, f \rangle_{[D(A^{1/2})]' \times D(A^{1/2})} \\
&= \langle \tilde{A}Gu, f \rangle_{[D(A^{1/2})]' \times D(A^{1/2})} = \langle \mathscr{A}^{1/2} Gu, \mathscr{A}^{1/2} f \rangle_H \\
&= \langle \mathscr{A}Gu, \mathcal{A}(\mathcal{A}^{-1}f) \rangle_H = \langle A^{-1/4} AGu, A^{-1/4} \mathcal{A}(\mathcal{A}^{-1}f) \rangle_{L^2(\Omega)} \\
&= \langle AGu, \mathcal{A}(\mathcal{A}^{-1}f) \rangle_{L^2(\Omega)} \\
&= \int_\Omega P^2(Gu) \overline{\mathcal{A}^{-1}f} \, dx + \int_\Gamma P(Gu) \frac{\partial \overline{\mathcal{A}^{-1}f}}{\partial \nu_{\mathcal{A}}} \, d\Gamma + \int_\Gamma (\mathcal{A}^{-1}f) \frac{\partial \overline{P(Gu)}}{\partial \nu_{\mathcal{A}}} \, d\Gamma \\
&= \left\langle u, \frac{\partial \mathcal{A}^{-1}f}{\partial \nu_{\mathcal{A}}} \right\rangle_U,
\end{aligned}$$

where we used Green's second formula and the definitions of G and $\mathcal{A}$. Since $C_0^\infty(\Gamma)$ is dense in $L^2(\Gamma)$, we actually obtain

$$B^*f = \left. \frac{\partial \mathcal{A}^{-1}f}{\partial \nu_{\mathcal{A}}} \right|_\Gamma, \ \forall f \in D(\mathscr{A}^{1/2}) = H_0^1(\Omega). \tag{2.18}$$

Finally, we cast the open-loop system (1.2) into the abstract form of a second-order collocated system in $\mathcal{H}$,

$$\begin{cases} \ddot{w} + \tilde{A}w = Bu, \\ y = B^* \dot{w}, \end{cases} \tag{2.19}$$

where $\tilde{A}$, B, and B^* are defined by (2.14), (2.17), and (2.18), respectively. System (2.19) was well studied in [2, 10].

3. Proof of Theorem 1.1. For brevity in notation, we shall denote by C_T, in subsequent sections, a positive constant depending only on the time T that may have different values in different contexts. First, we show that B_1 is admissible for $e^{i\mathscr{A}_1 t}$, the C_0-group generated by $i\mathscr{A}_1$ on H. Since the system (1.1) is collocated, B_1 is admissible for $e^{i\mathscr{A}_1 t}$ if and only if B_1^* is admissible for $e^{-i\mathscr{A}_1^* t} = e^{i\mathscr{A}_1 t}$ (see [29]). Therefore, we need to show the estimate

$$\int_0^T \left\| B_1^*(e^{i\mathscr{A}_1 t} v_0) \right\|_{L^2(\Gamma)}^2 \, dt \le C_T \|v_0\|_{H^{-1}(\Omega)}^2, \ \forall \, v_0 \in D(\mathscr{A}_1) = H_0^1(\Omega). \tag{3.1}$$

Set $z = \mathscr{A}_1^{-1}v$. Instead of (2.11), we consider the following z equation in the space $H_0^1(\Omega)$ that is derived from (2.9), (2.10), and (2.11):

$$
\begin{cases}
z_t(x,t) = iPz(x,t) - i(\Upsilon u(\cdot,t))(x), & (x,t) \in \Omega \times (0,T], \\
z(x,0) = z_0(x), & x \in \Omega, \\
z(x,t) = 0, & (x,t) \in \Gamma \times [0,T], \\
y(x,t) = -i\dfrac{\partial z(x,t)}{\partial \nu_{\mathcal{A}}}, & (x,t) \in \Gamma \times [0,T].
\end{cases}
\tag{3.2}
$$

Let $f(x,t) = -i(\Upsilon u(\cdot,t))(x)$. By the definition of the Dirichlet map Υ, we have

$$
\int_0^T \|f(\cdot,t)\|_{L^2(\Omega)}^2 \, dt \le C_T \int_0^T \|u(\cdot,t)\|_{L^2(\Gamma)}^2 \, dt.
\tag{3.3}
$$

By Lemma 4.1 of [15], there exists a C^2 vector field N on $\overline{\Omega}$ such that

$$
N(x) = \mu(x), \ x \in \Gamma; \ |N(x)|_g \le 1, \quad x \in \Omega.
\tag{3.4}
$$

Multiply both sides of the first equation in (3.2) by $N(\overline{z})$ and integrate over Ω, to give

$$
\begin{aligned}
0 &= \int_\Omega z_t N(\overline{z}) \, dx - i \int_\Omega Pz N(\overline{z}) \, dx - \int_\Omega f N(\overline{z}) \, dx \\
&= \int_\Omega z_t N(\overline{z}) \, dx + i \int_\Omega (\Delta_g z - (Dq)z) N(\overline{z}) \, dx - \int_\Omega f N(\overline{z}) \, dx \\
&= \int_\Omega z_t N(\overline{z}) \, dx + i \int_\Gamma \left|\frac{\partial z}{\partial \mu}\right|^2 d\Gamma - i \int_\Omega \langle \nabla_g z, \nabla_g (N(\overline{z})) \rangle_g \, dx \\
&\quad - i \int_\Omega (Dq)z N(\overline{z}) \, dx - \int_\Omega f N(\overline{z}) \, dx.
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
\int_\Gamma \left|\frac{\partial z}{\partial \mu}\right|^2 d\Gamma &= \operatorname{Re} \int_\Omega \langle \nabla_g z, \nabla_g (N(\overline{z})) \rangle_g dx \\
&\quad - \operatorname{Im} \int_\Omega z_t N(\overline{z}) \, dx + \operatorname{Re} \int_\Omega (Dq)z N(\overline{z}) dx + \operatorname{Im} \int_\Omega f N(\overline{z}) dx.
\end{aligned}
\tag{3.5}
$$

