NON-EQUILIBRIUM RELATIVISTIC TWO-PHASE FLOW WITH HEAT EXCHANGE

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Abstract. We consider two different systems of hyperbolic balance laws governing relativistic two-phase flows with general equations of state. The phases are assumed to be immiscible. In the first one the thermal equilibrium is attained through a relaxation procedure consistent with the second law of thermodynamics, while the second is a fully relaxed model in which the two phases reach an equal temperature instantaneously.

Finally, we deduce and compare the wave velocities for both the models: the hydrodynamical velocity of the relaxed system can never exceed that of the relaxation system.

1. Introduction. The development of models for compressible two-phase flows in both the classical and relativistic frameworks is really interesting in many scientific and engineering disciplines.

In the last few decades, several models have been developed both in classical [1, 2, 3, 10, 13, 23, 28, 29, 30, 31, 32, 33, 34, 35, 36, 40, 42, 43] and in relativistic [5, 6, 7, 8, 9, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 37, 38, 39, 41, 44] formulation, but none of these models has been universally accepted as a complete formulation for modelling two-phase flows.

Both the classical and the relativistic mathematical modelling have certain inherent difficulties that originate from the existence of interfaces among the phases or the fluids. These interfaces separate not only pure media, but also mixtures of materials in which wave dynamics is important. Such situations appear frequently in astrophysics, the physics of explosives, nuclear physics and many other applications.

The explained difficulties are concerned with the interaction between the fluids, which includes the transfer of mass, momentum and energy across the interfaces. The way in which such interfaces are treated is the key point of each model. To circumvent these difficulties, two classes of methods have been developed: the first class represents the models considering the interface as a sharp discontinuity; the second class corresponds to the models describing the interface as a diffused zone, such as the contact discontinuities in gas dynamics. The determination of thermodynamic flow variables in these zones is achieved on the basis of multiphase flow theory. The challenge is to derive physically and mathematically consistent thermodynamic laws for the mixture.

Following the ideas of Flåtten et al. [13], in the present paper we develop a new model for a relativistic two-phase diffusive flow with a single four-velocity and a single pressure, in which the two phases may coexist without being in thermal equilibrium. The mixture can be thought of as a continuum medium. By allowing the heat-transfer between the two phases, the mixture is driven towards the thermal equilibrium. This procedure is also referred to as a *relaxation procedure*, and such a system is also referred to as a *relaxation system*. As described in the next sections, the relaxation procedure adopted here is consistent with both the first and the second laws of thermodynamics.

Moreover, we consider the *relaxed limit* for the flow under investigation, in which the thermal equilibrium is supposed to be instantaneously reached. A very well-known relativistic flow model in the fully relaxed limit is the *homogeneous equilibrium relativistic model*, described and analysed for example in [15, 16].

In the model presented here, each fluid component still has its own particle number density, r_k, its specific internal energy, ε_k, and its energy density, ρ_k, [4, 27]

$$\rho_k = r_k(1 + \varepsilon_k) , \qquad k = 1, 2 . \tag{1.1}$$

In what follows, the units are such that the velocity of light is unitary: $c = 1$. Conversely, as mentioned above, we assume for the two phases of the fluid a single pressure, p, and a single four-velocity, u^α, which is supposed to be future pointing:

$$g_{\alpha\beta}u^\alpha u^\beta = 1 , \tag{1.2}$$

where $g_{\alpha\beta}$ are the covariant components of the Lorentz metric tensor with signature $+, -, -, -$.

The paper is organized as follows. In Sect. 2 we describe the two-phase relativistic flow and we derive the flow equations. In Sect. 3 we introduce the relaxation procedure involving individual temperatures, one for each phase of the mixture, and we derive the necessary and sufficient restrictions on the relaxation terms imposed by the first and second laws of thermodynamics. Moreover, we deduce the expressions for the source terms appearing in the flow equations. In order to obtain the closure of the relaxation system, in Sect. 4 we derive the evolution equation for both the pressure of the relativistic two-phase flow and for the volume fraction of a single phase of the mixture. In Sect. 5 we study the propagation of the weak discontinuity waves admitted by the relaxation system, obtaining the expression for the hydrodynamical mixture velocity. In Sect. 6 we analyse the fully relaxed system and we study the propagation of the weak discontinuity

waves admitted by such a system. Finally, the wave velocities obtained both for the relaxation and the relaxed systems are compared.

