QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXIII, NUMBER 3
SEPTEMBER 2015, PAGES 467–482
http://dx.doi.org/10.1090/qam/1380
Article electronically published on June 18, 2015

KURAMOTO OSCILLATORS WITH INERTIA: A FAST-SLOW DYNAMICAL SYSTEMS APPROACH

By

YOUNG-PIL CHOI (*Department of Mathematics, Imperial College London, London, SW7 2AZ, United Kingdom*),

SEUNG-YEAL HA (*Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 151-747, Republic of Korea*),

SUNGEUN JUNG (*Department of Mathematical Sciences, Seoul National University, Seoul 151-747, Republic of Korea*),

AND

MARSHALL SLEMROD (*Department of Mathematics, University of Wisconsin-Madison – and – Department of Mathematics, Weizmann Institute of Science, Rehobot 76100, Israel*)

Abstract. We present a fast-slow dynamical systems theory for a Kuramoto type model with inertia. The fast part of the system consists of N-decoupled pendulum equations with constant friction and torque as the phase of individual oscillators, whereas the slow part governs the evolution of order parameters that represent the amplitude and phase of the centroid of the oscillators. In our new formulation, order parameters serve as orthogonal observables in the framework of Artstein-Kevrekidis-Slemrod-Titi's unified theory of singular perturbation. We show that Kuramoto's order parameters become stationary regardless of the coupling strength. This generalizes an earlier result (Ha and Slemrod (2011)) for Kuramoto oscillators without inertia.

1. Introduction. The purpose of this paper is to extend the study of fast-slow dynamical systems theory for synchronization models, which started with [15, 16]. The Kuramoto model describes the synchronization process for many weakly coupled limit

Received August 19, 2013.
2010 *Mathematics Subject Classification.* Primary 91G80, 97M30.
Key words and phrases. Fast-slow dynamical system, Kuramoto oscillators, mean-field limit, order parameter, synchronization.
The work of the second author was partially supported by KRF-2011-0015388, and the work of the fourth author was supported by KMRS-KAIST.
E-mail address: young-pil.choi@imperial.ac.uk
E-mail address: syha@snu.ac.kr
WWW address: http://www.math.snu.ac.kr/~syha
E-mail address: mirinsil@naver.com
E-mail address: slemrod@math.wisc.edu

cycles produced by a complex Ginzburg-Landau model [19–21]. The Kuramoto oscillators can be visualized as point rotors on the unit circle $\mathbb{S}^1$ (see survey papers and books [1, 7, 26, 30, 31]). More precisely, let $x_j = e^{i\theta_j},\ \theta_j \in \mathbb{R}$, be the position of the j-th point rotor on the circle. Then the Kuramoto model with inertia reads as follows:

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad t > 0, \quad i = 1, \dots, N, \tag{1.1}$$

subject to the initial data:

$$(\theta_i, \dot{\theta}_i)\Big|_{t=0} = (\theta_{i0}, \omega_{i0}), \tag{1.2}$$

where m, K and Ω_i are constant inertia, the uniform positive coupling strength, and the intrinsic natural frequency of the i-th oscillator drawn from some distribution function $g = g(\Omega)$, respectively. For simplicity, we may assume that $\Omega_i \geq 0$ if necessary by taking $\theta_i \to \theta_i + (\min_{1\leq j\leq N} \Omega_j)t$.

The model (1.1) was first introduced by Ermentrout [14] for modeling the slow relaxation to the phase-locked states among fireflies, *Pteroptyx malaccae* (see [9] for rigorous justification). However the system (1.1) also appears in some mechanical models that describe an array of superconducting Josephson junctions [3, 11, 25, 37–41]. The second-order model (1.1) exhibits richer phenomena, such as a discontinuous first-order phase transition and hysteresis, etc. [1–3, 13, 17, 18, 32, 33], compared with the first-order Kuramoto model.

The self-consistent mean-field approach [19–21] initiated by Kuramoto uses the real order parameters r and ϕ to measure the degree of synchronization:

$$re^{i\phi} := \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}, \quad t \geq 0. \tag{1.3}$$

Note that the real order parameters r and ϕ are functions of all phases $\theta_i,\ i = 1, \cdots, N$, which are solutions of the system (1.1). Thus, they depend on K, N, and t implicitly, i.e., $r = r(K, t, N),\ \phi = \phi(K, t, N)$. In Kuramoto's derivation of the critical coupling strength K_c, he assumed that during the double limiting process,

$$t \to \infty \quad \text{and} \quad N \to \infty,$$

the order parameters r and ϕ approached constant states, which is not obvious a priori.

In this paper, we discuss a rigorous mathematical underpinning of Kuramoto's guess of the constant order parameters r and ϕ in an asymptotic limit via some Kuramoto type model (1.1) with inertia using the framework of Artstein-Kevrekidis-Slemrod-Titi's unified theory (AKST's theory) for a singular perturbation. We prove that the slow motion is just that r and ϕ are constants, whereas the fast motion is an uncoupled pendulum motion given constant torque and unit damping.

This paper has four sections after the Introduction. In Section 2, we briefly review AKST's theory of singular perturbation. In Section 3, we derive a fast-slow dynamical systems formulation of a Kuramoto type model with inertia and a small parameter $\varepsilon > 0$. In Section 4, we study the invariant measure for the fast system. Finally, Section 5 is devoted to the slow motion of the order parameters in the limit $\varepsilon \to 0$ using AKST's theory. In particular, we apply the Young measure approach to determine the evolution of order parameters and prove that they become stationary as $\varepsilon \to 0$.

2. Preliminaries. In this section, we briefly review the essence of AKST's unified approach [4, 28] to singular perturbations and averaging using Young measures for the reader's convenience. For more on classical singular perturbation and averaging, we refer to [5, 24, 27, 35].

2.1. *Invariant measures and Young measures.* In this part, we consider the basic concepts of invariant measures and Young measures, which we apply in later sections. For a detailed discussion, we refer to [6, 12, 23, 36].

A probability measure μ in the Euclidean space $\mathbb{R}^N$ is a σ-additive set function on the Borel σ-algebra $\mathcal{B}$ that consists of $\mathbb{R}^N$ with values in $[0,1]$ and $\mu(\mathbb{R}^N) = 1$. Thus, we specify $\mathcal{P}(\mathbb{R}^N)$ as the set of all probability measures on $\mathbb{R}^N$ endowed with weak convergence of measures [8].

Definition 2.1. Let μ be a probability measure defined on $\mathbb{R}^N$.

(1) The support of μ (denoted by $\mathrm{supp}(\mu)$) is the smallest closed set $C \subset \mathbb{R}^N$ such that $\mu(C) = 1$.

(2) μ is an invariant measure associated with the dynamical system

$$\frac{dx}{dt} = f(x), \quad f : \text{Lipshitz continuous}, \tag{2.1}$$

if and only if the solution $X(t, x_0)$ to (2.1) for x_0 in a neighborhood of $\mathrm{supp}(\mu)$ defined on some fixed interval I around $t = 0$ satisfies the relation

$$\mu(B) = \mu(X(t,B)), \quad \forall\, t \in I \quad \text{ and } B \in \mathcal{B}.$$

Next, we consider the definitions of Young measures and the convergence of a sequence of Young measures.

