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EDGE WAVES WITH LONGSHORE CURRENTS

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Abstract. We present an exact solution to the nonlinear governing equations for nonlinear edge waves with an underlying longshore current, propagating over a plane-sloping beach. By performing an analysis in Lagrangian variables we describe the flow characteristics in great detail.

1. Introduction. Edge waves are trapped waves near the shoreline on a sloping beach since their amplitude is maximal at the shoreline and decays rapidly offshore. These three-dimensional waves propagate along the straight shoreline, and the waveform is sinusoidal in the longshore. The edge waves play an important role in the dynamics of coastal zone and beach erosion processes (Leblond and Mysak [1]).

The edge wave phenomenon has been extensively studied and discussed in the mathematical literature within the framework of linear theory. Due to the small displacements associated with these waves, the governing equations for water waves or the shallow-water equations are linearized, and this simplification permits a thorough analysis (Stokes [2], Lamb [3], Ursell [4], Ehrenmark [5]). Nonlinear corrections to the linear theory have been given by Whitham [6]. He showed the existence of irrotational weakly nonlinear edge waves that propagate parallel to the shore using a formal Fourier series expansion for the full water-wave theory. A study of properties of nonlinear progressive edge waves based on the fact that the evolution is described by the nonlinear Schrödinger equation was performed by Yeh [7]. Mok and Yeh [8] studied the mass transport of nonlinear progressive edge waves.

Some exact solutions describing nonlinear edge waves in the Lagrangian framework were also obtained. Yih [9] and Mollo-Christensen [10] found that the deep water-wave solution discovered by Gerstner [11] can be adapted to construct edge waves propagating along a plane-sloping beach. However, this treatment of the edge wave problem provides only an implicit form for the free water surface. Constantin [12] presented an explicit edge wave solution. This flow has been a starting point for a number of interesting studies within the class of gravity or geophysical edge waves (cf. Ehrnstrom et al. [13], Henry

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and Mustafa [14], Johnson [15], Stuhlmeier [16], Matioc [17]). For a modern exposition of Gerstner's wave we refer to the discussion in Constantin [18] and Henry [19]; for extensions to equatorial surface geophysical flows see Constantin [20], Constantin and Germain [21] and Hsu [22], while extensions relevant for geophysical internal waves are described in the recent papers (Constantin [23, 24]) and (Hsu [25, 26]). We would only like to emphasize the special nature of Gerstner's flow by pointing out that all particles perform a circular motion. This is in stark contrast to the situation encountered beneath Stokes waves (periodic irrotational travelling waves); cf. Constantin [27], Henry [28], Constantin and Strauss [29].

The inclusion of longshore currents for the linear edge wave is described by Howd et al. [30]. The aim of the present paper is to extend the explicit exact solution in Constantin [12] to include an underlying uniform current. The solution is presented in Lagrangian coordinates by describing the circular path of each particle.

2. Governing equations. We take a plane beach and adopt a coordinate system as shown below, which shoreline being the x-axis and still sea in the region

$$R = \{(x, y, z) : x \in \mathbb{R},\ y \leq b_0,\ 0 \leq z \leq (b_0 - y)\tan\alpha\}$$

for some $b_0 \leq 0$; here $\alpha \in (0, \pi/2)$ defines the uniform slope. Let (u, v, w) be the velocity field, and the governing equations for the propagation of gravity waves are given by

$$\begin{cases} u_t + uu_x + vu_y + wu_z = -\frac{1}{\rho}P_x, \\ v_t + uv_x + vv_y + wv_z = -\frac{1}{\rho}P_y - g\sin\alpha, \\ w_t + uw_x + vw_y + ww_z = -\frac{1}{\rho}P_z - g\cos\alpha, \end{cases} \tag{2.1}$$

coupled with the equation of mass conservation

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0 \tag{2.2}$$

and with the incompressibility constraint

$$u_x + v_y + w_z = 0. \tag{2.3}$$

Here t stands for time, g=9.81 $\mathrm{m\,s^{-2}}$ is the (constant) gravitational acceleration, and ρ is the water's density, while P is the pressure. The boundary conditions which select the water-wave problem from all possible solutions of the equations (2.1) and (2.3) are (Constantin [18], Johnson [31]): (i) The dynamic boundary condition $P = P_0$ at the free surface, where P_0 is the constant atmosphere pressure, decouples the motion of the air from that of the water. (ii) The kinematic boundary condition at the free surface expresses the fact that the same particles always form the free water surface. (iii) The bottom is assumed to be impermeable and the normal velocity component at the sloping bed to be zero. The general description of the propagation of a water wave is encompassed by the equations (2.1) and (2.3) and the three boundary conditions (i)-(iii), a distinctive feature being that the free surface is not known and must be determined as a part of the solution.