2. Relativistic two-phase flow model. The relativistic two-phase flow model analysed in the present paper takes into account the balance laws for the particle number density of each phase:

$$\nabla_\alpha(X_k r_k u^\alpha) = 0 , \qquad k = 1, 2 , \tag{2.1}$$

and of the total energy-momentum conservation:

$$\nabla_\alpha T^{\alpha\beta} = 0 , \tag{2.2}$$

where the stress energy tensor, $T^{\alpha\beta}$, is given by

$$T^{\alpha\beta} = rf u^\alpha u^\beta - p g^{\alpha\beta} . \tag{2.3}$$

Here, r is the total particle number density, X_1 and X_2 are the volume fractions of the two phases of the mixture, such that $X_1 + X_2 = 1$, f is the total relativistic specific enthalpy

$$f = 1 + h = 1 + \varepsilon + \frac{p}{r} , \tag{2.4}$$

where $h = \varepsilon + p/r$ and ε represent, respectively, the "classical" specific enthalpy and the total energy density of the flow.

In order to determine suitable expressions for the bulk quantities r, ε and f, we introduce the mass fraction Y_k of the k-th phase, given by

$$Y_k = \frac{X_k r_k}{r} , \qquad k = 1, 2 . \tag{2.5}$$

Taking into account the balance laws (2.1), the mass fractions of both the phases satisfy the relation $Y_1 + Y_2 = 1$.

We define the total particle number density r and the total specific internal energy ε in terms of the mass and the volume fractions:

$$r = X_1 r_1 + X_2 r_2 = X r_1 + (1 - X) r_2 , \tag{2.6}$$

$$\varepsilon = Y_1 \varepsilon_1 + Y_2 \varepsilon_2 = Y \varepsilon_1 + (1 - Y) \varepsilon_2 , \tag{2.7}$$

where we set

$$X_1 = X, \qquad X_2 = 1 - X , \tag{2.8}$$

$$Y_1 = Y, \qquad Y_2 = 1 - Y . \tag{2.9}$$

Thus, the bulk quantities $\rho = r(1 + \varepsilon)$, i.e. the total energy density, f and rf can be easily expressed in terms of X_k and Y_k:

$$\rho = X_1 \rho_1 + X_2 \rho_2 , \tag{2.10}$$

$$f = Y_1 f_1 + Y_2 f_2 , \tag{2.11}$$

$$rf = X_1 r_1 f_1 + X_2 r_2 f_2 , \tag{2.12}$$

where $f_k = 1 + \varepsilon_k + p/r_k$, for $k = 1, 2$.

The balance law for the total particle number density is given by

$$\nabla_\alpha(r u^\alpha) = 0 , \tag{2.13}$$

where equations (2.1) and (2.6) are taken into account. Using (2.1), (2.5) and (2.13), we obtain the evolution equations for the mass fractions Y_k,

$$u^\alpha \partial_\alpha Y_k = 0 \; , \qquad k = 1, 2 \; . \tag{2.14}$$

By virtue of the expression (2.12), we can rewrite the total energy-momentum tensor (2.3) as

$$T^{\alpha\beta} = X_1 \left(r_1 f_1 u^\alpha u^\beta - p g^{\alpha\beta} \right) + X_2 \left(r_2 f_2 u^\alpha u^\beta - p g^{\alpha\beta} \right) \; , \tag{2.15}$$

and, thus, the evolution equation for the energy-momentum tensor of the k-th phase of the flow is

$$\nabla_\alpha \left[X_k \left(r_k f_k u^\alpha u^\beta - p g^{\alpha\beta} \right) \right] = F_k^\beta \; , \tag{2.16}$$

where the F_k^β represent the source term in the balance equation of each phase. According to the total energy-momentum conservation equation (2.2), the source terms F_k^β are such that

$$F_1^\beta = -F_2^\beta = F^\beta \; . \tag{2.17}$$

The projection along u^α, also referred to as the "energy evolution equation", and the spatial projection of the equation (2.16) for the fluid k are, respectively,

$$X_k r_k u^\alpha \left(\partial_\alpha \varepsilon_k + p \partial_\alpha \frac{1}{r_k} \right) - p u^\alpha \partial_\alpha X_k = u_\alpha F_k^\alpha \; , \tag{2.18}$$

$$X_k \left(r_k f_k u^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta} \partial_\alpha p \right) - p \gamma^{\alpha\beta} \partial_\alpha X_k = \gamma_\alpha^\beta F_k^\alpha \; , \tag{2.19}$$

where $\gamma^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ is the projection tensor onto the three-dimensional space orthogonal to u^α (i.e. the rest space of an observer moving with the four-velocity u^α).

