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DYNAMICAL STABILITY OF THE TRAIN OF SMOOTH SOLITARY WAVES TO THE GENERALIZED TWO-COMPONENT CAMASSA-HOLM SYSTEM

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Abstract. The present study is concerned with the stability of solitary waves for the generalized two-component Camassa-Holm system derived formally as a model in the shallow-water waves. Using the property of almost monotonicity and the local coercivity of the solitary-wave solution, it is shown that the train of N-smooth solitary waves of this system is dynamically stable to perturbations in energy space with a range of parameters.

1. Introduction. Consideration herein is the generalized two-component Camassa-Holm (g2CH) system arising from the shallow-water theory with nonzero constant vorticity [8, 28], namely,

$$\begin{cases} u_t - u_{xxt} - Au_x + 3uu_x - \sigma\left(2u_xu_{xx} + uu_{xxx}\right) + \rho\rho_x = 0, \quad t > 0, x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, \quad t > 0, x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u(x,t)$ is connected with the horizontal velocity and $\rho(x,t)$ is related to the free surface elevation from equilibrium (or scalar density) with the boundary conditions $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$. The scalar constant $A > 0$ features a linear underlying shear flow, which implies that the system (1.1) describes the interaction between surface gravity wave and a mean flow. The parameter σ, a real dimensionless constant, provides the competition in the fluid convection between nonlinear steepening and the amplification due to stretching.

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This system contains several classical shallow-water wave models. If $\rho \neq 0$ and $\sigma = 1$, it becomes the two-component Camassa-Holm (2CH) system [13, 28, 38, 40],

$$\begin{cases} u_t - u_{xxt} - Au_x + 3uu_x - (2u_x u_{xx} + uu_{xxx}) + \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{1.2}$$

System (1.2) is formally integrable [28, 40] since it can be written as a compatibility condition of two linear systems (Lax pair) and has a bi-Hamiltonian structure, as it can be obtained through recombining the tri-Hamiltonian structure of the Ito system [38]. This system is rigorously derived in the context of shallow water theory [13, 28]. The well-posedness for system (1.2) has been the subject of a number of works [23, 25–27]. It was also shown in [36, 44] that the system captures a smooth solitary wave with a single crest profile and with exponential decay at infinity. The stability issue of the solitary wave to the system had been studied in [44].

When $\rho = 0$, system (1.1) is reduced to a scalar equation,

$$u_t - u_{xxt} + 3uu_x - \sigma\left(2u_x u_{xx} + uu_{xxx}\right) = 0, \tag{1.3}$$

which models finite length, small amplitude nonlinear axisymmetric waves in a hyperelastic cylindrical rod [18]. In particular, setting $\rho = 0, \sigma = 0$ in system (1.1), it turns out to be the Benjamin-Bona-Mahony (BBM) equation [2]

$$u_t - u_{xxt} - Au_x + 3uu_x = 0, \tag{1.4}$$

which can be regarded as an improvement of the Korteweg-de Vries (KdV) equation for modeling long surface gravity waves of small amplitude even if it is not integrable and only has finitely many conservation laws [2, 37]. The proofs of the stability of solitary waves on the KdV equation and the BBM equation were given in [1, 2, 39]. Moreover, the stability results of the sum of N solitary waves for the generalized KdV equation and the generalized BBM equations were shown in [22, 34].

The most well-known relative of system (1.1) is the Camassa-Holm (CH) equation [6, 24] while letting $\rho = 0$ and $\sigma = 1$, namely,

$$u_t - u_{xxt} - Au_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1.5}$$

which is a model derived for the unidirectional propagation of long water waves in the shallow-water approximation to the Euler equations of inviscid incompressible fluid flow [6, 14, 29]. The equation is completely integrable using inverse scatter transformation [12] and inverse spectral transformation [10, 15], and it has bi-Hamiltonian structure [6, 38].

Much attention has been given to equation (1.5) for more than two decades because it inherits some properties of the KdV equation [30, 31, 35] but is not limited to the smooth solitary wave, which exactly answers the question raised by Whitham [42]. A breakthrough of the CH equation (1.5) is that, when $A = 0$, it admits peaked solitary waves [6, 32]. The wave profile of the so-called "peakon" is shaped like $u(t, x) = ce^{-|x-ct|}, c \neq 0$, which has largest amplitude c and speed c. The first derivative u_x is smooth except at the peak, where it has a jump discontinuity. Another important feature is that the CH equation has wave breaking phenomena [6, 11, 33], i.e. the solution remains bounded

while its slope becomes unbounded in finite time. In [4] and [5] the authors show that the solutions can be uniquely continued after breaking either global conservative or global dissipative weak solutions.

In view of the similar property between the CH equation and the KdV equation that solitons preserve their sizes and velocities as a result of collision [3, 7, 41, 43], it is natural to study the stability of these solitons with peaked traveling waves and wave breaking under certain well-posedness results. Unfortunately, the theory suitable for those models retaining smooth solitary waves is inapplicable to the CH equation with a peakon soliton. This gap is filled by using the estimate of conservation laws in [16]. Later on, the stability property on a single peakon is extended to multi-peakons and multi-antipeakon-peakons [19, 20]. For stability results on a smooth soliton of the CH equation with $A \neq 0$, we refer the readers to [17, 21] and related articles.

Our motivation in the present paper is to investigate the stability issue for multi-solitary waves to system (1.1) with an effect of the parameter σ. It was shown in [9] that there exist smooth solitary waves $\boldsymbol{\varphi}_c$ (defined in Definition 2.3, Section 2) of system (1.1) if the parameter $\sigma \leq 1$, $c > \frac{-A+\sqrt{A^2+4}}{2}$ and these single solitary waves are orbitally stable in the energy space by spectral analysis. Inspired by an approach in [21] to deal with stability of N-smooth solitary waves of the CH equation, our goal in this article is to study the orbital stability of a train of N-smooth solitary waves in the dynamics of the g2CH system.