By Lemma 2.2, one has

$$
\operatorname{Re}\langle \nabla_g z, \nabla_g (N(\overline{z})) \rangle_g = \operatorname{Re} DN(\nabla_g z, \nabla_g \overline{z}) + \frac{1}{2}\operatorname{div}_g(|\nabla_g z|_g^2 N) - \frac{1}{2}|\nabla_g z|_g^2 \operatorname{div}_g N.
\tag{3.6}
$$

Substitute (3.6) into (3.5), to obtain

$$
\begin{aligned}
\int_\Gamma \left|\frac{\partial z}{\partial \mu}\right|^2 d\Gamma &= \operatorname{Re} \int_\Omega DN(\nabla_g z, \nabla_g \overline{z}) dx \\
&\quad + \frac{1}{2}\int_\Gamma |\nabla_g z|_g^2 d\Gamma - \frac{1}{2}\int_\Omega |\nabla_g z|_g^2 \operatorname{div}_g N \, dx \\
&\quad - \operatorname{Im} \int_\Omega z_t N(\overline{z}) \, dx + \operatorname{Re} \int_\Omega (Dq)z N(\overline{z}) dx + \operatorname{Im} \int_\Omega f N(\overline{z}) dx.
\end{aligned}
\tag{3.7}
$$

Notice that for any $x \in \Gamma$, it has

$$\nabla_g z(x) = \langle \nabla_g z(x), \mu(x) \rangle_g \mu(x) + Y(x) = \frac{\partial z(x)}{\partial \mu} \mu(x) + Y(x)$$

for some $Y(x) \in \mathbb{R}^n_x$ with $\langle Y(x), \mu(x) \rangle_g = 0$, which implies that $Y(x) \cdot \nu(x) = 0$. This fact together with the boundary condtion $z|_\Gamma = 0$ gives

$$\begin{aligned}
|\nabla_g z(x)|_g^2 &= \langle \nabla_g z(x), \nabla_g \overline{z}(x) \rangle_g \\
&= \nabla_g z(x)(\overline{z}(x)) = \left| \frac{\partial z(x)}{\partial \mu} \right|^2 = \frac{1}{|\nu_{\mathcal{A}}(x)|_g^2} \left| \frac{\partial z(x)}{\partial \nu_{\mathcal{A}}} \right|^2, \ \forall \, x \in \Gamma,
\end{aligned} \tag{3.8}$$

where we used the fact $\mu = \frac{\nu_{\mathcal{A}}}{|\nu_{\mathcal{A}}|_g}$. In addition, by (1.3), (1.5), and (2.2), we have

$$\min_{x \in \Gamma} \frac{1}{|\nu_{\mathcal{A}}(x)|_g^2} \geq \frac{1}{b}. \tag{3.9}$$

By (3.8) and (3.9), making use of the Cauchy-Schwartz inequality and Lemma 2.3, we deduce, from (3.7), that

$$\begin{aligned}
\int_\Gamma \left| \frac{\partial z}{\partial \nu_{\mathcal{A}}} \right|^2 d\Gamma &\leq b \int_\Gamma \left| \frac{\partial z}{\partial \mu} \right|^2 d\Gamma \\
&= 2b\mathrm{Re} \int_\Omega DN(\nabla_g z, \nabla_g \overline{z}) \, dx - b \int_\Omega |\nabla_g z|_g^2 \mathrm{div}_g N \, dx - 2b\mathrm{Im} \int_\Omega z_t N(\overline{z}) \, dx \\
&\quad + 2b\mathrm{Re} \int_\Omega (Dq) z N(\overline{z}) \, dx + 2b\mathrm{Im} \int_\Omega f N(\overline{z}) \, dx \\
&\leq C \left(\int_\Omega |\nabla_g z|_g^2 \, dx + \int_\Omega |f|^2 \, dx \right) - 2b\mathrm{Im} \int_\Omega z_t N(\overline{z}) \, dx.
\end{aligned} \tag{3.10}$$

Now we compute the last term in (3.10). By the divergence formula and (3.2), it follows that

$$\begin{aligned}
\mathrm{div}_g(z_t \overline{z} N) &= z_t \overline{z} \mathrm{div}_g N + N(z_t) \overline{z} + N(\overline{z}) z_t \\
&= (iPz + f) \overline{z} \mathrm{div}_g N + \frac{d}{dt}(\overline{z} N(z)) - \overline{z}_t N(z) + N(\overline{z}) z_t \\
&= (iPz + f) \overline{z} \mathrm{div}_g N + \frac{d}{dt}(\overline{z} N(z)) + 2i\mathrm{Im} \, z_t N(\overline{z}).
\end{aligned}$$

Integrate the above equality over Ω, to give

$$\begin{aligned}
2i\mathrm{Im} \int_\Omega z_t N(\overline{z}) \, dx &= \int_\Omega [i(\Delta_g - Dq)z - f] \overline{z} \mathrm{div}_g N \, dx - \frac{d}{dt} \int_\Omega \overline{z} N(z) \, dx \\
&= -i \int_\Omega \langle \nabla_g z, \nabla_g(\overline{z} \mathrm{div}_g N) \rangle_g \, dx - i \int_\Omega Dq(z) \overline{z} \mathrm{div}_g N \, dx - \int_\Omega f \overline{z} \mathrm{div}_g N \, dx \\
&\quad - \frac{d}{dt} \int_\Omega \overline{z} N(z) \, dx.
\end{aligned}$$