3. The relaxation procedure and the derivation of the source terms. The aim of this section is the handling of the source terms F^β, given by (2.17), appearing in the balance equations (2.16), and (2.18), applying relaxation heat-transfer terms between the phases of the mixture.

We assume that the entropy S_k of each phase of the fluid is a function of both the internal energy ε_k and the specific volume $1/r_k$,

$$S_k = S_k \left(\varepsilon_k, r_k \right) \; , \qquad k = 1, 2 \; . \tag{3.1}$$

By thermodynamic argument, it follows that the derivatives of the entropy can be related to some observables

$$\left(\frac{\partial S_k}{\partial \varepsilon_k} \right)_{r_k} = \frac{1}{T_k} \; , \qquad \left(\frac{\partial S_k}{\partial r_k} \right)_{\varepsilon_k} = -\frac{p}{r_k^2 T_k} \; , \qquad k = 1, 2 \; , \tag{3.2}$$

where T_k is the temperature of the phase k. From the above equations (3.2), we derive the following thermodynamic differential relations

$$T_k dS_k = d\varepsilon_k + p d\frac{1}{r_k} \; , \qquad k = 1, 2 \; . \tag{3.3}$$

Thus, the following differential relations hold for the covariant derivative of S_k:

$$T_k u^\alpha \partial_\alpha S_k = u^\alpha \left(\partial_\alpha \varepsilon_k + p \partial_\alpha \frac{1}{r_k} \right) \; , \qquad k = 1, 2 \; . \tag{3.4}$$

We consider a relaxation procedure in which 1) the heat should be transferred between the phases with a rate proportional to the difference of their temperatures, and 2) the variation of the entropy of each phase should be due only to the heat-transfer terms. Mathematically,

$$X_1 r_1 T_1 u^\alpha \partial_\alpha S_1 = H_{12}(T_2 - T_1) \ , \tag{3.5}$$

$$X_2 r_2 T_2 u^\alpha \partial_\alpha S_2 = H_{21}(T_1 - T_2) \ , \tag{3.6}$$

where the relaxation coefficients, H_{12} and H_{21}, are further assumed to be independent from the temperatures T_1 and T_2.

In order to develop a model consistent with the first and the second laws of thermodynamics, we compute that the relaxation coefficients have to satisfy the constraint

$$H_{12} = H_{21} = H \geq 0 \ . \tag{3.7}$$

In order to demonstrate the condition (3.7), we first observe that the total specific entropy of the mixture S can be written in terms of the mass fractions of the two fluid phases,

$$S = Y_1 S_1 + Y_2 S_2 \ , \tag{3.8}$$

and then, taking into account equations (2.14), (3.5) and (3.6), the evolution equation for the total entropy S is given by

$$r u^\alpha \partial_\alpha S = H_{12} \frac{T_2 - T_1}{T_1} + H_{21} \frac{T_1 - T_2}{T_2} \ . \tag{3.9}$$

Equation (3.9) can be rewritten in a more convenient form,

$$\nabla_\alpha \left(r S u^\alpha \right) = H_{12} \frac{(T_2 - T_1)^2}{T_1 T_2} + (H_{12} - H_{21}) \frac{T_2 - T_1}{T_2} \ , \tag{3.10}$$

where the balance law for the total particle number (2.13) is used. Thus, (3.10) remains unconditionally non-negative if and only if the constraint (3.7) holds.

Introducing equations (3.4), (3.5) and (3.6) into the energy evolution equation (2.18), we obtain, under condition (3.7), that

$$p u^\alpha \partial_\alpha X + u^\alpha F_\alpha = H(T_2 - T_1) \ , \tag{3.11}$$

and thus, the source term F_α is equal to

$$F_\alpha = -p \partial_\alpha X + H(T_2 - T_1) u^\alpha \ . \tag{3.12}$$

4. Evolution equations for total pressure and volume fraction. In this section we derive the evolution equation for the pressure of the mixture, and then the evolution equation for the volume fraction X.