For integrability, we will adopt another form of the g2CH system (1.1), where $\eta \overset{\text{def}}{=} \rho - 1$, that is,

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma\left(2u_xu_{xx} + uu_{xxx}\right) + (1+\eta)\,\eta_x = 0, \\ \eta_t + \left((1+\eta)u\right)_x = 0. \end{cases} \tag{1.6}$$

In view of the conservation laws for system (1.6),

$$E(u,\eta) = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + u_x^2 + \eta^2\right)\, dx, \tag{1.7}$$

$$F(u,\eta) = \frac{1}{2}\int_{\mathbb{R}} \left(u^3 + \sigma u u_x^2 + 2u\eta + u\eta^2 - Au^2\right)\, dx, \tag{1.8}$$

we expect the orbital stability of N-smooth solitary waves of the g2CH system in the sense of the energy space $X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Note that existence for the smooth solitary waves and their stability naturally rely on the particular σ and A. By choosing $\sigma = 1$, the orbital stability of N-smooth solitary waves for system (1.2) is included while discussing the same topic on the g2CH system.

The N-smooth solitary waves $\sum_{i=1}^{N} \boldsymbol{\varphi}_{c_i}$ of system (1.6) can be shown to be orbitally stable in the energy space X, which is the principal result of the present paper.

THEOREM 1.1 (Main result). Suppose $\sigma \leq 1$. Let $c_1, c_2, \cdots, c_N$ be N speeds, such that $0 < \max\{\frac{-A+\sqrt{A^2+4}}{2}, \frac{A+\sqrt{A^2+2}}{2}\} < c_1 < \cdots < c_N$. Let $s > \frac{3}{2}$. If $M > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$, such that for any $\boldsymbol{u_0} = (u_0, \eta_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ it satisfies

$$\left\| \boldsymbol{u}_0 - \sum_{i=1}^{N} \boldsymbol{\varphi}(\cdot - z_j^0) \right\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} \leq \varepsilon, \tag{1.9}$$

for some $0 < \varepsilon < \varepsilon_0$ and $z_j^0 - z_{j-1}^0 \geq L$ with $L \geq L_0$, then for the corresponding solution $\boldsymbol{u} = (u, \eta) \in C([0,T]; H^s \times H^{s-1}) \cap C^1([0,T]; H^{s-1} \times H^{s-2})$ of the Cauchy problem for the g2CH system (1.6) with initial data $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$, there exist $\tilde{x}_1(t), \tilde{x}_2(t), \cdots, \tilde{x}_N(t) \in \mathbb{R}$, such that

$$\sup_{0<t<T} \left\| \boldsymbol{u}(t,\cdot) - \sum_{i=1}^{N} \boldsymbol{\varphi}(\cdot - \tilde{x}_i(t)) \right\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})} \leq M(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \tag{1.10}$$

where T depends only on initial data $\boldsymbol{u_0}$.

REMARK 1.2. It is noted that the assumptions in Theorem 1.1 ensure that the train of N-smooth solitary waves of different speeds is arranged in increasing order and sufficiently decoupled and as time goes by the phase between two solitary waves will be enlarged.

REMARK 1.3. In Theorem 1.1, the life-time span T may be infinite if $\sigma = 0$, since we have the global solution in that situation according to Proposition 2.2.

To establish the stability result for N-smooth solitary waves of the g2CH system, we develop the approaches employed in [21] for the CH equation and combine them with the general strategy initiated in [34] for the generalized KdV case. It is worth recalling that the general framework requires three principal ingredients: a modulation argument, almost monotonicity and local coercivity. Note that the stability issue of the multi-solitons of system (1.6) is more subtle than the CH case, because of the interaction between the two components u and η of solution $\boldsymbol{u}$ in system (1.6). To deal with these difficulties, one needs to provide a new and effective approach to control the superposition of the error terms with respect to the solution $\boldsymbol{u}$ and each smooth solitary wave $\boldsymbol{\varphi}_{c_i}$ in the local energy space. Intrigued by the classical method, we expect to find the orbit of N-smooth solitary waves by the modulation argument, i.e. for each instant t there exists a series of translations $\{\tilde{x}_i\}_{i=1}^N$ such that $\sum_{i=1}^N \boldsymbol{\varphi}_{c_i}(x - \tilde{x}_i)$ lies close to $u(x)$ in X-norm under certain assumptions. Fortunately, the implicit function theorem, which is crucial to the modulation argument, works regardless of the dimension of the space and also provides a new orthogonal condition connected with both elements in the remaining term of the solution. For the property of almost monotonicity, a more delicate work is needed to compare with the CH equation, not only because the energy function is in terms of both u and η, but also because it depends on the two parameters σ and $A > 0$. As to the local coercivity, we focus on the second differential operator of $cE - F$ around a solitary wave $\boldsymbol{\varphi}_c$ which is a 2×2 matrix. In order to apply the spectral analysis to this operator, restriction on σ is required. Hence, as long as one can extend the proof of these ingredients from a scalar to a vector, and choose the parameters properly, it is possible to establish Theorem 1.1.

The remainder of this paper is organized as follows. In Section 2, we recall some results on the existence and properties of solitary waves. Section 3 is the main part of this paper which contains two subsections. Firstly, we introduce a proposition which directly infers the main theorem and prove three properties on the modulation argument, the almost monotonicity for localized energy functionals and the local coercivity of the second differential operator around the solitary wave solution of this system. Secondly, extra three lemmas on the local estimate are introduced through the application of Taylor

expansion and conservation laws. Then we end with the proof of orbital stability of the train of N-smooth solitary waves for the g2CH system.