Definition 2.2 ([6, 8, 12, 34]). (1) ν is a Young measure if and only if $\nu : [a,b] \to \mathcal{P}(\mathbb{R}^N)$ is a measurable map.

(2) Let (ν_j) be a sequence of Young measures defined on the same interval $[a,b]$. The sequence (ν_j) converges to the limit Young measure ν_∞ if and only if

$$\int_a^b \int_{\mathbb{R}^N} h(\lambda,t)\nu_j(t)(d\lambda)dt \quad \to \quad \int_a^b \int_{\mathbb{R}^N} h(\lambda,t)\nu_\infty(t)(d\lambda)dt,$$

for every continuous real-valued function $h = h(\lambda, t)$.

Remark 2.1. 1. The Young measure was introduced by L. C. Young [42, 43] in a study of the calculus of variations with no minimizers in a classical sense, and then it was popularized by L. Tartar in a study of scalar hyperbolic conservation law [34].

2. The real-valued function $x = x(\cdot)$ in the interval $[a,b]$ can be viewed naturally as a Young measure, the value of which is a simply Dirac measure supported by the singleton $\{x(t)\}$. Thus, when we refer to the convergence of a sequence of functions in the sense of Young measures, we mean convergence in the sense of Definition 2.2 for the corresponding Dirac measure-valued functions. Thus, when we have a sequence of continuous functions uniformly bounded in j, $\{x_j(t)\}$, $t \in [a,b]$, its associated sequence of Young measures

is $\nu_j(t) = \delta(\lambda - x_j(t))$. If we select a separable test function $h(\lambda, t) = a(\lambda)b(t)$, then convergence of $x_j(\cdot)$ to the Young measure μ_∞ in the sense of Young measures means

$$\int_a^b b(t)a(x_j(t))dt \to \int_a^b b(t)\Big(\int_{\mathbb{R}^N} a(\lambda)\nu_\infty(t)(d\lambda)\Big)dt.$$

Therefore, the weak-$*$ limit in $L_\infty([a,b])$ of the sequence of functions $a(x_j(\cdot))$ is represented by the value

$$\int_{\mathbb{R}^N} a(\lambda)\nu_\infty(t)(d\lambda).$$

Next, we recall a fundamental theorem of Young measures from [6, 34].

THEOREM 2.1. Let (U^n) be a sequence of functions uniformly bounded in $L^\infty([0,1];\mathbb{R}^N)$. For any continuous function $V : \mathbb{R}^N \to \mathbb{R}$, there exists a family of probability measures $(\nu(t))$ and a subsequence U^{n_j} such that

$$V(U^{n_j}) \quad \overset{*}{\rightharpoonup} \quad \int_{\mathbb{R}^N} V(\lambda)\nu(t)(d\lambda) \qquad \text{in } \ L^\infty([0,1]).$$

2.2. *The AKST approach for singular perturbation.* In this part, we briefly review the approach presented in [4, 28]. The detailed theory can be found in the aforementioned references.

We consider a fast-slow dynamical system as follows:

$$\begin{aligned} &\frac{dU^\varepsilon}{dt} = \frac{F(U^\varepsilon)}{\varepsilon} + G(U^\varepsilon), \quad U^\varepsilon \in \mathbb{R}^N, \quad t > 0, \\ &U^\varepsilon(0) = U_{in}, \end{aligned} \tag{2.2}$$

where $F, G : \mathbb{R}^N \to \mathbb{R}^N$ are continuous functions denoting the fast and slow parts of the system (2.2), respectively, while ε is a small parameter.

The AKST's unified approach deals with a limiting process based on the dynamics of U^ε, when the small parameter approaches zero under some structural assumptions. Given suitable assumptions, the limit dynamics of the system (2.2) can be depicted as the evolution of the invariant measure of the fast part drifted by the slow part. The evolution of the invariant measure for the fast part can be characterized by the slow dynamics of the generalized moments of the invariant measure.

Let $V : \mathbb{R}^N \to \mathbb{R}$ be a bounded and continuous function, which is called (macroscopic) *measurement* or observer. For a given probability measure $\mu \in \mathcal{P}(\mathbb{R}^N)$, we set a generalized moment of μ as $\hat{V}(\mu)$:

$$\hat{V}(\mu) := \langle \mu, V\rangle = \int_{\mathbb{R}^N} V(\lambda)\mu(d\lambda).$$

Then, $\hat{V}$ is a bounded linear functional defined on $\mathcal{P}(\mathbb{R}^N)$, and we refer to $\hat{V}$ as an *observable.* Next, we define some classes of observables that are orthogonal with respect to the vector field generated by the fast part of (2.2).

DEFINITION 2.3 (Orthogonal observable). Let $(V, \hat{V})$ be the measurement and observable pair. The observable $\hat{V}(\mu)$ is called an *orthogonal observable* for the fast part of (2.2) if and only if the measurement V is the first integral of the fast system (2.3):

$$\frac{dU}{ds} = F(U), \qquad s := \frac{t}{\varepsilon}; \tag{2.3}$$

i.e., $V(U(s))$ is constant along the solution $U = U(s)$ of (2.3). In particular, if the measurement V is differentiable, this is equivalent to the following orthogonal relation:

$$\nabla_U V \cdot F(U) \equiv 0 \quad \text{along the solution } U = U(s) \text{ to (2.3)}.$$

Here $v \cdot w$ is the standard Euclidean inner product of two vectors $v, w \in \mathbb{R}^N$.

2.2.1. *The framework of AKST's theory.* We assume the following assumptions that guarantee the existence of the limit Young measure and the evolution of orthogonal observables.

- $(\mathcal{F}1)$ (Uniform boundedness of the overall system)
  There exists a compact set $\mathrm{H} \subset \mathbb{R}^N$ and a nontrivial interval I such that for any $0 < \varepsilon \ll 1$,
  $$U^\varepsilon(t) \in \mathrm{H}, \quad t \in \mathrm{I},$$
  where U^ε is the solution of the overall system (2.2).
- $(\mathcal{F}2)$ (Existence of a positively invariant set for the fast system)
  There exists a compact set $\mathrm{K} \subset \mathrm{H}$ that is positively invariant with respect to the fast part of (2.2).
- $(\mathcal{F}3)$ (Unique solvability) For any initial data $U_0 \in \mathrm{K}$, the overall system (2.2) and the fast system (2.3) are uniquely solvable.

For definiteness, we set the nontrivial interval I as the unit closed interval $[0, 1]$. Under the above assumptions, we can have evolution of orthogonal observables in the following theorem.