Assuming that the mixture is isobaric, e.g.

$$p = p_k(r_k, S_k) = p_k \left[r_k, \varepsilon_k(r_k, S_k) \right] \ , \quad k = 1, 2, \tag{4.1}$$

where p_k is the pressure of each phase, we write the following differential formula:

$$dp = dp_k = f_k \lambda_k^2 dr_k + \Gamma_k r_k T_k dS_k \ , \tag{4.2}$$

where λ_k represents the velocity of the single-phase hydrodynamical wave:

$$\lambda_k^2 = \left(\frac{\partial p}{\partial \rho_k}\right)_{S_k} = \frac{1}{f_k}\left(\frac{\partial p}{\partial r_k}\right)_{S_k} , \qquad k = 1, 2 , \tag{4.3}$$

and Γ_k is the Grüneisen coefficient of the k-th phase

$$\Gamma_k = \frac{1}{r_k}\left(\frac{\partial p}{\partial \varepsilon_k}\right)_{r_k} , \qquad k = 1, 2 . \tag{4.4}$$

From (4.2), by using (3.5) and (3.6), we obtain the following equations:

$$u^\alpha \partial_\alpha p = f_1 \lambda_1^2 u^\alpha \partial_\alpha r_1 + \frac{1}{X_1}\Gamma_1 H(T_2 - T_1) , \tag{4.5}$$

$$u^\alpha \partial_\alpha p = f_2 \lambda_2^2 u^\alpha \partial_\alpha r_2 + \frac{1}{X_2}\Gamma_2 H(T_1 - T_2) , \tag{4.6}$$

which using (2.1), can be rewritten as

$$\frac{X_1}{r_1 f_1 \lambda_1^2} u^\alpha \partial_\alpha p + \nabla_\alpha(X_1 u^\alpha) = \frac{\Gamma_1}{r_1 f_1 \lambda_1^2} H(T_2 - T_1) , \tag{4.7}$$

$$\frac{X_2}{r_2 f_2 \lambda_2^2} u^\alpha \partial_\alpha p + \nabla_\alpha(X_2 u^\alpha) = \frac{\Gamma_2}{r_2 f_2 \lambda_2^2} H(T_1 - T_2) . \tag{4.8}$$

From (4.7)-(4.8) we deduce the evolution equation for the pressure p,

$$u^\alpha \partial_\alpha p + \frac{1}{\omega}\nabla_\alpha u^\alpha = \frac{\xi}{\omega} H(T_2 - T_1) , \tag{4.9}$$

where

$$\omega = \frac{X_1}{r_1 f_1 \lambda_1^2} + \frac{X_2}{r_2 f_2 \lambda_2^2} , \tag{4.10}$$

$$\xi = \frac{\Gamma_1}{r_1 f_1 \lambda_1^2} - \frac{\Gamma_2}{r_2 f_2 \lambda_2^2} . \tag{4.11}$$

The evolution equation for the volume fraction $X_1 = X$ is easily obtained by introducing equation (4.9) into (4.7),

$$u^\alpha \partial_\alpha X + X(1 - X)\zeta \nabla_\alpha u^\alpha = \eta H(T_2 - T_1) , \tag{4.12}$$

where

$$\zeta = \frac{r_1 f_1 \lambda_1^2 - r_2 f_2 \lambda_2^2}{X_2 r_1 f_1 \lambda_1^2 + X_1 r_2 f_2 \lambda_2^2} , \tag{4.13}$$

$$\eta = \frac{X_2 \Gamma_1 + X_1 \Gamma_2}{X_2 r_1 f_1 \lambda_1^2 + X_1 r_2 f_2 \lambda_2^2} . \tag{4.14}$$

Therefore, the complete relaxation system describing the evolution of the two-phase relativistic flow under investigation is given by the following system of eight differential

equations in the eight independent field variables $u^\alpha, r_1, r_2, S_1, S_2, X$:

$$
\begin{cases}
rfu^\alpha\nabla_\alpha u^\beta - \gamma^{\alpha\beta}\partial_\alpha p = 0, \\[2mm]
\nabla_\alpha\left[Xr_1u^\alpha\right] = 0, \\[2mm]
\nabla_\alpha\left[(1-X)r_2u^\alpha\right] = 0, \\[2mm]
u^\alpha\partial_\alpha S_1 = \dfrac{1}{Xr_1T_1}H(T_2 - T_1), \\[2mm]
u^\alpha\partial_\alpha S_2 = \dfrac{1}{(1-X)r_2T_2}H(T_1 - T_2), \\[2mm]
u^\alpha\partial_\alpha X + X(1-X)\zeta\nabla_\alpha u^\alpha = \eta H(T_2 - T_1),
\end{cases}
\tag{4.15}
$$

where ω, ξ, ζ and η are given by (4.10), (4.11), (4.13) and (4.14). In addition to such a system (4.15), the evolution equation (4.9) for the pressure p of the mixture, $p = p_1(r_1, S_1) = p_2(r_2, S_2)$, holds.