NOTATION. The Lebesgue space $L^p(\mathbb{R}), 1 \le p \le \infty$, is denoted by L^p and the Sobolev space $H^s(\mathbb{R})$ is denoted by H^s with the functional space on $\mathbb{R}$. The corresponding norms are $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^s}$, respectively. The energy space is simply represented by $X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$. The inner product $(\cdot,\cdot)$ defined on X is $(\boldsymbol{u_1}, \boldsymbol{u_2})_X = (u_1, u_2)_{H^1(\mathbb{R})} + (\eta_1, \eta_2)_{L^2(\mathbb{R})}$, for $\boldsymbol{u_1} = (u_1, \eta_1), \boldsymbol{u_2} = (u_2, \eta_2) \in X$ with the norm $\|\boldsymbol{u}\|_X^2 = \|u\|_{H^1(\mathbb{R})}^2 + \|\eta\|_{L^2(\mathbb{R})}^2$.

2. Preliminary. In this section, we first recall the local well-posedness result for the Cauchy problem of the two-component CH system on the real line, that is,

$$\begin{cases} u_t - u_{xxt} - Au_x + 3uu_x - \sigma\left(2u_x u_{xx} + uu_{xxx}\right) + (1+\eta)\,\eta_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ \eta_t + \left((1+\eta)\,u\right)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ (u(x,0), \eta(x,0)) = (u_0(x), \eta_0(x))\,. \end{cases} \tag{2.1}$$

Denote

$$p(x) := \frac{1}{2} e^{-|x|}, \quad x \in \mathbb{R}.$$

Then for all $f \in L^2(\mathbb{R})$,

$$(1 - \partial_x^2)^{-1} f = p * f, \tag{2.2}$$

where '$*$' denotes the spatial convolution. Using this notation, system (1.6) can now be written in the following form:

$$\begin{cases} u_t + \sigma u u_x + \partial_x p * \left(-Au + \frac{3-\sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{1}{2}\left(1+\eta\right)^2\right) = 0, \\ \eta_t + \left((1+\eta)u\right)_x = 0. \end{cases} \tag{2.3}$$

Logically, prior to a discussion of stability as formulated above in terms of perturbations of the initial data there should be a theory for the initial-valued problem itself. This is a subject that has attracted a lot of attention and it is not our purpose to provide a survey of these results. The following local existence result suffices for the stability theory developed here (see [8]). More subtle results are available in some cases, but these do not concern us here.

PROPOSITION 2.1 ([8]). Let $(u_0, \eta_0) \in H^s \times H^{s-1}$, $s > \frac{3}{2}$. Then there exist a maximal time $T = T(u_0, \eta_0) > 0$ and a unique solution (u, η) of (2.1) in $C([0,T); H^s \times H^{s-1}) \cap C^1([0,T); H^{s-1} \times H^{s-2})$. Moreover, the solution depends continuously on the initial data and T is independent of s. In addition, the functionals $E(u,\eta)$ and $F(u,\eta)$ defined in (1.7) and (1.8) are independent of the existence time t.

In the case $\sigma = 0$, we have the following global existence of the solutions.

PROPOSITION 2.2 ([9]). Let $\sigma = 0$. If $(u_0, \eta_0) \in H^s \times H^{s-1}$ with $s > \frac{3}{2}$, then there exists a unique global solution (u, η) of (2.1) in $C\big([0,\infty); H^s \times H^{s-1}\big) \cap C^1\big([0,\infty); H^{s-1} \times H^{s-2}\big)$.

Moreover, the solution depends continuously on the initial data. In addition, the functionals $E(u,\eta)$ and $F(u,\eta)$ defined in (1.7) and (1.8) are independent of the existence time t.

Next, we provide the definition of a solitary wave of system (2.3) and the condition for the existence of solitary waves.

DEFINITION 2.3. A vector function is a solitary wave of (2.3) if it has the form

$$\boldsymbol{\varphi}_c(x,t) = (\varphi_c(x-ct), \psi_c(x-ct)) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}), \quad c \in \mathbb{R},$$

with φ_c and ψ_c vanishing at infinity along with their first and second derivatives.

One can check that a solitary wave of (2.3) satisfies

$$\begin{cases} \Big(-c\varphi + \frac{\sigma}{2}\varphi^2 + p * \big(-A\varphi + \frac{3-\sigma}{2}\varphi^2 + \frac{\sigma}{2}\varphi_x^2 + \frac{1}{2}(1+\psi)^2\big)\Big)_x = 0, \\ \Big(-c\psi + (1+\psi)\varphi\Big)_x = 0. \end{cases} \tag{2.4}$$

Integrating the above system and applying $\left(1-\partial_x^2\right)$ to the first equation, we have

$$\begin{cases} -(c+A)\varphi + c\varphi_{xx} + \frac{3}{2}\varphi^2 = \sigma\varphi\varphi_{xx} + \frac{\sigma}{2}\varphi_x^2 - \frac{1}{2}(1+\psi)^2 + \frac{1}{2}, \\ -c\psi + (1+\psi)\varphi = 0. \end{cases} \tag{2.5}$$

The following result on the existence of solitary waves of (2.3) was given in [9].

PROPOSITION 2.4 ([9]). Let $\sigma \le 1$ and assume $c > \frac{-A+\sqrt{A^2+4}}{2}$. Then there exists a smooth solitary wave $\boldsymbol{\varphi}_c = (\varphi_c, \psi_c)$ of (2.3), which decays exponentially to zero at infinity.

REMARK 2.5. The smooth solitary wave $\boldsymbol{\varphi}_c$ exists for either $c > A_1$ or $c < A_2$, where A_1 and A_2 are two roots of the equation $y^2 + Ay - 1 = 0$. Without loss of generality, we only consider the case $c > A_1 = \frac{-A+\sqrt{A^2+4}}{2}$ in this article.