THEOREM 2.2 ([4]). Suppose that the assumptions $(\mathcal{F}1) - (\mathcal{F}3)$ hold, and let $U^{\varepsilon_j}(\cdot)$ be the sequence of solutions for (2.2) defined on $[0, 1]$, which converges to the Young measure ν_0 in the sense of Young measures (see Theorem 2.1). Then we have

(1) For any orthogonal observable $\hat{V}(\cdot)$ of the system (2.2), the measurement $V(U^{\varepsilon_j}(t))$ weak-* converges to $\hat{V}(\nu_0(t))$:
$$\hat{V}(\nu_0(t)) = \int_K V(\lambda)\nu_0(t)(d\lambda).$$
(2) The weak-* limit $\hat{V}(\nu_0(t))$ satisfies
$$\hat{V}(\nu_0(t)) = V(U_{in}) + \int_0^t \int_K \nabla V(\lambda) \cdot G(\lambda)\nu_0(\tau)(d\lambda)d\tau. \tag{2.4}$$

Proof. The detailed proof of the theorem can be found in [4], but for the reader's convenience, we briefly sketch the proof below. Let $V : \mathbb{R}^N \to \mathbb{R}$ be a differentiable orthogonal observable in the fast-slow system (2.2). Along the solution to (2.2), we have

$$\begin{aligned}
\frac{dV(U^\varepsilon(\tau))}{d\tau} &= \nabla_U V(U^\varepsilon) \cdot \frac{dU^\varepsilon}{d\tau} \\
&= \frac{1}{\varepsilon}\nabla_U V(U^\varepsilon) \cdot F(U^\varepsilon) + \nabla_U V(U^\varepsilon) \cdot G(U^\varepsilon) \\
&= \nabla_U V(U^\varepsilon) \cdot G(U^\varepsilon).
\end{aligned}$$

We now integrate the above relation from 0 to t, which yields

$$V(U^\varepsilon(t)) - V(U_{in}) = \int_0^t \nabla_U V(U^\varepsilon(\tau)) \cdot G(U^\varepsilon(\tau))d\tau. \tag{2.5}$$

Note that since the solution $U^{(0)}$ of the fast system will be the dominant part of the overall solution U^ε for small ε, we can write the ansatz for U^ε as

$$U^\varepsilon(t) = U^{(0)}\Big(\frac{t}{\varepsilon}\Big) + \mathcal{O}(\varepsilon)$$

and substitute it into the integral relation (2.5) to obtain

$$V(U^{(0)}\Big(\frac{t}{\varepsilon}\Big)) - V(U_{in}) = \int_0^t \nabla_U V(U^{(0)}\Big(\frac{\tau}{\varepsilon}\Big)) \cdot G(U^{(0)}\Big(\frac{\tau}{\varepsilon}\Big))d\tau + \mathcal{O}(\varepsilon). \tag{2.6}$$

According to ($\mathcal{F}2$), the sequence $U^{(0)}\Big(\frac{t}{\varepsilon}\Big) \in K$ is the compact subset of $\mathbb{R}^N$ uniformly in $t \in [0,1]$. Thus, this sequence is compact in a weak-* topology due to the Banach-Alaoglu theorem; i.e., there exists a subsequence $U^{(0)}\Big(\frac{t}{\varepsilon_j}\Big)$ that weak-* converges to some limit function $\bar{U} \in L^\infty([0,1])$, i.e.,

$$U^{(0)}\Big(\frac{t}{\varepsilon_j}\Big) \quad \overset{*}{\rightharpoonup} \quad \bar{U} \qquad \text{as } j \to \infty.$$

In view of Theorem 2.1, we set the limit Young measure associated with the limit $L^\infty([0,1])$-function $\bar{U}$ by ν_0. On the other hand, the sequence $V\Big(U^{(0)}\Big(\frac{t}{\varepsilon_j}\Big)\Big)$ is also uniformly bounded, so it is also weak-* compact. Thus up to some subsequence (still labelled as ε_j), we have

$$V(U^{(0)}\Big(\frac{t}{\varepsilon_j}\Big)) \quad \overset{*}{\rightharpoonup} \quad \bar{V}. \tag{2.7}$$

Therefore, it follows from Theorem 2.1 that this weak-* limit $\bar{V}$ can be represented by the action of the limit Young measure ν_0 on the measurement V:

$$\bar{V} = \int_K V(\lambda)\nu_0(d\lambda). \tag{2.8}$$

Similarly, we have

$$\nabla_U V(U^{(0)}\Big(\frac{\tau}{\varepsilon_j}\Big)) \cdot G(U^{(0)}\Big(\frac{\tau}{\varepsilon_j}\Big)) \quad \overset{*}{\rightharpoonup} \quad \int_K \nabla_U V(\lambda) \cdot G(\lambda)\nu_0(\tau)(d\lambda) \quad \text{as } j \to \infty. \tag{2.9}$$

We now combine (2.6)–(2.9) to get the desired result. □

REMARK 2.2. It may be asked why we did not differentiate the above integral relation (2.4) to obtain an ordinary differential equation for $\hat{V}(\nu_0(t))$. First, we note that even if we could differentiate the integral relation, it would not yield an ordinary differential equation in the classical sense; i.e., since the Young measure $\mu_0(t)$ is determined by the generating sequence $\Big\{U^{(0)}\Big(\frac{\cdot}{\varepsilon}\Big)\Big\}$, which depends on the initial data U_{in}, the right hand side depends on the initial data. Second, the issue of differentiability has been covered in Theorem 6.5 of [4] . The sufficient conditions given there are that the Young measure μ_0 is uniquely determined by the initial data U_0 and, furthermore, that it is Lipschitz continuous as a function of the data U_0. In our system, the continuity of the measure μ_0 as a function of the data is not expected, so fortunately it will not be needed.

3. Derivation of fast-slow dynamics. In this section, we present a fast-slow reformulation for the system (2.2) in the double limiting process $N \to \infty$ and $t \to \infty$. This is done by an artificial truncation procedure which admittedly may eliminate some of the subtle features of the Kuramoto system. Specifically, as we show below, we truncate the system at N oscillators and then set $\varepsilon = \frac{1}{N}$ as a small parameter. We then rescale time t as $\frac{\tau}{\varepsilon}$ and study the dynamics of this system with the number of oscillators N *fixed.* Hence, instead of using $\frac{1}{\varepsilon}$, we are using a fixed finite number of oscillators, and tails of the Kuramoto oscillator sequence are neglected.