5. Weak discontinuity wave propagation for the relaxation system. In a domain $\mathcal{D}$ of the space-time V_4, we consider a regular hypersurface, Σ, not generated by the flow lines, with local equation $\varphi\left(x^\alpha\right) = 0$. As will be clear below, the hypersurface Σ is a space-like one, i.e. $L^\alpha L_\alpha < 0$, where $L_\alpha = \partial_\alpha\varphi$. In what follows, we denote by N_α the normalized vector

$$
N_\alpha = \frac{L_\alpha}{\sqrt{-L^\beta L_\beta}}, \qquad N_\alpha N^\alpha = -1.
\tag{5.1}
$$

We consider a particular class of solutions of system (4.15), namely, weak discontinuity waves Σ, on which the independent field variables $u^\alpha, r_1, r_2, S_1, S_2$ and X are continuous, but jump discontinuities may occur in their normal derivatives. In this case, if Q denotes any of the fields, then there exists [4, 27] the distribution δQ, with support Σ, such that

$$
\overline{\delta}\left[\nabla_\alpha Q\right] = N_\alpha \delta Q,
\tag{5.2}
$$

where $\overline{\delta}$ is the Dirac measure defined by φ with Σ as support and square brackets denote the discontinuity, δ being an operator of infinitesimal discontinuity; δ behaves like a derivative insofar as algebraic manipulations are concerned.

From system (4.15), we obtain the following linear homogeneous system in the distributions $\delta u^\alpha, \delta r_1, \delta r_2, \delta S_1, \delta S_2$ and δX:

$$\begin{cases} rfL\delta u^\alpha - \gamma^{\alpha\beta} N_\beta \delta p = 0, \\[2mm] X r_1 N_\alpha \delta u^\alpha + XL\delta r_1 + r_1 L \delta X = 0, \\[2mm] (1-X) r_2 N_\alpha \delta u^\alpha + (1-X)L\delta r_2 - r_2 L \delta X = 0, \\[2mm] L\delta S_1 = 0, \\[2mm] L\delta S_2 = 0, \\[2mm] L\delta X + X(1-X)\zeta N_\alpha \delta u^\alpha = 0, \end{cases} \tag{5.3}$$

where $L = u^\alpha N_\alpha$, and from equation (4.9) we obtain the following equation for the distribution δp:

$$\omega L\delta p + N_\alpha \delta u^\alpha = 0 . \tag{5.4}$$

Due to the unitary character of u^α, we also have that

$$u_\alpha \delta u^\alpha = 0 . \tag{5.5}$$

In what follows, we investigate the normal speeds of propagation of the various waves with respect to an observer moving with the flow velocity u^α. The normal speed λ_Σ of propagation of the wave front Σ, described by a time-like word line having tangent vector field u^α, that is, the velocity in the time direction u^α, is given by [4, 27]

$$\lambda_\Sigma^2 = \frac{L^2}{1+L^2} . \tag{5.6}$$

The local causality condition, i.e. the requirement that the characteristic hypersurface Σ has to be time-like or null (or equivalently that the normal N_α be space-like or null, $g^{\alpha\beta} N_\alpha N_\beta \leq 0$) is equivalent to the condition $0 \leq \lambda_\Sigma^2 \leq 1$.

From the above system (5.3), we first obtain the solution in the case $L = 0$, which represents a wave moving with the mixture. The corresponding discontinuities are characterized by the conditions

$$\delta p = 0 , \qquad N_\alpha \delta u^\alpha = 0 ; \tag{5.7}$$

this implies that system (5.3) admits six independent eigenvectors corresponding to $L = 0$ in the space of the field variables.

From now on, we suppose $L \neq 0$. Thus, by $(5.3)_{4,5}$, we obtain that $\delta S_1 = \delta S_2 = 0$. Equation $(5.3)_1$ multiplied by N_α gives us

$$rfL N_\alpha \delta u^\alpha + l^2 \delta p = 0 , \tag{5.8}$$

where $l^2 = 1 + L^2$. As a consequence, equations $(5.3)_2$ and (5.8) represent a linear homogeneous system, in the two scalar distributions $N_\alpha \delta u^\alpha$ and δp, which may have non-trivial solutions only if the determinant of the coefficient vanishes, i.e.