Therefore,

$$-2b\text{Im}\int_0^T\int_\Omega z_t N(\overline{z})\,dxdt$$

$$=\int_0^T\int_\Omega \langle\nabla_g z,\nabla_g(\overline{z}\text{div}_g(N))\rangle_g\,dxdt+\int_0^T\int_\Omega Dq(z)\overline{z}\text{div}_g(N)\,dxdt \qquad (3.11)$$

$$+i\int_0^T\int_\Omega f\overline{z}\text{div}_g(N)\,dxdt+i\int_\Omega \overline{z}N(z)\,dx\Big|_0^T.$$

Using (3.11) and integrating both sides of (3.10) over $[0,T]$, we obtain

$$\int_0^T\int_\Gamma\left|\frac{\partial z}{\partial\nu_{\mathcal{A}}}\right|^2 d\Gamma dt \le C_T\left(\|z\|_{L^2(0,T;H^1(\Omega))}^2\right. \tag{3.12}$$
$$\left.+\|f\|_{L^2(\Omega\times(0,T))}^2+\|z\|_{L^\infty(0,T;H^1(\Omega))}^2\right).$$

Let $f=-i\Upsilon u=0$ in (3.2). For any $z_0\in D(\mathscr{A}_1)=H_0^1(\Omega)$, it has

$$e^{i\mathscr{A}_1 t}z_0\in C([0,T];D(\mathscr{A}_1))\cap C^1((0,T];H),$$

and hence, by (3.12), we have

$$\int_0^T\int_\Gamma\left|\frac{\partial(e^{i\mathscr{A}_1 t}z_0)}{\partial\nu_{\mathcal{A}}}\right|^2 d\Gamma dt \le C_T\|z_0\|_{D(\mathscr{A}_1)}^2,\ \forall\ z_0\in D(\mathscr{A}_1). \tag{3.13}$$

This implies

$$\int_0^T\int_\Gamma\left|\frac{\partial(e^{i\mathscr{A}_1 t}\mathscr{A}_1^{-1}v_0)}{\partial\nu_{\mathcal{A}}}\right|^2 d\Gamma dt \le C_T\|v_0\|_{H^{-1}(\Omega)}^2,\ \forall\ v_0\in D(\mathscr{A}_1). \tag{3.14}$$

By the definition of B_1^*, (3.13) is just (3.1). Therefore, B_1 is admissible.

Next, we show the boundedness of input-output map for the system (3.2), that is, for some (and hence for all) $T>0$, the solution to (3.2) with $z_0=0$ satisfies

$$\int_0^T\int_\Gamma\left|\frac{\partial z(x,t)}{\partial\nu_{\mathcal{A}}}\right|^2 d\Gamma dt \le C_T\int_0^T\int_\Gamma |u(x,t)|^2\,d\Gamma dt,\ \forall\ u\in L^2(0,T;L^2(\Gamma)). \tag{3.15}$$

Noticing that the solution to (3.2) with $z_0=0$ is given by

$$z(x,t)=\int_0^t\left[e^{i\tilde{A}_1(t-s)}f(\cdot,s)\right](x)ds=-i\int_0^t\left[e^{i\tilde{A}_1(t-s)}\Upsilon u(\cdot,s)\right](x)ds,$$

and the admissibility just verified, we have [30]

$$\tilde{A}_1 z(x,t)\ =-i\int_0^t\left[e^{i\tilde{A}_1(t-s)}\tilde{A}_1\Upsilon u(\cdot,s)\right](x)ds$$

$$=\int_0^t\left[e^{i\tilde{A}_1(t-s)}B_1 u(\cdot,s)\right](x)ds\in C([0,T];H).$$

It then follows that

$$z\in C([0,T];H_0^1(\Omega)). \tag{3.16}$$

This together with (3.3) and (3.12) shows that (3.15) is valid. The proof is complete. $\square$

REMARK 3.1. From Proposition 4.1 of [23], (3.15) implies the admissibility of B (so is for B^*), but we do not use this fact here.

4. Proof of Theorem 1.2. By definition of the well-posedness (see [6]), Theorem 1.2 is equivalent to saying that B is admissible (so is B^*), and the solution to (1.2) with zero initial data satisfies

$$\|y\|^2_{L^2(0,T;L^2(\Gamma))} \leq C_T \|u\|^2_{L^2(0,T;L^2(\Gamma))}, \ \forall \, u \in L^2(0,T;L^2(\Gamma)). \tag{4.1}$$

Introduce the following transform

$$z(t) = \mathcal{A}^{-1}w(t) \in C(0,T;V). \tag{4.2}$$

Instead of (1.2), we consider the following system in $V \times H^1_0(\Omega)$:

$$\begin{cases} z_{tt}(x,t) + P^2 z(x,t) = \Upsilon u(x,t), & (x,t) \in \Omega \times (0,T] \triangleq Q, \\ z(x,0) = 0, \ z_t(x,0) = 0, & x \in \Omega, \\ z(x,t) = 0, \ Pz(x,t) = 0, & (x,t) \in \Gamma \times (0,T] \triangleq \Sigma, \\ y(x,t) = \dfrac{\partial z_t}{\partial \nu_{\mathcal{A}}}(x,t), & (x,t) \in \Sigma, \end{cases} \tag{4.3}$$

where the operator Υ is defined by (2.5), and the validity of the first equation in (4.3) is attributed to the fact

$$\mathcal{A}^{-1}\tilde{A}Gu = \tilde{A}^{1/2}Gu = \mathcal{A}Gu = \Upsilon u, \ \forall \, u \in L^2(\Gamma).$$

Therefore, Theorem 1.2 is valid if and only if B is admissible, and for some (and hence for all) $T > 0$, there exists a $C_T > 0$ such that the solution to (4.3) satisfies

$$\int_\Sigma \left| \frac{\partial z_t}{\partial \nu_{\mathcal{A}}}(x,t) \right|^2 d\Sigma \leq C_T \int_\Sigma |u(x,t)|^2 d\Sigma. \tag{4.4}$$