Consider the following system of second-order ODEs:

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} a_j \sin(\theta_j - \theta_i), \quad i = 1, \cdots, N,\ t > 0, \tag{3.1}$$

where $a_j \in \{0, 1\}$ is the interaction weight indicating the impact of the j-th oscillator. The second-order system (3.1) can be rewritten as the first-order system:

$$\begin{aligned} \dot{\theta}_i &= \omega_i, \quad i = 1, \cdots, N,\ t > 0, \qquad (\theta_i, \omega_i) \in \mathbb{R}^2, \\ \dot{\omega}_i &= \frac{1}{m}\Big(-\omega_i + \Omega_i + \frac{K}{N}\sum_{j=1}^{N} a_j \sin(\theta_j - \theta_i)\Big). \end{aligned}$$

Next, we introduce the weighted Kuramoto order parameter $(r, \phi) \in \mathbb{R}_+ \times \mathbb{R}$:

$$re^{i\phi} := \frac{1}{N}\sum_{j=1}^{N} a_j e^{i\theta_j}. \tag{3.2}$$

Note that r is always bounded, i.e., $0 \le r \le 1$. We next divide (3.2) by $e^{i\theta_i}$ to get the equation

$$re^{i(\phi - \theta_i)} = \frac{1}{N}\sum_{j=1}^{N} a_j e^{i(\theta_j - \theta_i)},$$

and we compare the real and imaginary parts of the above relation to find

$$\begin{aligned} r\cos(\phi - \theta_i) &= \frac{1}{N}\sum_{j=1}^{N} a_j \cos(\theta_j - \theta_i), \\ r\sin(\phi - \theta_i) &= \frac{1}{N}\sum_{j=1}^{N} a_j \sin(\theta_j - \theta_i). \end{aligned}$$

We differentiate equation (3.2) with respect to t to get

$$\dot{r}e^{i\phi} + ire^{i\phi}\dot{\phi} = \frac{i}{N}\sum_{j=1}^{N} a_j e^{i\theta_j}\dot{\theta}_j.$$

We divide the resulting equation by $e^{i\phi}$ to find

$$\dot{r} + ir\dot{\phi} = -\frac{1}{N}\sum_{j=1}^{N} a_j \sin(\theta_j - \phi)\dot{\theta}_j + \frac{i}{N}\sum_{j=1}^{N} a_j \cos(\theta_j - \phi)\dot{\theta}_j. \tag{3.3}$$

We now take the real and imaginary parts of (3.3) to obtain

$$\dot{r} = -\frac{1}{N}\sum_{j=1}^{N} a_j \sin(\theta_j - \phi)\dot{\theta}_j,$$
$$\dot{\phi} = \frac{1}{rN}\sum_{j=1}^{N} a_j \cos(\theta_j - \phi)\dot{\theta}_j.$$

Thus, we obtain a coupled system for $(\theta_i, \omega_i, r, \phi)$:

$$\begin{aligned} \dot{\theta}_i &= \omega_i, \quad i = 1, \cdots, N, \ t > 0, \\ \dot{\omega}_i &= \frac{1}{m}\Big(- \omega_i + \Omega_i - Kr\sin(\theta_i - \phi)\Big), \end{aligned}$$

and

$$\dot{r} = -\frac{1}{N}\sum_{j=1}^{N} a_j \sin(\theta_j - \phi)\omega_j,$$
$$\dot{\phi} = \frac{1}{rN}\sum_{j=1}^{N} a_j \cos(\theta_j - \phi)\omega_j.$$

We want to study the long-time dynamics and mean-field limit simultaneously, so we introduce the stretched time variable $t = \frac{\tau}{\varepsilon}$ where $0 \le \tau \le 1$. Thus, the system becomes

$$\begin{aligned} \frac{d\theta_i}{d\tau} &= \frac{\omega_i}{\varepsilon}, \quad i = 1, \cdots, N, \ \tau > 0, \\ \frac{\omega_i}{d\tau} &= \frac{1}{m\varepsilon}\Big(- \omega_i + \Omega_i - Kr\sin(\theta_i - \phi)\Big), \\ \frac{dr}{d\tau} &= -\frac{1}{\varepsilon N}\sum_{j=1}^{N} a_j \sin(\theta_j - \phi)\omega_j, \\ \frac{d\phi}{d\tau} &= \frac{1}{r\varepsilon N}\sum_{j=1}^{N} a_j \cos(\theta_j - \phi)\omega_j. \end{aligned}$$

We then perform the following steps. We set

$$\varepsilon N = 1,$$

so that in principle as $\varepsilon \to 0$, we will get an infinite set of equations. However to prevent this event and maintain our system as finite dimensional, we set a truncation on a_j:

$$a_j = \begin{cases} 1, & j \le M, \\ 0, & j > M. \end{cases}$$

Thus, we have a coupled fast-slow system:

$$\begin{aligned}
\frac{d\theta_i}{d\tau} &= \frac{\omega_i}{\varepsilon}, \quad i = 1, \cdots, M, \ \tau > 0, \\
\frac{d\omega_i}{d\tau} &= \frac{1}{m\varepsilon}\Big(-\omega_i + \Omega_i - Kr\sin(\theta_i - \phi)\Big), \\
\frac{dr}{d\tau} &= -\sum_{j=1}^{M} \sin(\theta_j - \phi)\omega_j, \\
\frac{d\phi}{d\tau} &= \frac{1}{r}\sum_{j=1}^{M} \cos(\theta_j - \phi)\omega_j.
\end{aligned} \tag{3.4}$$

Note that the system (3.4) is the sum of the fast (3.5) and slow (3.6) parts respectively. For $i \in \{1, \cdots, M\}$,

$$\begin{aligned}
&\frac{d\theta_i}{dt} = \omega_i, \quad \frac{d\omega_i}{dt} = \frac{1}{m}\Big(-\omega_i + \Omega_i - Kr\sin(\theta_i - \phi)\Big), \quad t > 0, \\
&\frac{dr}{dt} = 0, \quad \frac{d\phi}{dt} = 0,
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\frac{d\theta_i}{d\tau} = 0, \quad \frac{d\omega_i}{d\tau} = 0, \quad \tau > 0, \\
&\frac{dr}{d\tau} = -\sum_{j=1}^{M} \sin(\theta_j - \phi)\omega_j, \quad \frac{d\phi}{d\tau} = \frac{1}{r}\sum_{j=1}^{M} \cos(\theta_j - \phi)\omega_j.
\end{aligned} \tag{3.6}$$

Since θ_j only enters the right hand side of (3.5)–(3.6) via sine and cosine functions, without loss of generality we may restrict θ_j to the interval $[-\pi, \pi]$ and identify all θ_i mod 2π as the same θ_i.

4. Invariant measure for the fast system.

4. Invariant measure for the fast system. In this section, we study the invariant measure of the fast system, which has a support in the ω-limit set of the fast system.

Consider an autonomous two-dimensional system in $\mathbb{R}^2$:

$$\begin{aligned}
\frac{dx}{dt} &= y, \quad t > 0, \\
\frac{dy}{dt} &= \frac{1}{m}\Big(-y + \Omega - Kr_0 \sin(x - \phi_0)\Big),
\end{aligned} \tag{4.1}$$

subject to initial data:

$$(x, y)(0) = (x_0, y_0), \tag{4.2}$$

where $r_0 \in (0, 1], \phi_0 \in \mathbb{R}$ and $\Omega \geq 0$ are constants.

Note that the system (4.1) is periodic in x and (θ_i, ω_i) for the fast system (3.5) satisfies (4.1) with $\Omega = \Omega_i$ for $i = 1, \cdots, M$, while the equilibrium solutions to (4.1) should satisfy the relation

$$y = 0, \qquad \Omega = Kr_0 \sin(x - \phi_0). \tag{4.3}$$

This means that if the system (4.1) has equilibrium solutions, then they are the equilibrium solutions to the system (4.1) without inertia, i.e., $m = 0$. Thus, the presence of inertia does not affect the structure of the equilibrium solutions, although it is known in [9] that inertia can modulate the relaxation speed toward the phase-locked states depending on the relative size of m and K.