Finally, we introduce those results associated with the stability of the single smooth solitary wave of (2.3). Beginning with the dual space of X: $X^* = H^{-1}(\mathbb{R}) \times L^2(\mathbb{R})$, we can define a natural isomorphism I from X to X^*:

$$I = \begin{pmatrix} 1-\partial_x^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then combining the map I and the definition of the $H^1(\mathbb{R})$ norm and $L^2(\mathbb{R})$ norm, the pair $\langle\cdot,\cdot\rangle$ between X and X^* can be presented as

$$\langle I\boldsymbol{u}, \boldsymbol{v}\rangle = (u,v)_{H^1(\mathbb{R})} + (\eta, w)_{L^2(\mathbb{R})},$$

where $\boldsymbol{u} = (u,\eta) \in X, \boldsymbol{v} = (v,w) \in X^*$.

According to the definition of the norm in X, the quantity $E(\boldsymbol{u})$, which is an invariant functional of (2.1) with respect to time, can be written as

$$E(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{u},\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u}\|_X^2, \quad u \in X.$$

Furthermore, taking advantage of the conservation law $F(\boldsymbol{u})$, system (2.3) has following abstract Hamiltonian form:

$$\partial_t \boldsymbol{u} = JF'(\boldsymbol{u}), \tag{2.6}$$

where J is a closed skew symmetric operator given by

$$J = \begin{pmatrix} -\partial_x(1-\partial_x^2)^{-1} & 0 \\ 0 & -\partial_x \end{pmatrix},$$

and $F'(\boldsymbol{u}) : X \to X^*$ is the variational derivative of F in X at $\boldsymbol{u}$.

Attention is now given to the definition of orbital stability of a single solitary wave.

DEFINITION 2.6. The solitary wave $\boldsymbol{\varphi}_c$ of (2.3) is stable in X if for any $\varepsilon > 0$, there exists $\delta > 0$ and $s_0 \in \mathbb{R}$ such that for any $\boldsymbol{u}_0 \in X$ satisfying

$$\|\boldsymbol{u}_0 - \boldsymbol{\varphi}_c(\cdot - s_0)\|_X < \delta,$$

and if $\boldsymbol{u} \in C([0,T); X)$ for some $0 < T \leq \infty$ is a solution of (2.3) with $\boldsymbol{u}(0) = \boldsymbol{u}_0$, then

$$\inf_{s\in\mathbb{R}} \|\boldsymbol{u}(t) - \boldsymbol{\varphi}_c(\cdot - s)\|_X < \varepsilon, \qquad \forall\, t \in [0,T).$$

Otherwise, the solitary wave $\boldsymbol{\varphi}_c$ is said to be unstable in X.

One key to the stability of a single solitary wave is the coercivity of H_c which is the second differential operator of $cE - F$ around single solitary wave $\boldsymbol{\varphi}_c$. Here we rephrase their results and refer the readers to [9] for details.

LEMMA 2.7 ([9]). Assume $\sigma \leq 1$ and $c > A_1$. Let $\boldsymbol{\varphi}_c(x,t)$ be a smooth solitary wave of (2.3). Then there exists a constant $k = k(c) > 0$ such that

$$\langle H_c(\boldsymbol{\psi}), \boldsymbol{\psi}\rangle \geq k \|\boldsymbol{\psi}\|_X^2, \tag{2.7}$$

for all $\boldsymbol{\psi} \in X$ satisfying $(\boldsymbol{\varphi}_c, \boldsymbol{\psi}) = (\boldsymbol{\varphi}_c', \boldsymbol{\psi}) = 0$.

LEMMA 2.8 ([9]). Let $\sigma \leq 1$ and $c > A_1$. All those smooth solitary waves of (2.3) are orbitally stable in the energy space X.

REMARK 2.9. The exact form of H_c in (2.7) is

$$H_c = cE''(\boldsymbol{\varphi}_c) - F''(\boldsymbol{\varphi}_c) = \begin{pmatrix} L_c & -(1+\eta) \\ -(1+\eta) & c-\varphi \end{pmatrix},$$

where

$$L_c = -\partial_x\left((c - \sigma\varphi)\partial_x\right) - 3\varphi + \sigma\varphi_{xx} + c + A.$$

The proofs of Lemma 2.7 and Lemma 2.8 are based on the spectrum analysis of H_c and the convexity of $d(c) = cE(\boldsymbol{\varphi}_c) - F(\boldsymbol{\varphi}_c)$, both requiring $\sigma \leq 1$. In the following section, we will show the property of local coercivity via Lemma 2.7. Therefore, $\sigma \leq 1$ is a prerequisite to establish the stability of the train of N-smooth solitary waves.

3. Orbital stability of the trains of N solitary waves. The goal of this section is to show the orbital stability of the trains of N-smooth solitary waves with the assumption on the initial profile which guarantees the existence of a solution. For $\alpha > 0$ and $L > 0$, we define the following neighborhood of N-smooth solitary waves $\sum_{i=1}^N \boldsymbol{\varphi}_{c_i}$ with spatial shifts x_i that satisfies $x_i - x_{i-1} \geq L$ for every $i = 1, \cdots, N$,

$$U(\alpha, L) = \left\{ \boldsymbol{u} \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \inf_{x_i - x_{i-1} > L} \left\| \boldsymbol{u} - \sum_{i=1}^N \boldsymbol{\varphi}_{c_i}(\cdot - x_i) \right\|_X < \alpha \right\}.$$

We want to prove that there exist $M > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ such that for any $L > L_0$, $0 < \varepsilon < \varepsilon_0$ and any initial data $\boldsymbol{u}_0 \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), s > \frac{3}{2}$ satisfies initial condition (1.9). Then the corresponding solution $\boldsymbol{u}(t)$ belongs to $U\left(M(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{L}{2}\right)$ for all $t \in [0, T)$, where T is the maximal existence time and M is independent of time t. By the continuity of the map $t \mapsto \boldsymbol{u}(t)$ from $[0, T)$ into $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ where $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \subset H^1(\mathbb{R}) \times L^2(\mathbb{R})$, one can demonstrate Theorem 1.1 as long as the following proposition is proved.