Step 1 (Energy identity). Let N be the C^2 vector field on $\overline{\Omega}$ satisfying (3.4). Multiply the both sides of (4.3) by $N(P\overline{z})$, to give

$$\int_Q z_{tt} N(P\overline{z}) \, dQ + \int_Q P^2 z N(P\overline{z}) \, dQ - \int_Q \Upsilon u N(P\overline{z}) \, dQ = 0. \tag{4.5}$$

Compute the first term on the left-hand side of (4.5) to yield

$$\int_Q z_{tt} N(P\overline{z}) \, dQ = \int_\Omega z N(P\overline{z}) \, dx \Big|_0^T - \int_Q z_t N(P\overline{z_t}) \, dQ$$

$$= \int_\Omega z N(P\overline{z}) \, dx \Big|_0^T - \int_Q [N(z_t P\overline{z_t}) - N(z_t)P\overline{z_t}] \, dQ$$

$$= \int_\Omega z N(P\overline{z}) \, dx \Big|_0^T - \int_\Sigma z_t(P\overline{z_t})\langle N,\mu \rangle_g \, d\Sigma + \int_Q z_t(P\overline{z_t})\mathrm{div}_g N \, dQ + \int_Q N(z_t)P\overline{z_t} \, dQ$$

$$= \int_\Omega z N(P\overline{z}) \, dx \Big|_0^T + \int_Q z_t(P\overline{z_t})\mathrm{div}_g N \, dQ + \int_Q N(z_t)P\overline{z_t} \, dQ.$$

Since

$$\text{Re} \int_Q z_t(P\overline{z_t})\text{div}_g N \, dQ = -\text{Re} \int_Q z_t \text{div}_g N \Delta_g \overline{z_t} \, dQ + \text{Re} \int_Q z_t Dq(\overline{z_t}) \, dQ$$

$$= -\text{Re} \int_\Sigma z_t \text{div}_g N \frac{\partial \overline{z_t}}{\partial \mu} \, d\Sigma + \text{Re} \int_Q \langle \nabla_g(z_t \text{div}_g N), \nabla_g \overline{z_t} \rangle_g \, dQ + \text{Re} \int_Q z_t Dq(\overline{z_t}) \, dQ$$

$$= \int_Q \text{div}_g N |\nabla_g z_t|_g^2 \, dQ + \text{Re} \int_Q z_t \nabla_g(\text{div}_g N)(\overline{z_t}) \, dQ + \text{Re} \int_Q z_t Dq(\overline{z_t}) \, dQ$$

$$= \int_Q \text{div}_g N |\nabla_g z_t|_g^2 \, dQ + \frac{1}{2} \int_Q \nabla_g(\text{div}_g N)(|z_t|^2) \, dQ + \frac{1}{2} \int_Q Dq(|z_t|^2) dQ,$$

and

$$\text{Re} \int_Q N(z_t) P\overline{z_t} \, dQ = -\text{Re} \int_Q N(z_t) \Delta_g \overline{z_t} \, dQ + \text{Re} \int_Q N(z_t) Dq(\overline{z_t}) \, dQ$$

$$= -\text{Re} \int_\Sigma N(z_t) \frac{\partial \overline{z_t}}{\partial \mu} \, d\Sigma + \text{Re} \int_Q \langle \nabla_g \overline{z_t}, \nabla_g(N(z_t)) \rangle_g \, dQ + \text{Re} \int_Q N(z_t) Dq(\overline{z_t}) \, dQ$$

$$= -\int_\Sigma \left| \frac{\partial z_t}{\partial \mu} \right|^2 d\Sigma + \text{Re} \int_Q DN(\nabla_g z_t, \nabla_g \overline{z_t}) \, dQ + \frac{1}{2} \int_Q \text{div}_g(|\nabla_g z_t|_g^2 N) \, dQ$$

$$- \frac{1}{2} \int_Q |\nabla_g z_t|_g^2 \text{div}_g N \, dQ + \text{Re} \int_Q N(z_t) Dq(\overline{z_t}) \, dQ$$

$$= -\frac{1}{2} \int_\Sigma \left| \frac{\partial z_t}{\partial \mu} \right|^2 d\Sigma + \text{Re} \int_Q DN(\nabla_g z_t, \nabla_g \overline{z_t}) \, dQ - \frac{1}{2} \int_Q |\nabla_g z_t|_g^2 \text{div}_g N \, dQ$$

$$+ \text{Re} \int_Q N(z_t) Dq(\overline{z_t}) \, dQ,$$

where the formulae (3.6) and (3.8) were used for the computation of the above identity, it follows that

$$\text{Re} \int_Q z_{tt} N(P\overline{z}) \, dQ = \text{Re} \int_\Omega z N(P\overline{z}) \, dx \Big|_0^T + \frac{1}{2} \int_Q \text{div}_g N |\nabla_g z_t|_g^2 \, dQ$$

$$+ \frac{1}{2} \int_Q \nabla_g(\text{div}_g N)(|z_t|^2) \, dQ + \frac{1}{2} \int_Q Dq(|z_t|^2) \, dQ - \frac{1}{2} \int_\Sigma \left| \frac{\partial z_t}{\partial \mu} \right|^2 d\Sigma \qquad (4.6)$$

$$+ \text{Re} \int_Q DN(\nabla_g z_t, \nabla_g \overline{z_t}) \, dQ + \text{Re} \int_Q N(z_t) Dq(\overline{z_t}) \, dQ.$$