$$\mathcal{H} \equiv \omega rfL^2 - l^2 = 0 , \tag{5.9}$$

which corresponds to the hydrodynamical waves propagating in such a two-fluid relaxation system. Their speed of propagation is given by

$$\lambda_\Sigma^2 = \frac{1}{\omega r f} = \frac{1}{r f \left\{ \dfrac{X_1}{r_1 f_1 \lambda_1^2} + \dfrac{X_2}{r_2 f_2 \lambda_2^2} \right\}} \ , \tag{5.10}$$

and the local causality condition, which ensures their spatial orientation, is given by

$$0 < \frac{1}{\omega r f} \leq 1 \ . \tag{5.11}$$

Moreover, from (5.10), the following relation holds:

$$\frac{1}{r f \lambda_\Sigma^2} = \frac{X_1}{r_1 f_1 \lambda_1^2} + \frac{X_2}{r_2 f_2 \lambda_2^2} \ . \tag{5.12}$$

The associated discontinuities can be written in terms of $\psi = n_\alpha \delta u^\alpha$:

$$\begin{cases} \delta u^\alpha = -\psi n^\alpha \ , \\[2mm] \delta r_1 = -r_1 \dfrac{l}{L} \left[1 - \zeta(1 - X) \right] \psi \ , \qquad \delta r_2 = -r_2 \dfrac{l}{L} \left[1 + X\zeta \right] \psi \ , \\[2mm] \delta S_1 = 0 \ , \qquad \delta S_2 = 0 \ , \\[2mm] \delta X = -\dfrac{-(1 - X)X\zeta}{L} l\psi, \qquad \delta p = -\dfrac{l}{\omega L}\psi, \end{cases} \tag{5.13}$$

where n^α is the unitary space-like four-vector defined by

$$n_\alpha = \frac{1}{l} \left(N_\alpha - L u_\alpha \right) \ . \tag{5.14}$$

If the above condition (5.11), characterizing the space-like orientations of the surfaces, is verified, the differential equations (4.15), governing the motion of the two-fluid relaxation system, represent a (not strictly) hyperbolic system. In fact, all velocities (eigenvalues) are real, and there exists a complete set of eigenvectors in the space of field variables, i.e. eight independent eigenvectors (six from $L = 0$ and two from $\mathcal{H}_1 = 0$), for the eight independent field variables u^α, r_1, r_2, S_1, S_2 and X.

6. Relaxed system. We now consider the system obtained by letting the relaxation coefficient H tend to infinity; i.e., we achieve instantaneous thermal equilibrium among the two phases of the mixture. Assuming the same temperature for the two phases, $T_1 = T_2$, the entropy of the mixture, given by equation (3.8), satisfies the characteristic equation

$$u^\alpha \partial_\alpha S = 0, \tag{6.1}$$

where relations (2.13) and (3.10) have been taken into account.

The relaxed system may be expressed in a convenient form as

$$
\begin{cases}
r f u^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta} \partial_\alpha p = 0, \\[2mm]
\nabla_\alpha (r u^\alpha) = 0, \\[2mm]
u^\alpha \partial_\alpha S = 0, \\[2mm]
u^\alpha \partial_\alpha Y = 0.
\end{cases}
\tag{6.2}
$$

Moreover, by virtue of equations (2.5), (2.6), (2.7), (3.3) and (3.8), and setting a common value for the temperatures of the two phases, $T = T_1 = T_2$, we can write the following generalized Gibbs equation:

$$
d\varepsilon = T dS - p d\frac{1}{r} - \left\{ \left(\varepsilon_2 + \frac{p}{r_2} - T S_2 \right) - \left(\varepsilon_1 + \frac{p}{r_1} - T S_1 \right) \right\} dY ,
\tag{6.3}
$$

and, thus, we can suppose that the equation of state for the pressure of the mixture is given by $p = p(r, S, Y)$. Thus, (6.2) is a system of six differential equations in the six independent field variables u^α, r, S and Y.

In order to study a weak discontinuity waves propagation compatible with the relaxed system (6.2), we apply the equation (5.2) to the system (6.2). Thus we obtain the following linear homogeneous system in the distributions δu^α, δr, δS and δY:

$$
\begin{cases}
r f L \delta u^\beta - \gamma^{\alpha\beta} N_\alpha \left\{ \left(\dfrac{\partial p}{\partial r} \right)_{S,Y} \delta r + \left(\dfrac{\partial p}{\partial S} \right)_{r,Y} \delta S + \left(\dfrac{\partial p}{\partial Y} \right)_{r,S} \delta Y \right\} = 0, \\[3mm]
L \delta r + r N_\alpha \delta u^\alpha = 0, \\[2mm]
L \delta S = 0, \\[2mm]
L \delta Y = 0.
\end{cases}
\tag{6.4}
$$

From the above equations (6.4), if $L = 0$, we have $N_\alpha \delta u^\alpha = \delta p = 0$ and thus system (6.4) admits four independent eigenvectors in the space of the field variables.