PROPOSITION 3.1. Let $\sigma, c_1, c_2, \cdots, c_N$ meet the assumption in Theorem 1.1. There exist $M > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\boldsymbol{u}_0 \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > \frac{3}{2}$, if

$$\left\| \boldsymbol{u}_0 - \sum_{i=1}^{N} \boldsymbol{\varphi}(\cdot - z_j^0) \right\|_X \leq \varepsilon,$$

for some $0 < \varepsilon < \varepsilon_0$ and $z_j^0 - z_{j-1}^0 \geq L$ with $L \geq L_0$ and if for some $0 < t^* < T$, the solution $\boldsymbol{u} \in C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2})$ of Cauchy problem (2.1) with initial value $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$ satisfies

$$\boldsymbol{u}(t) \in U\left(M(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{L}{2}\right), \quad \forall t \in [0, t^*]. \tag{3.1}$$

Then

$$\boldsymbol{u}(t^*) \in U\left(\frac{M}{2}(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{2L}{3}\right), \tag{3.2}$$

where M, L_0 and ε_0 are independent of t^*.

We briefly prove that Proposition 3.1 implies the stability result in Theorem 1.1. Let M, L_0, ε_0 be chosen as in Proposition 3.1 and let $\boldsymbol{u}_0$ satisfy the initial condition (1.9). Then, by continuity of $\boldsymbol{u}(t) \in H^s \times H^{s-1}$, we have $\boldsymbol{u}(t) \in U\left(M(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{L}{2}\right)$ for $t \in [0, \kappa]$ for some $\kappa > 0$. Let

$$t^* = \sup\left\{ t \geq 0, \boldsymbol{u}(t') \in U\left(M(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{L}{2}\right), \forall t' \in [0, t] \right\}.$$

Assume $t^* < T$. According to Proposition 3.1, we have that $\forall t \in [0, t^*], \boldsymbol{u}(t) \in U\left(\frac{M}{2}(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{2L}{3}\right)$. By continuity, there exists $\tau > 0$ such that $\forall t \in [0, t^* + \tau], \boldsymbol{u}(t) \in U\left(\frac{M}{2}(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{2L}{3}\right)$, which contradicts the definition of t^*. Hence, we conclude $t^* \geq T$ which implies the stability consequence of the trains of N-smooth solitary waves.

3.1. *Modulation argument, almost monotonicity and local coercivity.* In this subsection, three properties of system (2.3) are established. We start with the modulation argument that as long as $\boldsymbol{u}$ stays in the neighborhood $U(\alpha, \frac{L}{2})$ of the sum of N modulated solitary waves, where $\alpha = O(\sqrt{\varepsilon} + L^{-\frac{1}{8}})$, it can be decomposed as the sum of N modulated solitary waves plus a vector function $\boldsymbol{v} = (v, w)$ which is an infinitesimal in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$; that is,

$$\begin{cases} u(t, x) = \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \tilde{x}_i(t)) + v(t, x), \\ \eta(t, x) = \sum_{i=1}^{N} \psi_{c_i}(\cdot - \tilde{x}_i(t)) + w(t, x). \end{cases}$$

LEMMA 3.2. Let $\alpha_0 > 0$ and $L_0 > 0$. For any $0 < \alpha < \alpha_0$ and $L > L_0$, if the initial data $\boldsymbol{u}_0 = (u_0, \eta_0)$ satisfies the assumption (1.9) given in Theorem 1.1 and the solution $\boldsymbol{u} \in U(\alpha, \frac{L}{2})$ on $[0, t^*]$, then there exist C^1 functions $\tilde{x}_i(t) : [0, t^*] \to \mathbb{R}, i = 1, \cdots, N$, and $t \in [0, t^*]$, such that

$$\int_{\mathbb{R}} v(t)(1 - \partial_x^2)\partial_x R_i(t)\, dx + \int_{\mathbb{R}} w(t)\partial_x P_i(t)\, dx = 0, \tag{3.3}$$

where $\boldsymbol{R}_i(t,x) = (R_i(t,x), P_i(t,x)) = (\varphi_{c_i}(x - \tilde{x}_i(t)), \psi_{c_i}(x - \tilde{x}_i(t))) = \boldsymbol{\varphi}_i(t,x)$,

$$\|\boldsymbol{v}(t)\|_X \le O(\sqrt{\alpha}), \tag{3.4}$$

$$\left|\dot{\tilde{x}}_i(t) - c_i\right| \le O(\sqrt{\alpha}) + O(L^{-1}), \tag{3.5}$$

$$\tilde{x}_i(t) - \tilde{x}_{i-1}(t) \ge \frac{3L}{4} + \frac{(c_i - c_{i-1})}{2} \cdot t, \quad i \ge 2, \tag{3.6}$$

where $\dot{\tilde{x}}_i(t)$ means the derivative of $\tilde{x}_i(t)$ respect to time t.