By Green's first formula, (3.6) and (3.8), the second term on the left-hand side of (4.5) is computed as

$$\text{Re} \int_Q P^2 z N(P\overline{z}) \, dQ = \text{Re} \int_Q \langle \nabla_g(Pz), \nabla_g(N(P\overline{z})) \rangle_g \, dQ$$

$$-\text{Re} \int_\Sigma \frac{\partial Pz}{\partial \mu} N(P\overline{z}) \, d\Sigma = \text{Re} \int_Q DN(\nabla_g(Pz), \nabla_g(P\overline{z})) \, dQ$$

$$+\frac{1}{2} \int_Q \text{div}_g(|\nabla_g(Pz)|_g^2 N) \, dQ - \frac{1}{2} \int_Q |\nabla_g(Pz)|_g^2 \text{div}_g N \, dQ \tag{4.7}$$

$$-\int_\Sigma \left| \frac{\partial Pz}{\partial \mu} \right|^2 d\Sigma = \text{Re} \int_Q DN(\nabla_g(Pz), \nabla_g(P\overline{z})) \, dQ$$

$$-\frac{1}{2} \int_Q |\nabla_g(Pz)|_g^2 \text{div}_g N \, dQ - \frac{1}{2} \int_\Sigma \left| \frac{\partial Pz}{\partial \mu} \right|^2 d\Sigma.$$

Combine (4.5), (4.6), and (4.7), to obtain

$$\frac{1}{2} \int_\Sigma \left(\left| \frac{\partial z_t}{\partial \mu} \right|^2 + \left| \frac{\partial Pz}{\partial \mu} \right|^2 \right) d\Sigma = R_1 + R_2 + b_T, \tag{4.8}$$

where

$$R_1 = \text{Re} \int_Q DN(\nabla_g(Pz), \nabla_g(P\overline{z})) \, dQ + \text{Re} \int_Q DN(\nabla_g z_t, \nabla_g \overline{z_t}) \, dQ$$

$$+\frac{1}{2} \int_Q Dq(|z_t|^2) \, dQ + \frac{1}{2} \int_Q \text{div}_g N \left(|\nabla_g z_t|_g^2 - |\nabla_g(Pz)|_g^2 \right) \, dQ \tag{4.9}$$

$$+\frac{1}{2} \int_Q \nabla_g(\text{div}_g N)(|z_t|^2) \, dQ + \text{Re} \int_Q N(z_t) Dq(\overline{z_t}) \, dQ,$$

$$R_2 = -\int_Q \Upsilon u N(P\overline{z}) \, dQ, \tag{4.10}$$

$$b_T = \text{Re} \int_\Omega z N(P\overline{z}) \, dx \Big|_0^T. \tag{4.11}$$

Step 2 (Estimate of R_1). It is easy to show

$$\begin{cases} z_{tt} + \mathscr{A} z = 0, \\ z(0) = z_0, \ z_t(0) = z_1, \\ y = B^* \mathcal{A} z_t, \end{cases} \tag{4.12}$$

where $\mathscr{A}$ and $\mathcal{A}$ are given by (2.13) and (1.4), respectively, which is the dual system of (2.16), associates with a C_0-group solution in the space $V \times H_0^1(\Omega)$. That is to say, for any $(z_0, z_1) \in V \times H_0^1(\Omega)$, the corresponding solution to (4.12) satisfies $(z, z_t) \in V \times H_0^1(\Omega)$, and depends continuously on (z_0, z_1). By this fact, letting $\Upsilon u = 0$ in (4.8) and using (3.8) and (3.9), we obtain from (4.8) that

$$\int_\Sigma \left| \frac{\partial z_t}{\partial \nu_\mathcal{A}} \right|^2 d\Sigma \leq C_T \|(z_0, z_1)\|_{V \times H_0^1(\Omega)}^2, \tag{4.13}$$

which is equivalent to saying that for any initial data $(w_0, w_1) \in H_0^1(\Omega) \times H$, the solution to the system (1.2) with $u = 0$ satisfies

$$\int_\Sigma \left| \frac{\partial \mathcal{A}^{-1} w_t}{\partial \nu_\mathcal{A}} \right|^2 d\Sigma \leq C_T \|(w_0, w_1)\|_{H_0^1(\Omega) \times H}^2. \tag{4.14}$$

This shows that B^* is admissible; so is B [29]. In other words,

$$u \mapsto \{w, w_t\} \text{ is continuous from } L^2(\Sigma) \to C([0, T]; H_0^1(\Omega) \times H). \tag{4.15}$$

By (4.15), we have

$$(z, z_t) \in C([0, T]; V \times H_0^1(\Omega)) \text{ continuously in } u \in L^2(\Sigma). \tag{4.16}$$

Hence

$$R_1 \leq C_T \|u\|_{L^2(\Sigma)}^2, \ \forall \, u \in L^2(\Sigma), \tag{4.17}$$

where we used Lemma 2.3.

Step 3 (Estimate of R_2 and b_T). From (4.10) and (4.11), by virtue of (4.16), we readily obtain

$$R_2 + b_T \leq C_T \|u\|_{L^2(\Sigma)}^2, \ \forall \, u \in L^2(\Sigma). \tag{4.18}$$

Step 4 (Final estimate). Combining (4.17), (4.18), and (4.8), and using (3.8) and (3.9), we finally obtain

$$\int_\Sigma \left| \frac{\partial z_t}{\partial \nu_\mathcal{A}}(x, t) \right|^2 d\Sigma + \int_\Sigma \left| \frac{\partial Pz}{\partial \nu_\mathcal{A}}(x, t) \right|^2 d\Sigma \leq C_T \int_\Sigma |u(x, t)|^2 d\Sigma. \tag{4.19}$$

This concludes (4.4). This together with (4.15) completes the proof of Theorem 1.2. $\square$

REMARK 4.1. It was realized in Corollary A.2 of [22] that (4.1), the boundedness of the input-output map, implies the admissibility of B (so is for B^*), but the proof there is incorrect. A simple proof was given in [4]. But here we do not need this fact.