In the case $L \neq 0$, equations (6.4)$_{3,4}$ give $\delta S = \delta Y = 0$. By composing the equation (6.4)$_1$ with N_β we obtain

$$
r f L N_\alpha \delta u^\alpha + l^2 \left(\frac{\partial p}{\partial r} \right)_{S,Y} \delta r = 0 ,
\tag{6.5}
$$

which, together with the equation (6.4)$_2$, represents a linear homogeneous system in the two scalar distributions $N_\alpha \delta u^\alpha$ and δr. This system has non-trivial solutions iff

$$
\tilde{\mathcal{H}} = f L^2 - l^2 \left(\frac{\partial p}{\partial r} \right)_{S,Y} = 0 ,
\tag{6.6}
$$

which corresponds to the hydrodynamical waves. Their speeds of propagation are given by

$$
\tilde{\lambda}_\Sigma^2 = \frac{1}{f} \left(\frac{\partial p}{\partial r} \right)_{S,Y} .
\tag{6.7}
$$

Since the pressure of such a system is given by

$$p(r, S, Y) = p\left[\rho(r, S, Y), S, Y\right] \; , \tag{6.8}$$

and taking into account that

$$\left(\frac{\partial \rho}{\partial r}\right)_{S,Y} = f \; , \tag{6.9}$$

we obtain

$$\left(\frac{\partial p}{\partial r}\right)_{S,Y} = \left(\frac{\partial p}{\partial \rho}\right)_{S,Y} \left(\frac{\partial \rho}{\partial r}\right)_{S,Y} = f \left(\frac{\partial p}{\partial \rho}\right)_{S,Y} \; . \tag{6.10}$$

Thus the speeds (6.7) can be rewritten as

$$\tilde{\lambda}_\Sigma^2 = \left(\frac{\partial p}{\partial \rho}\right)_{S,Y} \; . \tag{6.11}$$

As we shall demonstrate in the following, the mixture hydrodynamical velocity of the relaxed system is always less than the corresponding velocity of the relaxation system, e.g.

$$0 < \tilde{\lambda}_\Sigma^2 \leq \lambda_\Sigma^2 \leq 1 \; , \tag{6.12}$$

and by the local causality condition (5.11), we can conclude that the wave surfaces for the relaxed system are time-like.

The associated discontinuities to the distribution in the system (6.4), written in terms of $\tilde{\psi} = n_\alpha \delta u^\alpha$, are

$$\begin{cases} \delta u^\alpha = -\tilde{\psi} n^\alpha \; , \\[2mm] \delta r = -r \dfrac{l}{L} \tilde{\psi} \; , \\[2mm] \delta S = 0 \; , \\[2mm] \delta Y = 0 \; , \end{cases} \tag{6.13}$$

where n^α is the unitary space-like four-vector given by (5.14).

Since the condition characterizing the space-like orientations of the surface is verified, the governing equations represent a (not strictly) hyperbolic system. All velocities (eigenvalues) are real, and there exists a complete set of eigenvectors in the space of field variables, i.e. six independent eigenvectors (four from $L = 0$ and two from $\tilde{\mathcal{H}} = 0$), for the six independent field variables u^α, r, S, Y.

In order to prove inequality (6.12), we rewrite any thermodynamic relation in terms of the parameters ζ_k defined as

$$\zeta_k = \left(\frac{\partial T}{\partial p}\right)_{S_k} = -\frac{1}{r_k^2}\left(\frac{\partial r_k}{\partial S_k}\right)_p \; , \qquad k = 1, 2 \; . \tag{6.14}$$

Thus we obtain that

$$\left(\frac{\partial S_k}{\partial p}\right)_T = \frac{1}{r_k^2}\left(\frac{\partial r_k}{\partial T}\right)_p = -\frac{\zeta_k \hat{C}_k}{T} \; , \qquad k = 1, 2 \; , \tag{6.15}$$

where the specific heat capacity $\hat{C}_k$ of each phase of the mixture is given by

$$\hat{C}_k = T\left(\frac{\partial S_k}{\partial T}\right)_p , \qquad k = 1, 2 . \tag{6.16}$$