Proof. We use the standard modulation argument to discover the translations of N-smooth solitary waves. Let $Z = (z_1, \cdots, z_N) \in \mathbb{R}^N$ be fixed such that $z_i - z_{i-1} > \frac{L}{2}$ and set $\boldsymbol{R}_Z(\cdot) = \sum_{i=1}^N \boldsymbol{\varphi}_{c_i}(\cdot - z_i)$. For $0 < \delta_0 < 1$, we define the function

$$Y : \prod_{i=1}^N (-\delta_0, \delta_0) \times B_{H^1}(R_Z, \delta_0) \times B_{L^2}(P_Z, \delta_0) \to \mathbb{R}^N,$$

$$(y_1, \cdots, y_N, u, \eta) \longmapsto \left(Y^1(y_1, \cdots, y_N, u, \eta), \cdots, Y^N(y_1, \cdots, y_N, u, \eta)\right),$$

with

$$\begin{aligned} Y^j(y_1, \cdots, y_N, u, \eta) = &\int_{\mathbb{R}} \left(u - \sum_{i=1}^N \varphi_{c_i}(\cdot - z_i - y_i)\right)(1 - \partial_x^2)\partial_x \varphi_{c_j}(\cdot - z_j - y_j)\, dx \\ &+ \int_{\mathbb{R}} \left(\eta - \sum_{i=1}^N \psi_{c_i}(\cdot - z_i - y_i)\right)\partial_x \psi_{c_j}(\cdot - z_j - y_j)\, dx, \end{aligned}$$

where $B_{H^1}(R_Z, \delta_0)$ is the ball in $H^1(\mathbb{R})$ with center R_Z and radius δ_0, and $B_{L^2}(P_Z, \delta_0)$ is the ball in $L^2(\mathbb{R})$ with center P_Z and radius δ_0. To apply the implicit function theorem, two facts are needed. One fact is that function Y should be a C^1 mapping which can be proved by dominated convergence theorem. The other fact is that the matrix of all first-order partial derivatives of function Y at $(0, \cdots, 0, R_Z, P_Z)$ should be invertible. For $j = 1, \cdots, N$,

$$\begin{aligned} \frac{\partial Y^j}{\partial y_j}(y_1, \cdots, y_N, u, \eta) = &\int_{\mathbb{R}} \left(u_x - \sum_{i=1, i\ne j}^N \partial_x \varphi_{c_i}(\cdot - z_i - y_i)\right)(1 - \partial_x^2)\partial_x \varphi_{c_j}(\cdot - z_j - y_j)\, dx \\ &+ \int_{\mathbb{R}} \left(\eta_x - \sum_{i=1, i\ne j}^N \partial_x \psi_{c_i}(\cdot - z_i - y_i)\right)\partial_x \psi_{c_j}(\cdot - z_j - y_j)\, dx, \end{aligned}$$

$$\begin{aligned} \frac{\partial Y^j}{\partial y_i}(y_1, \cdots, y_N, u, \eta) = &\int_{\mathbb{R}} \partial_x \varphi_{c_i}(\cdot - z_i - y_i)(1 - \partial_x^2)\partial_x \varphi_{c_j}(\cdot - z_j - y_j)\, dx \\ &+ \int_{\mathbb{R}} \partial_x \psi_{c_i}(\cdot - z_i - y_i)\partial_x \psi_{c_j}(\cdot - z_j - y_j)\, dx, \quad \text{where} \quad i \ne j, \end{aligned}$$

$$\frac{\partial Y^j}{\partial u}(y_1,\cdots,y_N,u,\eta)=\int_{\mathbb{R}}(1-\partial_x^2)\partial_x\varphi_{c_j}(\cdot-z_j-y_j)\,dx,$$

$$\frac{\partial Y^j}{\partial \eta}(y_1,\cdots,y_N,u,\eta)=\int_{\mathbb{R}}\partial_x\psi_{c_j}(\cdot-z_j-y_j)\,dx.$$

Hence,

$$\frac{\partial Y^j}{\partial y_j}(0,\cdots,0,R_Z,P_Z)=\|\partial_x\varphi_{c_j}\|^2_{H^1(\mathbb{R})}+\|\partial_x\psi_{c_j}\|^2_{L^2(\mathbb{R})},$$

$$\begin{aligned}\frac{\partial Y^j}{\partial y_i}(0,\cdots,0,R_Z,P_Z)=&\left(\partial_x\varphi_{c_i}(\cdot-z_i),\partial_x\varphi_{c_j}(\cdot-z_j)\right)_{H^1(\mathbb{R})}\\&+\left(\partial_x\psi_{c_i}(\cdot-z_i),\partial_x\psi_{c_j}(\cdot-z_j)\right)_{L^2(\mathbb{R})}.\end{aligned}$$

Furthermore, there exists $L_0>0$ such that if $L>L_0$ large enough, we have

$$D_{(y_1,\cdots,y_N)}Y(0,\cdots,0,R_z,P_Z)=D+P\neq 0,$$

where D is an invertible diagonal matrix with

$$\|D^{-1}\|\leq\min_{j=1,\cdots,N}\left(\|\partial_x\varphi_{c_j}\|^2_{H^1(\mathbb{R})}+\|\partial_x\psi_{c_j}\|^2_{L^2(\mathbb{R})}\right)^{-1}$$

and P is a matrix with

$$\|P\|\leq O(e^{-\frac{L}{4}}),$$

which implies $D_{(y_1,\cdots,y_N)}Y(0,\cdots,0,R_Z,P_Z)$ is invertible. Therefore, by the implicit function theorem, there exist $0<\beta_0<\delta_0$ and C^1 functions $(y_1(u,\eta),\cdots,y_N(u,\eta))$ from $B(\boldsymbol{R}_Z,\delta_0)$ to a neighborhood of $(0,\cdots,0)$ which are uniquely determined, such that

$$Y(y_1(u,\eta),\cdots,y_N(u,\eta),u,\eta)=0,\quad\forall(u,\eta)\in B_{H^1}(R_Z,\beta_0)\times B_{L^2}(P_Z,\beta_0).$$