5. Proofs of Theorems 1.4 and 1.5. It is known from the Appendix of [10] that the transfer functions of the systems (2.11) and (2.19) are, respectively,

$$H_1(\lambda) = B_1^*(\lambda - i\tilde{A}_1)^{-1} B_1 \tag{5.1}$$

and

$$H(\lambda) = \lambda B^*(\lambda^2 + \tilde{A})^{-1} B, \tag{5.2}$$

where $\tilde{A}_1$, B_1, and B_1^* are given by (2.7), (2.9), and (2.10); $\tilde{A}$, B, and B^* are given by (2.14), (2.17), and (2.18), respectively. Moreover, from the well-posedness claimed by Theorems 1.1 and 1.2, it follows that there exist constants $M, \alpha, \beta > 0$ such that [8]

$$\sup_{\text{Re}\lambda \geq \alpha} \|H_1(\lambda)\|_{\mathcal{L}(U)} < M \tag{5.3}$$

and

$$\sup_{\text{Re}\lambda \geq \beta} \|H(\lambda)\|_{\mathcal{L}(U)} < M. \tag{5.4}$$

PROPOSITION 5.1. Theorem 1.4 is valid if for any $u \in C_0^\infty(\Gamma)$ and $\varepsilon > 0$, the solution v_ε to the following equation

$$\begin{cases} \varepsilon^{-1} v_\varepsilon(x) - i P v_\varepsilon(x) = 0, & x \in \Omega, \\ v_\varepsilon(x) = u(x), & x \in \Gamma, \end{cases} \tag{5.5}$$

satisfies

$$\lim_{\varepsilon \to 0+} \int_\Gamma \left| \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_\mathcal{A}} \right|^2 d\Gamma = 0. \tag{5.6}$$

Proof. We need only show that $H_1(\lambda)u$ converges to zero in the strong topology of U along the positive axis (see [32]), that is,

$$\lim_{\lambda \in \mathbb{R}, \lambda \to +\infty} H_1(\lambda)u = 0, \ \forall \, u \in L^2(\Gamma). \tag{5.7}$$

Due to (5.3) and a density argument, it suffices to show that (5.7) is satisfied for all $u \in C_0^\infty(\Gamma)$. To this end, let $u \in C_0^\infty(\Gamma)$. Denote by

$$v_\lambda = (\lambda - i \tilde{A}_1)^{-1} B_1 u. \tag{5.8}$$

Then along the line from (2.5) to (2.9), we conclude that v_λ satisfies

$$\begin{cases} \lambda v_\lambda(x) - i P v_\lambda(x) = 0, & x \in \Omega, \\ v_\lambda(x) = u(x), & x \in \Gamma. \end{cases} \tag{5.9}$$

It follows from (2.10), (5.8), and (5.1) that

$$(H_1(\lambda)u)(x) = -i \frac{\partial(\mathcal{A}^{-1} v_\lambda)}{\partial \nu_\mathcal{A}}(x), \ \forall \, x \in \Gamma. \tag{5.10}$$

Since $u \in C_0^\infty(\Gamma)$, there exists a unique classical solution to (5.9). Let $\tilde{v} \in C^\infty(\Omega)$ be the unique solution to the boundary value problem

$$\begin{cases} P\tilde{v}(x) = 0, & x \in \Omega, \\ \tilde{v}(x) = u(x), & x \in \Gamma. \end{cases}$$

Then (5.9) can be written as

$$\begin{cases} i\lambda \, v_\lambda(x) - P(v_\lambda(x) - \tilde{v}(x)) = 0, & x \in \Omega, \\ v_\lambda(x) - \tilde{v}(x) = 0, & x \in \Gamma, \end{cases}$$

or equivalently,

$$i \, \lambda \mathcal{A}^{-1} v_\lambda = v_\lambda - \tilde{v}.$$

Hence, (5.10) becomes

$$(H_1(\lambda)u)(x) = -\frac{1}{\lambda} \frac{\partial v_\lambda}{\partial \nu_\mathcal{A}}(x) + \frac{1}{\lambda} \frac{\partial \tilde{v}}{\partial \nu_\mathcal{A}}(x), \ \forall x \in \Gamma.$$

Letting $v_\varepsilon = v_\lambda$ with $\varepsilon = \frac{1}{\lambda}$, we obtain the required result. □

Proof of Theorem 1.4. By Proposition 5.1, we only need to prove (5.6).

Let N be the C^2 vector field on $\overline{\Omega}$ satisfying (3.4). Multiply both sides of the first equation of (5.5) by $N(\overline{v}_\varepsilon)$ and integrate over Ω. Using Green's first formula and the fact that $N(\overline{v}_\varepsilon) = \langle \nabla_g \overline{v}_\varepsilon, N \rangle_g = \frac{\partial \overline{v}_\varepsilon}{\partial \mu}$ on Γ, we obtain