Following the idea of Flåtten, Morin and Munkejord [13], we consider the differential

$$D = \frac{1}{r_1}d(X_1 r_1) + \frac{1}{r_2}d(X_2 r_2) = \frac{X_1}{r_1}dr_1 + \frac{X_2}{r_2}dr_2 , \tag{6.17}$$

which in a more explicit form is

$$D = \frac{X_1}{r_1}\left[\left(\frac{\partial r_1}{\partial p}\right)_{S_1}dp + \left(\frac{\partial r_1}{\partial S_1}\right)_p dS_1\right] + \frac{X_2}{r_2}\left[\left(\frac{\partial r_2}{\partial p}\right)_{S_2}dp + \left(\frac{\partial r_2}{\partial S_2}\right)_p dS_2\right] , \tag{6.18}$$

and, by virtue of (4.3), (5.12) and (6.14), it can be written as

$$D = \frac{1}{rf\lambda_\Sigma^2}dp - (X_1 r_1 \zeta_1 dS_1 + X_2 r_2 \zeta_2 dS_2) . \tag{6.19}$$

Taking into account that the entropy of each phase of the mixture is a function $S_k = S_k(p, T)$, and using the thermodynamic relations (6.15) and (6.16), the differential (6.19) can also be written as

$$D = \left[\frac{1}{rf\lambda_\Sigma^2} + \frac{\zeta_1^2 C_1 + \zeta_2^2 C_2}{T}\right]dp - \frac{\zeta_1 C_1 + \zeta_2 C_2}{T}dT , \tag{6.20}$$

where the extensive heat capacity C_k of each phase of the mixture is given by

$$C_k = X_k r_k \hat{C}_k = rY_k \hat{C}_k , \qquad k = 1, 2 . \tag{6.21}$$

Furthermore, using (6.15) and (6.16), the differential of the total entropy S of the mixture, defined in (3.8), is given by

$$dS = Y_1 dS_1 + Y_2 dS_2 + O(dY)$$
$$= \frac{Y_1 \hat{C}_1 + Y_2 \hat{C}_2}{T}dT - \frac{Y_1 \zeta_1 \hat{C}_1 + Y_2 \zeta_2 \hat{C}_2}{T}dp + O(dY) . \tag{6.22}$$

Introducing the relation (6.22) into the expression of the differential (6.20) we obtain

$$D = \left[\frac{1}{rf\lambda_\Sigma^2} + \frac{C_1 C_2(\zeta_1 - \zeta_2)^2}{T(C_1 + C_2)}\right]dp - r\frac{\zeta_1 C_1 + \zeta_2 C_2}{C_1 + C_2}dS + O(dY) . \tag{6.23}$$

Moreover, by virtue of relation (2.5) between the mass and the volume fractions, the differential D, defined in (6.17), can also be written as

$$D = \frac{1}{r_1}d(X_1 r_1) + \frac{1}{r_2}(X_2 r_2) = \frac{1}{r_1}d(Y_1 r_1) + \frac{1}{r_2}d(Y_2 r)$$
$$= \left(\frac{Y_1}{r_1} + \frac{Y_2}{r_2}\right)dr + O(dY)$$
$$= \frac{1}{r}\left\{\left(\frac{\partial r}{\partial p}\right)_{S,Y}dp + \left(\frac{\partial r}{\partial S}\right)_{p,Y}dS\right\} + O(dY)$$
$$= \frac{1}{rf\tilde{\lambda}_\Sigma^2}dp + \frac{1}{r}\left(\frac{\partial r}{\partial S}\right)_{p,Y}dS + O(dY) . \tag{6.24}$$

By comparing relations (6.23) and (6.24), the mixture hydrodynamical velocity of the relaxed system, $\tilde{\lambda}_\Sigma^2$, can be written in terms of the mixture hydrodynamical velocity of the relaxation system, λ_Σ^2, as

$$\frac{1}{\tilde{\lambda}_\Sigma^2} - \frac{1}{\lambda_\Sigma^2} = \frac{rf}{T(C_1 + C_2)} C_1 C_2 (\zeta_1 - \zeta_2)^2 \ . \tag{6.25}$$

It is easily observed that the previous difference is non-negative only if the extensive heat capacities of the two components of the mixture are positive:

$$C_1 > 0, \qquad C_2 > 0 \ , \tag{6.26}$$

and the equality holds iff ζ_1 and ζ_2 are equal.

Thus, under the condition (6.26), the following inequality holds:

$$0 < \tilde{\lambda}_\Sigma^2 \le \lambda_\Sigma^2 \le 1 \ ; \tag{6.27}$$

i.e. the mixture hydrodynamical velocity of the relaxed system, $\tilde{\lambda}_\Sigma^2$, is space-like, since the mixture hydrodynamical velocity of the relaxation system, λ_Σ^2, satisfies the condition (5.11), which ensures its spatial orientation.

In conclusion, we can say that the absolute value of the velocities of the hydrodynamical waves of the relaxed system can never exceed those of the relaxation system.

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