Moreover, there exists $K_0>0$ such that if $(u,\eta)\in B_{H^1}(R_Z,\beta_0)\times B_{L^2}(P_Z,\beta_0)$ with $0<\beta<\beta_0$ the following holds:

$$\sum_{i=1}^N|y_i(\boldsymbol{u})|=\sum_{i=1}^N|y_i(u,\eta)|\leq K_0\beta,\tag{3.7}$$

where K_0 and β_0 depend on c_1 and L_0. For $(u,\eta)\in B_{H^1}(R_Z,\beta_0)\times B_{L^2}(P_Z,\beta_0)$, setting $\tilde{x}_i(\boldsymbol{u})=z_i+y_i(\boldsymbol{u})$ and $\beta_0\leq\min\{\frac{L_0}{8K_0},\delta_0\}$ infers

$$\tilde{x}_i(\boldsymbol{u})-\tilde{x}_{i-1}(\boldsymbol{u})=z_i-z_{i-1}+y_i(\boldsymbol{u})-y_{i-1}(\boldsymbol{u})\geq\frac{L}{2}-2K_0\beta_0\geq\frac{L}{4}.\tag{3.8}$$

Then, we define the modulation of $\boldsymbol{u}=(u,\eta)\in U(\alpha,\frac{L}{2})$ for $L>L_0$ and $0<\alpha<\alpha_0$ at a fixed time t. Indeed, for $0<\alpha<\alpha_0$, $U(\alpha,\frac{L}{2})$ can be covered as follows:

$$U\left(\alpha,\frac{L}{2}\right)\subset\bigcup_{Z\in\mathbb{R}^N,z_i-z_{i-1}>\frac{L}{2}}B_{H^1}(R_Z,2\alpha)\times B_{L^2}(P_Z,2\alpha).$$

Additionally, the modulation of $\boldsymbol{u}$ is uniquely defined due to the uniqueness in the implicit function theorem.

Thus, we define the modulation of the solution $\boldsymbol{u}(t) = (u(t), \eta(t))$ of system (2.3) satisfying $\boldsymbol{u}(t) \in U(\alpha, \frac{L}{2})$ for all $t \in [0, t^*]$ by setting $i = 1, \cdots, N$ and

$$\tilde{x}_i(t) = \tilde{x}_i(\boldsymbol{u}(t)), \quad \boldsymbol{v}(t) = \boldsymbol{u}(t) - \sum_{i=1}^{N} \boldsymbol{\varphi}_{c_i}(\cdot - \tilde{x}_i(t)),$$

where $\boldsymbol{v}$ satisfies the orthogonal condition

$$\int_{\mathbb{R}} v(t)(1 - \partial_x^2)\partial_x R_i(t)\, dx + \int_{\mathbb{R}} w(t)\partial_x P_i(t)\, dx = 0.$$

According to the translation $\tilde{x}_i(t)$ defined above, using (3.7), triangle inequality and the smoothness of $\boldsymbol{\varphi}_{c_i}$, the following estimate holds:

$$\begin{aligned}
\|\boldsymbol{v}(t)\|_X \leq & \|\boldsymbol{u}(t) - \sum_{i=1}^{N} \boldsymbol{\varphi}_{c_i}(\cdot - z_i)\|_X + \sum_{i=1}^{N} \|\boldsymbol{\varphi}_{c_i}(\cdot - z_i) - \boldsymbol{\varphi}_{c_i}(\cdot - z_i - y_i(\boldsymbol{u}(t)))\|_X \\
\leq & \alpha + \sum_{i=1}^{N} \Bigg(4E(\boldsymbol{u}) - 2\int_{\mathbb{R}} \varphi_{c_i}(\cdot - z_i)\varphi_{c_i}(\cdot - z_i - y_i(\boldsymbol{u}(t)))\, dx \\
& - 2\int_{\mathbb{R}} \partial_x\varphi_{c_i}(\cdot - z_i)\partial_x\varphi_{c_i}(\cdot - z_i - y_i(\boldsymbol{u}(t)))\, dx \\
& - 2\int_{\mathbb{R}} \psi_{c_i}(\cdot - z_i)\psi_{c_i}(\cdot - z_i - y_i(\boldsymbol{u}(t)))\, dx \Bigg)^{\frac{1}{2}} \\
\leq & \alpha + O(\sum_{i=1}^{N} |y_i(\boldsymbol{u})|^{\frac{1}{2}}) \\
\leq & O(\sqrt{\alpha}).
\end{aligned}$$

Our attention is now turned to the speed of $\tilde{x}_i(t)$. In order to show it stays close to c_i, we differentiate the orthogonality condition with respect to t,

$$\begin{aligned}
\left| \int_{\mathbb{R}} v_t(1 - \partial_x^2)\partial_x R_i\, dx + \int_{\mathbb{R}} w_t \partial_x P_i\, dx \right| &= \left| \dot{\tilde{x}}_i \left(\int_{\mathbb{R}} v(1 - \partial_x^2)\partial_x^2 R_i\, dx + \int_{\mathbb{R}} w\partial_x^2 P_i\, dx \right) \right| \\
&\leq \left| \dot{\tilde{x}}_i \right| O(\|\boldsymbol{v}\|_X) \\
&\leq \left| \dot{\tilde{x}}_i - c_i \right| O(\|\boldsymbol{v}\|_X) + O(\|\boldsymbol{v}\|_X).
\end{aligned} \tag{3.9}$$