$$
\begin{aligned}
0 \;&=\; \int_\Omega \varepsilon^{-1} v_\varepsilon N(\overline{v}_\varepsilon)\,dx - i\int_\Omega P v_\varepsilon N(\overline{v}_\varepsilon)\,dx \\
&=\; \varepsilon^{-1}\int_\Omega v_\varepsilon N(\overline{v}_\varepsilon)\,dx - i\int_\Omega (-\Delta_g + Dq)v_\varepsilon N(\overline{v}_\varepsilon)\,dx \\
&=\; \varepsilon^{-1}\int_\Omega v_\varepsilon N(\overline{v}_\varepsilon)\,dx + i\int_\Gamma \left|\frac{\partial v_\varepsilon}{\partial \mu}\right|^2 d\Gamma - i\int_\Omega \langle \nabla_g v_\varepsilon, \nabla_g N(\overline{v}_\varepsilon)\rangle_g\,dx \\
&\quad -i\int_\Omega Dq(v_\varepsilon)N(\overline{v}_\varepsilon)\,dx.
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
\int_\Gamma \left|\frac{\partial v_\varepsilon}{\partial \mu}\right|^2 d\Gamma \;&=\; \mathrm{Re}\int_\Omega \langle \nabla_g v_\varepsilon, \nabla_g N(\overline{v}_\varepsilon)\rangle_g\,dx - \varepsilon^{-1}\mathrm{Im}\int_\Omega v_\varepsilon N(\overline{v}_\varepsilon)\,dx \\
&\quad +\mathrm{Re}\int_\Omega Dq(v_\varepsilon)N(\overline{v}_\varepsilon)\,dx.
\end{aligned} \tag{5.11}
$$

Substitute (3.6) into (5.11), and make use of the divergence theorem, (3.8) and Lemma 2.3 to give

$$
\begin{aligned}
\int_\Gamma \left|\frac{\partial v_\varepsilon}{\partial \mu}\right|^2 d\Gamma \;&=\; 2\mathrm{Re}\int_\Omega DN(\nabla_g v_\varepsilon, \nabla_g \overline{v}_\varepsilon)\,dx - \int_\Omega |\nabla_g v_\varepsilon|_g^2 \mathrm{div}_g(N)\,dx \\
&\quad -2\varepsilon^{-1}\mathrm{Im}\int_\Omega v_\varepsilon N(\overline{v}_\varepsilon)\,dx + 2\mathrm{Re}\int_\Omega Dq(v_\varepsilon)N(\overline{v}_\varepsilon)\,dx \\
&\leq\; C\int_\Omega |\nabla_g v_\varepsilon|_g^2\,dx - 2\varepsilon^{-1}\mathrm{Im}\int_\Omega v_\varepsilon N(\overline{v}_\varepsilon)\,dx \\
&\leq\; C\int_\Omega |\nabla_g v_\varepsilon|_g^2\,dx + \varepsilon^{-3/2}\int_\Omega |v_\varepsilon|^2\,dx + \varepsilon^{-1/2}\int_\Omega |\nabla_g v_\varepsilon|_g^2\,dx
\end{aligned} \tag{5.12}
$$

where C is a positive constant independent of ε.

Next, multiply both sides of the first equation of (5.5) by $\overline{v}_\varepsilon$ and integrate over Ω by parts to obtain

$$
\begin{aligned}
0 \;&=\; \varepsilon^{-1}\int_\Omega |v_\varepsilon|^2\,dx - i\int_\Omega P v_\varepsilon \overline{v}_\varepsilon\,dx \\
&=\; \varepsilon^{-1}\int_\Omega |v_\varepsilon|^2\,dx + i\int_\Gamma \frac{\partial v_\varepsilon}{\partial \nu_\mathcal{A}}\overline{u}\,d\Gamma - i\int_\Omega |\nabla_g v_\varepsilon|_g^2\,dx.
\end{aligned} \tag{5.13}
$$

Compare the imaginary part of the identity (5.13) and multiply by $\varepsilon^{3/2}$ to give

$$
\varepsilon^{3/2}\int_\Omega |\nabla_g v_\varepsilon|_g^2\,dx \leq \frac{\varepsilon^{1/2}}{2}\int_\Gamma |u|^2\,d\Gamma + \frac{\varepsilon^{5/2}}{2}\int_\Gamma \left|\frac{\partial v_\varepsilon}{\partial \nu_\mathcal{A}}\right|^2 d\Gamma. \tag{5.14}
$$

The same treatment to the real part of the identity (5.13) yields

$$\varepsilon^{1/2} \int_\Omega |v_\varepsilon|^2 \, dx \le \frac{\varepsilon^{1/2}}{2} \int_\Gamma |u|^2 \, d\Gamma + \frac{\varepsilon^{5/2}}{2} \int_\Gamma \left| \frac{\partial v_\varepsilon}{\partial \nu_\mathcal{A}} \right|^2 d\Gamma. \tag{5.15}$$

Combining (5.12), (5.14), and (5.15), and making use of (3.8) and (3.9), we obtain

$$\int_\Gamma \left| \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_\mathcal{A}} \right|^2 d\Gamma \quad \le C b \varepsilon^2 \int_\Omega |\nabla_g v_\varepsilon|_g^2 \, dx + b \varepsilon^{1/2} \int_\Gamma |u|^2 \, d\Gamma$$

$$+ b \varepsilon^{1/2} \int_\Gamma \left| \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_\mathcal{A}} \right|^2 d\Gamma \le \left(C b \frac{\varepsilon}{2} + b \varepsilon^{1/2} \right) \int_\Gamma |u|^2 \, d\Gamma \tag{5.16}$$

$$+ \left(\frac{Cb}{2} \varepsilon^{1/2} + b \right) \varepsilon^{1/2} \int_\Gamma \left| \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_\mathcal{A}} \right|^2 d\Gamma.$$

This implies that

$$\lim_{\varepsilon \to 0+} \int_\Gamma \left| \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_\mathcal{A}} \right|^2 d\Gamma = 0.$$

The proof is complete. □

PROPOSITION 5.2. Theorem 1.5 is valid if for any $u \in C_0^\infty(\Gamma)$ and $\varepsilon > 0$, the solution w_ε to the equation

$$\begin{cases} w_\varepsilon(x) + \varepsilon^2 P^2 w_\varepsilon(x) = 0, & x \in \Omega, \\ w_\varepsilon(x) = 0, \; P w_\varepsilon(x) = u(x), & x \in \Gamma \end{cases} \tag{5.17}$$

satisfies

$$\lim_{\varepsilon \to 0+} \int_\Gamma \left| \varepsilon \frac{\partial (P w_\varepsilon)}{\partial \nu_\mathcal{A}} \right|^2 d\Gamma = 0. \tag{5.18}$$