Our fast system (4.1) is analogous to the equation for a damped pendulum with an applied constant torque (in our case $\frac{\Omega}{m}$), and the qualitative behavior of its solution has already been addressed in previous studies (e.g. [22, 31]). In particular, Levi et al. [22] made an extensive classification of the dynamic behavior of solutions depending on the relative sizes of Ω and Kr_0. Below, we list the results of Levi et al., which have been slightly modified in our setting without proofs.

4.1. *Subcritical regime* ($Kr_0 > \Omega$). In these regimes, x satisfies the same trigonometric equation as that of the Kuramoto model without inertia. Thus it follows from [10] that the explicit formula can be given as follows:

$$\begin{aligned} t\sqrt{(Kr_0)^2-\Omega^2} &= \log\left|\frac{\Omega\tan\frac{x(t)-\phi_0}{2}-Kr_0-\sqrt{(Kr_0)^2-\Omega^2}}{\Omega\tan\frac{x(t)-\phi_0}{2}-Kr_0+\sqrt{(Kr_0)^2-\Omega^2}}\right| \\ &- \log\left|\frac{\Omega\tan\frac{x_0-\phi_0}{2}-Kr_0-\sqrt{(Kr_0)^2-\Omega^2}}{\Omega\tan\frac{x_0-\phi_0}{2}-Kr_0+\sqrt{(Kr_0)^2-\Omega^2}}\right|. \end{aligned}$$

However, the above explicit formula is not useful for finding the invariant measure for the fast system. The equilibria can be found explicitly as follows: for $n = 0, \pm1, \pm2, \cdots$,

$$\begin{aligned} (x^e_{1n}, y^e_{1n}) &= (\phi_0 + \sin^{-1}\left(\frac{\Omega}{Kr_0}\right) + 2n\pi, 0) \quad \text{(stable node)}, \\ (x^e_{2n}, y^e_{2n}) &= (\phi_0 + \pi - \sin^{-1}\left(\frac{\Omega}{Kr_0}\right) + 2n\pi, 0) \quad \text{(unstable saddle)}. \end{aligned}$$

The large-time behavior of the general solution can be seen in the following proposition.

PROPOSITION 4.1 ([22]). Suppose that the Ω and K satisfy

$$0 \le \frac{\Omega}{Kr_0} < 1.$$

Then there exists a positive number $\sigma_* = \sigma_*\left(\frac{\Omega}{Kr_0}\right) > 0$ such that

(1) For $0 < \frac{1}{Kr_0} < \sigma_*$, there exists an exponentially stable running periodic orbit.
(2) For $\frac{1}{Kr_0} > \sigma_*$, every orbit tends to one of the equilibria.
(3) For $\frac{1}{Kr_0} = \sigma_*$, the state space is split into two regions; i.e., all orbits in the upper region tend to the boundary between the two regions, whereas all orbits in the lower region tend to one of the equilibria.

REMARK 4.1. For case (3), the boundary separating two regions is the stable manifold connecting neighboring saddle points, i.e., the running periodic orbit. Proposition 4.1 implies that the ω-limit sets consist of equilibria or running periodic orbits.

4.2. *Critical regime* ($Kr_0 = \Omega$). In this regime, x also satisfies the same trigonometric equation as that of the Kuramoto model without inertia. Thus it follows from [10] that we can get the explicit formula

$$t = \frac{2}{\Omega\tan\frac{x_0-\phi_0}{2} - Kr_0} - \frac{2}{\Omega\tan\frac{x(t)-\phi_0}{2} - Kr_0}.$$

Again, the above explicit formula is not very useful for finding the invariant measure for the fast system. For the critical coupling case, two equilibria x_1^e and x_2^e in subcritical case coalesce and become

$$(x_{3n}^e, y_{3n}^e) = (\phi_0 + 2\tan^{-1}\left(\frac{Kr_0}{\Omega}\right) + n\pi, 0), \quad n = 0, \pm 1, \cdots .$$

The large-time behavior of the general solution can be summarized as follows.

Proposition 4.2 ([22]). Suppose that the Ω and K satisfy

$$\frac{\Omega}{Kr_0} = 1.$$

Then there exists a positive number $\sigma_* = \sigma_*\left(\frac{\Omega}{Kr_0}\right) > 0$ such that

(1) For $0 < \frac{1}{Kr_0} < \sigma_*$, there exist an exponentially stable running periodic orbit and orbits that approach the running periodic orbit or equilibria.
(2) For $\frac{1}{Kr_0} > \sigma_*$, every orbit tends to one of the equilibria.
(3) For $\frac{1}{Kr_0} = \sigma_*$, the state space is split into two regions; i.e., all orbits in the upper region tend to the boundary between the two regions, whereas all orbits in the lower region tend to one of the equilibria. The phase boundary consists of an orbit connecting two neighboring equilibria (continued 2π-periodically).

4.3. *Supercritical regime* ($Kr_0 < \Omega$). First, note that in the supercritical case we have no equilibrium points. If the orbits of (4.1) were bounded, we could apply the standard Poincaré-Bendixson Theorem, but unfortunately this is not the case. Hence we recall some results of Levi et al. [22] on the existence of running periodic orbits. The existence of a running periodic orbit and its global stability can be summarized in the following proposition. For this, we introduce two positive numbers $L^{\pm}$ as follows:

$$L^+ := 2(\Omega + Kr_0), \quad L^- := \frac{\Omega - Kr_0}{2}.$$

Proposition 4.3 ([22]). Suppose that Ω and K satisfy the relation

$$\Omega > Kr_0.$$

Then there exists a unique running periodic orbit $\mathcal{P}$ to the system (4.1)–(4.2) in the strip $\mathbb{R} \times [L^-, L^+]$, and it is globally asymptotically stable in the sense that all orbits will be drawn to $\mathcal{P}$ as $t \to \infty$.

Remark 4.2. As an immediate corollary of Proposition 4.3, for any point $z_0 = (x_0, y_0) \in \mathbb{R}^2$,

$$\omega(z_0) = \mathcal{P},$$

and the invariant measure for the fast system (4.1) is supported on the inverse image of a periodic orbit:

$$\nu(d\lambda) = \frac{1}{T} dx^{-1}(d\lambda),$$

where T is the period of the periodic orbit and x^{-1} denotes the inverse of x. For the zero inertia case $m = 0$, the period T is explicitly computable.

5. Limit dynamics of order parameters. In this section, we study the evolution of the order parameters as an application of AKST's theory to the fast-slow system (3.4). Recall that the limit measure ν_0 in Theorems 2.1 and 2.2 is an invariant measure of the fast system which was characterized in the previous section.