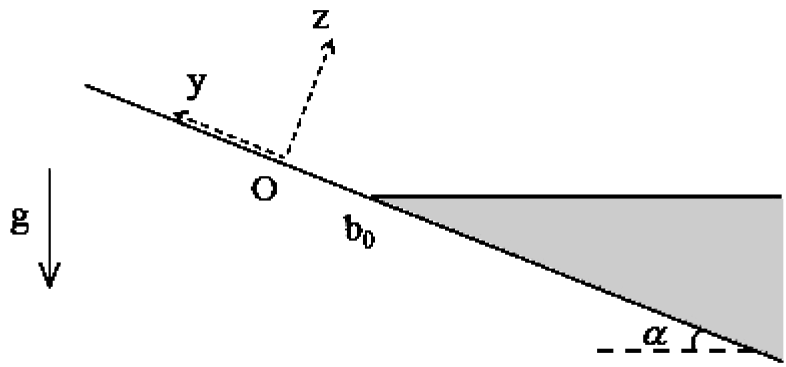

FIG. 1. Cross section of the still sea.

3. An exact solution in the Lagrangian description. In this section, we will present the exact solution of the edge wave problem. This solution represents edge waves travelling parallel to the shoreline which has a constant speed of propagation c_0. Throughout this paper, the Lagrangian positions (x, y, z) of the fluid particles are given as functions of labeling variables (a, b, c) and time t by

$$\left\{\begin{array}{l} x = a - Ut - \frac{1}{k} e^{k(b-c)} \sin[k(a + c_0 t)], \\ y = b - c + \frac{1}{k} e^{k(b-c)} \cos[k(a + c_0 t)], \\ z = c + c\tan\alpha - \frac{\tan\alpha}{2k} e^{2kb_0}\left(1 - e^{-2kc(1+\cot\alpha)}\right) \end{array}\right. \tag{3.1}$$

where $k > 0$ is a fixed wave number. U is the uniform longshore current. The quantities a, b, c do not express the initial coordinates of a particle, but are simply labeling variables serving to identify a particle. Our aim is to prove that the motion (3.1) is dynamically possible and that we can associate with it an expression for the hydrodynamical pressure P such that the governing equations and boundary conditions are all satisfied. The resulting free surface of the water at $c = (b_0 - b)\tan\alpha$ is the edge wave we are looking for. The map (3.1) is a diffeomorphism from the still water region R to the water region bounded below by the rigid bed $z = 0$ and above by the free water surface, which is parametrized by

$$\left\{\begin{array}{l} x = a - Ut - \frac{1}{k} e^{kb(1+\tan\alpha) - kb_0\tan\alpha} \sin[k(a + c_0 t)], \\ y = b(1 + \tan\alpha) - kb_0\tan\alpha + \frac{1}{k} e^{kb(1+\tan\alpha) - kb_0\tan\alpha} \cos[k(a + c_0 t)], \\ z = (b - b_0)(1 + \tan\alpha)\tan\alpha - \frac{\tan\alpha}{2k} e^{2kb_0}\left(1 - e^{2k(b-b_0)(1+\tan\alpha)}\right), \end{array}\right. \tag{3.2}$$

with $a \in \mathbb{R}$, $b \le b_0$ and $t \ge 0$. Indeed, observing that $(a, b, c) \mapsto (a, b - c, c)$ defines a diffeomorphism of $\mathbb{R}^3$, it suffices to show that the mapping

$$\begin{pmatrix} a \\ b' \\ c \end{pmatrix} \mapsto \left\{\begin{array}{l} a - Ut - \frac{1}{k} e^{kb'} \sin[k(a + c_0 t)], \\ b' + \frac{1}{k} e^{kb'} \cos[k(a + c_0 t)], \\ c + c\tan\alpha - \frac{\tan\alpha}{2k} e^{2kb_0}\left(1 - e^{-2kc(1+\cot\alpha)}\right), \end{array}\right. \tag{3.3}$$

with $a \in \mathbb{R}$, $b' \le b_0 - c(1+\cot\alpha)$ and $t \ge 0$, is a diffeomorphism at every fixed value of t (Constantin [12], Henry [19]). For notational convenience let us choose

$$\xi = k(b-c), \theta = k(a - c_0 t).$$

Then the Jacobian matrix of the transformation (3.1) is given by

$$\begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{pmatrix} = \begin{pmatrix} 1 - e^{\xi}\cos\theta & -e^{\xi}\sin\theta & 0 \\ -e^{\xi}\sin\theta & 1 + e^{\xi}\cos\theta & 0 \\ e^{\xi}\sin\theta & -1 - e^{\xi}\cos\theta & z'(c) \end{pmatrix}. \tag{3.4}$$

The determinant of the matrix equals $(1 - e^{2\xi})z'(c)$. Due to the time-independence of the determinant, the fluid flow is volume-preserving, so that the Lagrangian form of the equation of continuity is fulfilled. From (3.1) we compute the velocity and acceleration of a particle as

$$\begin{cases} u = \frac{Dx}{Dt} = -U - c_0 e^{\xi}\cos\theta, \\ v = \frac{Dy}{Dt} = -c_0 e^{\xi}\sin\theta, \\ w = \frac{Dz}{Dt} = 0, \end{cases} \quad \begin{cases} \frac{Du}{Dt} = kc_0^2 e^{\xi}\sin\theta, \\ \frac{Dv}{Dt} = -kc_0^2 e^{\xi}\cos\theta, \\ \frac{Dw}{Dt} = 0. \end{cases} \tag{3.5}$$