Proof. As in the beginning of proof of Proposition 5.1, (1.7) is equivalent to, in the strong topology of $L^2(\Gamma)$, that

$$\lim_{\lambda \in \mathbb{R}, \lambda \to +\infty} H(\lambda) u = 0, \; \forall \, u \in L^2(\Gamma), \tag{5.19}$$

where $H(\lambda)$ is given by (5.2). Due to (5.4) and a density argument, it suffices to show that (5.19) is satisfied for all $u \in C_0^\infty(\Gamma)$. Now let $u \in C_0^\infty(\Gamma)$. Denote by

$$w_\lambda(x) = ((\lambda^2 + \tilde{A})^{-1} B u)(x), \; \forall \, x \in \Omega.$$

Then $w_\lambda(x)$ satisfies

$$\begin{cases} \lambda^2 w_\lambda(x) + P^2 w_\lambda(x) = 0, & x \in \Omega, \\ w_\lambda(x) = 0, \; P w_\lambda(x) = u(x), & x \in \Gamma, \end{cases} \tag{5.20}$$

and the transfer function $H(\lambda)$ can be written as

$$(H(\lambda)u)(x) = \lambda B^* w_\lambda = \lambda \frac{\partial \mathcal{A}^{-1} w_\lambda}{\partial \nu_\mathcal{A}}(x), \; \forall \, x \in \Gamma. \tag{5.21}$$

Since $u \in C_0^\infty(\Gamma)$, there exists a unique classical solution to (5.20). Let $\tilde{w} \in C^\infty(\Omega)$ be the unique solution to the following boundary value problem

$$\begin{cases} P^2 \tilde{w}(x) = 0, & x \in \Omega, \\ \tilde{w}(x) = 0, \; P \tilde{w}(x) = u(x), & x \in \Gamma. \end{cases}$$

Then (5.20) becomes

$$\begin{cases} \lambda^2 w_\lambda(x) + P^2(w_\lambda(x) - \tilde{w}(x)) = 0, & x \in \Omega, \\ w_\lambda(x) - \tilde{w}(x) = 0, \ P(w_\lambda(x) - \tilde{w}(x)) = 0, & x \in \Gamma, \end{cases}$$

or equivalently,

$$\lambda^2 \mathcal{A}^{-1} w_\lambda = P w_\lambda - P \tilde{w}. \tag{5.22}$$

Substitute (5.22) into (5.21), to obtain

$$(H(\lambda)u)(x) = \frac{1}{\lambda} \frac{\partial(Pw_\lambda)}{\partial \nu_\mathcal{A}}(x) - \frac{1}{\lambda} \frac{\partial(P\tilde{w})}{\partial \nu_\mathcal{A}}(x).$$

Since $P\tilde{w}$ is independent of λ, letting $w_\varepsilon(x) = w_\lambda(x)$ with $\varepsilon = 1/\lambda$, we conclude the required result. $\qquad \square$

Proof of Theorem 1.5. By Proposition 5.2, it suffices to prove (5.18).

We decompose u into the real part and the imaginary part. Let $u = u_1 + iu_2$, where u_1 and u_2 are real functions in $C_0^\infty(\Gamma)$. Then the solution w_ε to (5.17) can be uniquely decomposed as $w_\varepsilon = w_{1\varepsilon} + iw_{2\varepsilon}$ corresponding to the decomposition of u, where the real function $w_{k\varepsilon}$ verifies the equation

$$\begin{cases} w_{k\varepsilon}(x) + \varepsilon^2 P^2 w_{k\varepsilon}(x) = 0, & x \in \Omega, \\ w_{k\varepsilon}(x) = 0, \ Pw_{k\varepsilon}(x) = u_k(x), & x \in \Gamma, \end{cases} \tag{5.23}$$

for $k = 1, 2$. Then (5.18) is equivalent to

$$\lim_{\varepsilon \to 0+} \int_\Gamma \left| \varepsilon \frac{\partial(Pw_{k\varepsilon})}{\partial \nu_\mathcal{A}} \right|^2 d\Gamma = 0, \ \forall \, k = 1, 2.$$

Therefore, it is sufficient to show (5.18) for real functions $u \in C_0^\infty(\Gamma)$. Now assume that $u \in C_0^\infty(\Gamma)$ is a real function. Denote by

$$\varphi_\varepsilon(x) = \varepsilon^{-1} w_\varepsilon(x) + iPw_\varepsilon(x), \ \forall \, x \in \Omega. \tag{5.24}$$

Then φ_ε satisfies the equation

$$\begin{cases} \varepsilon^{-1}\varphi_\varepsilon(x) - iP\varphi_\varepsilon(x) = 0, & x \in \Omega, \\ \varphi_\varepsilon(x) = -iu(x), & x \in \Gamma. \end{cases} \tag{5.25}$$

While (5.25) is the same type with that of (5.5). It follows from (5.6) and (5.24) that

$$0 = \lim_{\varepsilon \to 0+} \int_\Gamma \left| \varepsilon \frac{\partial \varphi_\varepsilon}{\partial \nu_\mathcal{A}} \right|^2 d\Gamma = \lim_{\varepsilon \to 0+} \int_\Gamma \left| \frac{\partial w_\varepsilon}{\partial \nu_\mathcal{A}} \right|^2 d\Gamma + \lim_{\varepsilon \to 0+} \int_\Gamma \left| \varepsilon \frac{\partial(Pw_\varepsilon)}{\partial \nu_\mathcal{A}} \right|^2 d\Gamma.$$

Therefore, (5.18) is valid. The proof is complete. $\qquad \square$

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