First, we define

$$\begin{aligned} U &:= (\theta, \omega, r, \phi) \in \mathbb{R}^{2M+2}, \quad F(U) := (\omega, f(\omega), 0, 0), \\ G(U) &:= \Big(0, \cdots, 0, 0, \cdots, 0, -\sum_{j=1}^M \sin(\theta_j - \phi)\omega_j, \frac{1}{r}\sum_{j=1}^M \cos(\theta_j - \phi)\omega_j\Big), \end{aligned}$$

where $f(\omega) = (f_1(\omega), \cdots, f_M(\omega))$ is defined by

$$f_i(\omega) := \frac{1}{m}\Big(-\omega_i + \Omega_i - Kr\sin(\theta_i - \phi)\Big).$$

The system (3.4) can be written in a compact form:

$$\frac{dU^\varepsilon}{d\tau} = \frac{F(U^\varepsilon)}{\varepsilon} + G(U^\varepsilon). \tag{5.1}$$

Recall that the assumption of $(\mathcal{F}2)$ in Section 2.2 requires that the unique solution of

$$\frac{dU^{(0)}}{dt} = F(U^{(0)}), \quad U^{(0)}(0) = U_{in}, \quad t := \frac{\tau}{\varepsilon} \tag{5.2}$$

lies in a compact subset of $\mathbb{R}^{2M+2}$. In fact, it follows from Lemma 4.2 that

$$U^{(0)} \in K.$$

The theory assumes $(\mathcal{F}1) - (\mathcal{F}3)$, and the overall system (5.1) has unique solutions on some finite interval, say $0 \le \tau \le 1$. This is certainly true for the initial data in $\mathbf{H}(\delta)$:

$$\mathbf{H}(\delta) := \{(\theta, \omega, r, \phi) \ : \ \theta \in \mathbb{R}^M, \quad \omega \in \mathbb{R}^M, \quad 0 < \delta \le r, \quad \phi \in \mathbb{R}\}.$$

The next step is to identify an *orthogonal* measurement $V(U)$ for which

$$\nabla V(U) \cdot F(U) = 0,$$

which is equivalent to

$$\sum_{i=1}^M \Big[\frac{\partial V(U)}{\partial \theta_i}\omega_i + \frac{1}{m}\frac{\partial V(U)}{\partial \omega_i}\Big(-\omega_i + \Omega_i - Kr\sin(\theta_i - \phi)\Big)\Big] = 0.$$

It is easy to see that $V(U) = \tilde{V}(r, \phi)$ will suffice as an orthogonal measurement. In particular, we can select V from the projection map of the $(2M+1)$-th component or $(2M+2)$-th component as our two measurements, which of course yield orthogonal observables, i.e.,

$$V(U) = r, \phi.$$

By Theorem 2.1, the solution of (5.1) with initial data $(\theta, \omega, r, \phi) \in \mathbf{H}(\delta)$, defined on $0 \le \tau \le 1$, will have a convergent subsequence $U^{\varepsilon_j}(\cdot)$ that converges to a Young measure $\nu_0(\cdot)$ on $[0, 1]$ in the sense of Young measures. The value of the limit Young measure is an invariant measure of the fast system (5.2). The oscillators of the fast system (5.2) are decoupled, so the invariant measure $\nu_0(\tau)$ is a product measure:

$$\begin{aligned}\nu_0(\tau)(d\lambda) = \nu_1(\tau)(d\lambda_1, d\lambda_{M+1}) \otimes \cdots \otimes \nu_M(\tau)(d\lambda_M, d\lambda_{2M})\\ \otimes \nu_{2M+1}(\tau)(d\lambda_{2M+1}) \otimes \nu_{2M+2}(\tau)(d\lambda_{M+2}),\end{aligned} \tag{5.3}$$

where $\lambda = (\lambda_1, \cdots, \lambda_{2M+2})$ and each $\nu_i(\tau)(d\lambda_i, d\lambda_{M+i})$, $1 \le i \le M$, is itself an invariant probability measure for the i-th oscillator equation of (3.4). Since $\frac{dr}{d\tau} = 0$ and $\frac{d\phi}{d\tau} = 0$ in the fast system,

$$\nu_{2M+1}(\tau) = \delta(\lambda_{2M+1} - r(\tau)), \qquad \nu_{2M+2}(\tau) = \delta(\lambda_{2M+2} - \phi(\tau)).$$

Furthermore, by Theorem 2.2, we can use the orthogonal observables associated with any measurement V that satisfy the integral equation

$$\hat{V}(\nu_0(\tau)) = V(U_{in}) + \int_0^\tau \int_{\mathbb{R}^{2M+2}} \nabla V(\lambda) \cdot G(\lambda) \nu_0(s)(d\lambda) ds. \tag{5.4}$$

For the choices $V(U) = r$, $V(U) = \phi$, (5.4) yields the evolution of r and ϕ:

$$\begin{aligned} r(\tau) &= r(0) \\ &- \sum_{j=1}^M \int_0^\tau \int_{\mathbb{R}^{2M}} \sin(\lambda_j - \phi(s))\lambda_{M+j}\ \nu_1(s)(d\lambda_1, d\lambda_{M+1}) \otimes \cdots \\ &\otimes \nu_M(s)(d\lambda_M, d\lambda_{2M}) ds, \\ \phi(\tau) &= \phi(0) \\ &+ \sum_{j=1}^M \int_0^\tau \int_{\mathbb{R}^{2M}} \frac{1}{r(s)} \cos(\lambda_j - \phi(s))\lambda_{M+j}\ \nu_1(s)(d\lambda_1, d\lambda_{M+1}) \otimes \cdots \\ &\otimes \nu_M(s)(d\lambda_M, d\lambda_{2M}) ds. \end{aligned}$$

Next, we substitute (5.3) into the equations above to obtain

$$\begin{aligned} r(\tau) &= r(0) - \sum_{j=1}^M \int_0^\tau \int_{\mathbb{R}^2} \sin(\lambda_j - \phi(s))\lambda_{M+j}\ \nu_j(s)(d\lambda_j, d\lambda_{M+j}) ds, \\ \phi(\tau) &= \phi(0) + \sum_{j=1}^M \int_0^\tau \int_{\mathbb{R}^2} \frac{1}{r(s)} \cos(\lambda_j - \phi(s))\lambda_{M+j}\ \nu_j(s)(d\lambda_j, d\lambda_{M+j}) ds. \end{aligned} \tag{5.5}$$

The system (5.5) produces what is usually known as an amplitude equation for r, although it is actually the coupled system (5.5) that determines r.