Since

$$\begin{pmatrix} P_a \\ P_b \\ P_c \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} \tag{3.6}$$

and in view of (3.4), (2.1) can be written as

$$P_x = -\rho k c_0^2 e^{\xi}\sin\theta, \tag{3.7}$$

$$P_y = \rho k c_0^2 e^{\xi}\cos\theta - \rho g \sin\alpha, \tag{3.8}$$

$$P_z = -\rho g\cos\alpha, \tag{3.9}$$

we have that (3.6) is the same as

$$P_a = -\rho(kc_0^2 - g\sin\alpha)e^{\xi}\sin\theta, \tag{3.10}$$

$$P_b = \rho(kc_0^2 - g\sin\alpha)e^{\xi}\cos\theta + \rho k c_0^2 e^{2\xi} - \rho g\sin\alpha, \tag{3.11}$$

$$P_c = -\rho(kc_0^2 - g\sin\alpha)e^{\xi}\cos\theta - \rho k c_0^2 e^{2\xi} - \rho g\cos\alpha + \rho g(\cos\alpha + \sin\alpha)e^{2kb_0}e^{-2kc(1+\cot\alpha)}. \tag{3.12}$$

Now, the gradient of the expression

$$\begin{aligned} P = P_0 + \rho\frac{(kc_0^2 - g\sin\alpha)}{k}e^{\xi}\cos\theta + \frac{\rho g\sin\alpha}{2k}e^{2\xi} - \rho g(c\cos\alpha + (b-b_0)\sin\alpha) \\ -\frac{\rho g\sin\alpha}{2k}e^{2kb_0}e^{-2kc(1+\cot\alpha)} \end{aligned} \tag{3.13}$$

with respect to the labeling variables is precisely the right-hand side of (3.10)-(3.12). Since the boundary condition (i) requires the pressure to be time independent on the surface, there can be no terms containing θ and the dispersion relation

$$kc_0^2 - g\sin\alpha = 0 \tag{3.14}$$

must hold. Thus the propagation speed of the edge waves is $c_0 = \sqrt{g\sin\alpha}/k$. This means that the uniform longshore current does not have an influence on the dispersion relation of the edge wave. Therefore the flow determined by (3.1) satisfies the governing equations (2.1). At the free surface $c = (b_0 - b)\tan\alpha$, we have $P = P_0$ so that the dynamic boundary condition (i) is satisfied. The kinematic boundary condition (ii) at the free surface is also satisfied as at any instance the free surface (3.2) is the image of the still water surface $c = (b_0 - b)\tan\alpha$ under (3.1) for all $t \geq 0$. Since the velocity of field is given by (3.4), the fluid velocity has no normal component to the beach when z=0, which proves (iii). The proof that (3.1) is an explicit solution to the governing equations for water waves with uniform longshore current on a sloping beach is completed.

4. Discussion.

4.1. *Vorticity.* In this section we will compute the vorticity of the flow (3.1). From (3.4) we get

$$\begin{pmatrix} \frac{\partial a}{\partial x} & \frac{\partial b}{\partial x} & \frac{\partial c}{\partial x} \\ \frac{\partial a}{\partial y} & \frac{\partial b}{\partial y} & \frac{\partial c}{\partial y} \\ \frac{\partial a}{\partial z} & \frac{\partial b}{\partial z} & \frac{\partial c}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{1+e^{\xi}\cos\theta}{1-e^{2\xi}} & \frac{e^{\xi}\sin\theta}{1-e^{2\xi}} & 0 \\ \frac{e^{\xi}\sin\theta}{1-e^{2\xi}} & \frac{1-e^{\xi}\cos\theta}{1-e^{2\xi}} & 0 \\ 0 & \frac{1}{z'(c)} & \frac{1}{z'(c)} \end{pmatrix}. \tag{4.1}$$

Thus the vorticity can be straightforwardly calculated as

$$\omega = (w_y - v_z, u_z - w_x, v_x - u_y) = -\begin{pmatrix} 0, & 0, & \frac{2kc_0e^{2\xi}}{1-e^{2\xi}} \end{pmatrix}. \tag{4.2}$$

We note that since the current is constant, it does not impact on the vorticity of the flow, and the prescription of the vorticity matches that of Constantin [12].

4.2. *Run-up.* The run-up is obtained by setting z=0 in (3.2) so that we have

$$\begin{cases} x = a - Ut - \frac{1}{k}e^{kb_0}\sin[k(a + c_0t)], \\ y = b_0 + \frac{1}{k}e^{kb_0}\cos[k(a + c_0t)], \\ z = 0. \end{cases} \tag{4.3}$$

The above formula shows that while the uniform longshore current influences the parametric representation of the run-up, it does not affect the amplitude of the edge wave.

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