• Supercritical regime ($|\Omega_i| > Kr$): In this case, there are no equilibrium solutions to the system (1.1). Note that the solution $\dot{\theta}(t)$ has only one sign after some time; i.e., there exists T^* such that either $\dot{\theta}(t) \ge 0$ or $\dot{\theta}(t) \le 0$ for $t \ge T^*$ based on the argument in (Step 2) of the proof of Lemma 4.2. We conclude with the slow evolution for r and ϕ in the case of $|\Omega_i| > Kr$. In this case, the invariant measure is supported on the inverse image of the periodic orbit:

$$\nu_i(d\lambda_i, d\lambda_{M+i}) = \frac{1}{T_i} d\theta_i^{-1}(\lambda_i), \qquad \text{where } \theta_i^{-1} \text{ denotes the inverse of } \theta_i.$$

Unlike the case with zero inertia, however, we have no explicit representation for the period T_i. However, the explicit representation of T_i is not needed for the following estimate:

$$\begin{aligned}\mathcal{I} := \int_{\mathbb{R}^2} &\sin(\lambda_i - \phi)\lambda_{M+i}\nu_i(s)(d\lambda_i, d\lambda_{M+i}) \\ &= \frac{1}{T_i}\int_{\mathbb{R}} \sin(\lambda_i - \phi)\dot{\lambda}_i \; d\theta_i^{-1}(\lambda_i), \quad \text{by } \dot{\lambda}_i = \lambda_{M+i} \\ &= \frac{1}{T_i}\int_0^{T_i} \sin(\theta_i(s_i) - \phi)\dot{\theta}_i(s_i) ds_i, \quad \text{by } s_i := \theta_i^{-1}(\lambda_i) \\ &= \frac{1}{T_i}\int_0^{T_i} \Big(\frac{d}{ds_i}\cos(\theta_i(s_i) - \phi)\Big) ds_i \\ &= 0, \quad \text{by the periodicity of } \theta_i.\end{aligned} \tag{5.6}$$

Similarly, we have

$$\mathcal{J} := \int_{\mathbb{R}^2} \cos(\lambda_i - \phi)\lambda_{M+i}\nu_i(s)(d\lambda_i, d\lambda_{M+i}) = 0.$$

• Subcritical and critical regimes ($|\Omega_i| \le Kr$): It follows from the results in Section 4 that the support of an invariant measure for the fast system is the union of the equilibria and the running periodic orbit. The contribution of $\mathcal{I}$ and $\mathcal{J}$ with invariant measure, the support of which is the running periodic orbit, will be zero based on the same calculation as the supercritical case. However, we note that the invariant measure situated on the stable equilibria has the form

$$\nu_i(s)(d\lambda_i, d\lambda_{M+i}) = \delta(\lambda_i - \theta_{in}^e(s)) \otimes \delta(\lambda_{M+i}).$$

Thus if we insert the above ansatz into $\mathcal{I}$ and $\mathcal{J}$ and use the fact that

$$\int_{\mathbb{R}} \lambda_{M+i}\delta(\lambda_{M+i})(d\lambda_{M+i}) = 0,$$

then we have

$$\mathcal{I} = 0, \qquad \mathcal{J} = 0.$$

Therefore, for any cases we have

$$r(s) = r(0), \quad \phi(s) = \phi(0), \quad 0 \le s \le 1.$$

THEOREM 5.1. The limiting dynamics for the Kuramoto system (3.4) as $\varepsilon \to 0$ is given as follows:

$$r(\tau) = \text{const}, \qquad \phi(\tau) = \text{const}, \qquad 0 \le \tau \le 1.$$

REMARK 5.1. 1. The results of Theorem 5.1 can be interpreted as follows. Since $t = \frac{\tau}{\varepsilon}$, if τ is bounded, then as $\varepsilon \to 0+$, $t \to \infty$. Theorem 5.1 shows that if we look at our original unscaled system for the t-interval $[0, \frac{1}{\varepsilon}]$ and map the graph of $r(t), \varphi(t)$ onto the fixed scaled τ-interval $[0, 1]$, the graphs of r and ϕ will be constant. Specifically, the order parameters r and ϕ are indeed constant as a function of τ in the limit as $\varepsilon \to 0$.

2. The complete synchronization problem of Kuramoto oscillators with inertia was studied in [9, 13], while the slow-fast dynamical systems theory for the flocking and synchronization models without inertia were investigated in [15, 16].

References

[1] J. A. Acebrón, L. L. Bonilla, C. J. P. Pérez Vicente, F. Ritort, and R. Spigler, *The Kuramoto model: A simple paradigm for synchronization phenomena*, Rev. Mod. Phys. **77** (2005), 137-185.

[2] J. A. Acebrón, L. L. Bonilla, and R. Spigler, *Synchronization in populations of globally coupled oscillators with inertial effects*, Phys. Rev. E (3) **62** (2000), no. 3, 3437–3454, DOI 10.1103/PhysRevE.62.3437. MR1788951 (2002e:34059)

[3] J. A. Acebrón and R. Spigler, *Adaptive frequency model for phase-frequency synchronization in large populations of globally coupled nonlinear oscillators.* Phys. Rev. Lett. **81** (1998), 2229-2332.

[4] Zvi Artstein, Ioannis G. Kevrekidis, Marshall Slemrod, and Edriss S. Titi, *Slow observables of singularly perturbed differential equations*, Nonlinearity **20** (2007), no. 11, 2463–2481, DOI 10.1088/0951-7715/20/11/001. MR2361254 (2008j:34087)

[5] Zvi Artstein and Alexander Vigodner, *Singularly perturbed ordinary differential equations with dynamic limits*, Proc. Roy. Soc. Edinburgh Sect. A **126** (1996), no. 3, 541–569, DOI 10.1017/S0308210500022903. MR1396278 (97g:34073)

[6] J. M. Ball, *A version of the fundamental theorem for Young measures*, PDEs and continuum models of phase transitions (Nice, 1988), Lecture Notes in Phys., vol. 344, Springer, Berlin, 1989, pp. 207–215, DOI 10.1007/BFb0024945. MR1036070 (91c:49021)

[7] Neil J. Balmforth and Roberto Sassi, *A shocking display of synchrony*, Bifurcations, patterns and symmetry. Phys. D **143** (2000), no. 1-4, 21–55, DOI 10.1016/S0167-2789(00)00095-6. MR1783383 (2001g:82007)

[8] Patrick Billingsley, *Convergence of probability measures*, John Wiley & Sons Inc., New York, 1968. MR0233396 (38 #1718)

[9] Young-Pil Choi, Seung-Yeal Ha, and Seok-Bae Yun, *Complete synchronization of Kuramoto oscillators with finite inertia*, Phys. D **240** (2011), no. 1, 32–44, DOI 10.1016/j.physd.2010.08.004. MR2740100 (2012b:34093)

[10] Young-Pil Choi, Seung-Yeal Ha, Sungeun Jung, and Yongduck Kim, *Asymptotic formation and orbital stability of phase-locked states for the Kuramoto model*, Phys. D **241** (2012), no. 7, 735–754, DOI 10.1016/j.physd.2011.11.011. MR2897541

[11] B. C. Daniels, S. T. Dissanayake and B. R. Trees, *Synchronization of coupled rotators: Josephson junction ladders and the locally coupled Kuramoto model.* Phys. Rev. E. **67** (2003), 026216.

[12] Constantine M. Dafermos, *Hyperbolic conservation laws in continuum physics*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 325, Springer-Verlag, Berlin, 2005. MR2169977 (2006d:35159)

[13] Florian Dörfler and Francesco Bullo, *On the critical coupling for Kuramoto oscillators*, SIAM J. Appl. Dyn. Syst. **10** (2011), no. 3, 1070–1099, DOI 10.1137/10081530X. MR2837521 (2012i:34043)

[14] G. Bard Ermentrout, *Synchronization in a pool of mutually coupled oscillators with random frequencies*, J. Math. Biol. **22** (1985), no. 1, 1–9, DOI 10.1007/BF00276542. MR802731 (86m:92010)

[15] Seung-Yeal Ha, Sungeun Jung, and Marshall Slemrod, *Fast-slow dynamics of planar particle models for flocking and swarming*, J. Differential Equations **252** (2012), no. 3, 2563–2579, DOI 10.1016/j.jde.2011.09.014. MR2860630 (2012k:34119)

[16] Seung-Yeal Ha and Marshall Slemrod, *A fast-slow dynamical systems theory for the Kuramoto type phase model*, J. Differential Equations **251** (2011), no. 10, 2685–2695, DOI 10.1016/j.jde.2011.04.004. MR2831709 (2012h:37157)

[17] H. Hong, M. Y. Choi, J. Yi, and K.-S. Soh, *Inertial effects on periodic synchronization in a system of coupled oscillators*, Phys. Rev. E. **59** (1999), 353-363.

[18] H. Hong, Gun Sang Jeon, and M. Y. Choi, *Spontaneous phase oscillation induced by inertia and time delay*, Phys. Rev. E (3) **65** (2002), no. 2, 026208, 5, DOI 10.1103/PhysRevE.65.026208. MR1908304

[19] Y. Kuramoto, *Chemical oscillations, waves, and turbulence*, Springer Series in Synergetics, vol. 19, Springer-Verlag, Berlin, 1984. MR762432 (87e:92054)

[20] Yoshiki Kuramoto, *Self-entrainment of a population of coupled non-linear oscillators* (Kyoto Univ., Kyoto, 1975), Lecture Notes in Phys., 39, Springer, Berlin, 1975, pp. 420–422. MR0676492 (58 #32705)

[21] Y. Kuramoto and I. Nishikawa, Onset of collective rhythms in large populations of coupled oscillators, in *Cooperative dynamics in complex physical systems*, edited by H. Takayama. 300-306 (1988).

[22] M. Levi, F. C. Hoppensteadt, and W. L. Miranker, *Dynamics of the Josephson junction*, Quart. Appl. Math. **36** (1978/79), no. 2, 167–198. MR0484023 (58 #3972)

[23] V. V. Nemytskii and V. V. Stepanov, *Qualitative theory of differential equations*, Princeton Mathematical Series, No. 22, Princeton University Press, Princeton, N.J., 1960. MR0121520 (22 #12258)

[24] Robert E. O'Malley Jr., *Singular perturbation methods for ordinary differential equations*, Applied Mathematical Sciences, vol. 89, Springer-Verlag, New York, 1991. MR1123483 (92i:34071)

[25] K. Park and M. Y. Choi, *Synchronization in networks of superconducting wires*, Phys. Rev. B **56** (1997), 387-394.

[26] Arkady Pikovsky, Michael Rosenblum, and Jürgen Kurths, *Synchronization*, A universal concept in nonlinear sciences. Cambridge Nonlinear Science Series, vol. 12, Cambridge University Press, Cambridge, 2001. MR1869044 (2002m:37001)

[27] J. A. Sanders and F. Verhulst, *Averaging methods in nonlinear dynamical systems*, Applied Mathematical Sciences, vol. 59, Springer-Verlag, New York, 1985. MR810620 (87d:34065)

[28] Marshall Slemrod, *Averaging of fast-slow systems*, Coping with complexity: model reduction and data analysis, Lect. Notes Comput. Sci. Eng., vol. 75, Springer, Berlin, 2011, pp. 1–7, DOI 10.1007/978-3-642-14941-2_1. MR2757569 (2012c:34170)

[29] J. J. Stoker, *Nonlinear vibrations in mechanical and electrical systems*, Interscience Publishers, Inc., New York, N.Y., 1950. MR0034932 (11,666a)

[30] Steven H. Strogatz, *Norbert Wiener's brain waves*, Frontiers in mathematical biology, Lecture Notes in Biomathematics, 100, Springer-Verlag, 1994, pp. 122–138, DOI 10.1007/978-3-642-50124-1_7. MR1348659

[31] Steven H. Strogatz, *From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators*, Phys. D **143** (2000), no. 1-4, 1–20, DOI 10.1016/S0167-2789(00)00094-4. Bifurcations, patterns and symmetry. MR1783382 (2001g:82008)

[32] H. A. Tanaka, A. J. Lichtenberg, and S. Oishi, *First order phase transitions resulting from finite inertia in coupled oscillators systems*, Phys. Rev. Lett. **78** (1997), 2104.

[33] H. A. Tanaka, A. J. Lichtenberg, and S. Oishi, *Self-synchronization of coupled oscillators with hysteretic responses*, Physica D **100** (1997), 279-300.

[34] L. Tartar, *Compensated compactness and applications to partial differential equations*, Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV, Res. Notes in Math., vol. 39, Pitman, Boston, Mass., 1979, pp. 136–212. MR584398 (81m:35014)

[35] A. N. Tihonov, *On systems of differential equations containing parameters* (Russian), Mat. Sbornik N.S. **27(69)** (1950), 147–156. MR0036902 (12,181d)

[36] Michel Valadier, *A course on Young measures*, Workshop on Measure Theory and Real Analysis (Italian) (Grado, 1993), Rend. Istit. Mat. Univ. Trieste **26** (1994), suppl., 349–394 (1995). MR1408956 (97k:28009)

[37] S. Watanabe and J. W. Swift, *Stability of periodic solutions in series arrays of Josephson junctions with internal capacitance*, J. Nonlinear Sci. **7** (1997), no. 6, 503–536, DOI 10.1007/s003329900038. MR1474640 (98f:34051)

[38] S. Watanabe, and S. H. Strogatz, *Constants of motion for superconducting Josephson arrays*, Physica D **74** (1994), 197-253.

[39] K. Wiesenfeld, R. Colet, and S. H. Strogatz, *Synchronization transitions in a disordered Josephson series arrays*, Phys. Rev. Lett. **76** (1996), 404-407.

[40] K. Wiesenfeld, R. Colet, and S. H. Strogatz, *Frequency locking in Josephson arrays: connection with the Kuramoto model*, Phys. Rev. E **57** (1988), 1563-1569.

[41] K. Wiesenfeld and J. W. Swift, *Averaged equations for Josephson junction series arrays*, Phys. Review E **51** (1995), 1020-1025.

[42] L. C. Young, *Lectures on the calculus of variations and optimal control theory*, foreword by Wendell H. Fleming, W. B. Saunders Co., Philadelphia, 1969. MR0259704 (41 #4337)

[43] L. C. Young, *Generalized curves and the existence of an attained absolute minimum in the calculus of variations*, Comtes Rendus de la Societe des Sciences et des Lettres de Varsovie, Classe III, **30** (1937), pp. 212-234.