Substituting $\boldsymbol{u}$ by $\boldsymbol{v} + \sum_{i=1}^{N} \boldsymbol{R}_i$ in the system (2.3) leads to

$$\begin{aligned}
& \big(1 - \partial_x^2\big) v_t + \sum_{i=1}^{N} (1 - \partial_x^2)\partial_t R_i \\
& \quad = -\frac{\sigma}{2}(1 - \partial_x^2)\partial_x \left((v + \sum_{i=1}^{N} R_i)^2 \right) - \partial_x \Bigg(-A(v + \sum_{i=1}^{N} R_i) + \frac{3 - \sigma}{2}(v + \sum_{i=1}^{N} R_i)^2 \\
& \qquad + \frac{\sigma}{2}(v_x + \sum_{i=1}^{N} \partial_x R_i)^2 + (w + \sum_{i=1}^{N} P_i) + \frac{1}{2}(w + \sum_{i=1}^{N} P_i)^2 \Bigg),
\end{aligned} \tag{3.10}$$

and

$$w_t+\sum_{i=1}^{N}\partial_t P_i=-\partial_x\bigg((v+\sum_{i=1}^{N}R_i)+(v+\sum_{i=1}^{N}R_i)(w+\sum_{i=1}^{N}P_i)\bigg). \tag{3.11}$$

Since $\boldsymbol{R}_i$ satisfies (2.4), i.e.

$$\begin{aligned}(1-\partial_x^2)\partial_t R_i+(\dot{\tilde{x}}_i-c_i)(1-\partial_x^2)\partial_x R_i=&-\frac{\sigma}{2}(1-\partial_x^2)\partial_x(R_i^2)+A\partial_x R_i-\frac{3-\sigma}{2}\partial_x(R_i^2)\\&-\frac{\sigma}{2}(\partial_x R_i)^2-\partial_x P_i-P_i\partial_x P_i,\end{aligned} \tag{3.12}$$

and

$$\partial_t P_i+(\dot{\tilde{x}}_i-c_i)\partial_x P_i+\partial_x R_i+\partial_x(R_iP_i)=0, \tag{3.13}$$

combining (3.10)-(3.13), we infer that $\boldsymbol{v}(t)=(v(t),w(t))$ satisfies the following conditions on $[0,t^*]$:

$$\begin{aligned}&\big(1-\partial_x^2\big)v_t-\sum_{i=1}^{N}(\dot{\tilde{x}}_i-c_i)(1-\partial_x^2)\partial_x R_i\\&\quad=-\frac{\sigma}{2}(1-\partial_x^2)\partial_x\left[(v+\sum_{i=1}^{N}R_i)^2-\sum_{i=1}^{N}R_i^2\right]-\partial_x\left[\frac{3-\sigma}{2}\left((v+\sum_{i=1}^{N}R_i)^2-\sum_{i=1}^{N}R_i^2\right)\right.\\&\qquad\left.-Av+\frac{\sigma}{2}\left((v_x+\sum_{i=1}^{N}\partial_x R_i)^2-\sum_{i=1}^{N}(\partial_x R_i)^2\right)+w+\frac{1}{2}(w+\sum_{i=1}^{N}P_i)^2-\frac{1}{2}\sum_{i=1}^{N}P_i^2\right]\end{aligned} \tag{3.14}$$

and

$$w_t-\sum_{i=1}^{N}(\dot{\tilde{x}}_i-c_i)\partial_x P_i=-\partial_x\bigg(v+(v+\sum_{i=1}^{N}R_i)(w+\sum_{i=1}^{N}P_i)-\sum_{i=1}^{N}R_iP_i\bigg). \tag{3.15}$$

Take the L^2-inner product for (3.14) with $\partial_x R_i$ and (3.15) with $\partial_x P_i$, respectively. Then use integration by parts, the exponential decay of $\boldsymbol{R_i}$ and its derivatives to simplify these two equations. By (3.4),(3.8) and (3.9), letting α_0 small enough and $L_0>\sigma^2$ large enough, there holds

$$|\dot{\tilde{x}}_i-c_i|(\|\partial_x\boldsymbol{R}_i\|_X+O(\alpha))\le O(\sqrt{\alpha})+O(e^{-L}).$$

Consequently, we demonstrate (3.5).

Furthermore, by the assumption $z_j^0-z_{j-1}^0\ge L$ given in Theorem 1.1, it is deduced that

$$\begin{aligned}\tilde{x}_i(t)-\tilde{x}_{i-1}(t)&=\tilde{x}_i(0)-\tilde{x}_{i-1}(0)+\big(\dot{\tilde{x}}_i(s)-\dot{\tilde{x}}_{i-1}(s)\big)\,t\\&=\tilde{x}_i(0)-\tilde{x}_{i-1}(0)+\big(\dot{\tilde{x}}_i(s)-c_i+c_{i-1}-\dot{\tilde{x}}_{i-1}(s)\big)\,t+(c_i-c_{i-1})t\\&\ge\frac{3L}{4}+\frac{1}{2}(c_i-c_{i-1})t.\end{aligned}$$

Hence, we complete the proof of Lemma 3.2. □

Next, we prove the almost monotonicity of the functionals which describe the energy at the right of ith bump, for $i = 1, 2, \cdots, N$. To construct these functionals, we begin with the introduction of weight functions.

Let Ψ be a C^∞ function, such that

$$\begin{cases} 0 < \Psi(x) < 1,\ \Psi'(x) > 0, \quad x \in \mathbb{R}, \\ |\Psi'''(x)| \le 10\Psi'(x), \quad x \in [-1, 1] \end{cases} \tag{3.16}$$

and

$$\Psi(x) = \begin{cases} e^{-|x|}, & x < -1, \\ 1 - e^{-|x|}, & x > 1. \end{cases} \tag{3.17}$$

Set $\Psi_K = \Psi(\cdot/K)$, $K > 0$. Define the weight functions $\Phi_i = \Phi_i(t, x)$ by

$$\Phi_1 = 1 - \Psi_{2,K}, \qquad \Phi_N = \Psi_{N,K}, \qquad \Phi_i = \Psi_{i,K} - \Psi_{i+1,K}, \quad i = 2, \cdots, N-1, \tag{3.18}$$

where $\Psi_{i,K} = \Psi_K(x - y_i(t))$ with $y_i(t)$ defined by

$$y_1 = -\infty, \quad y_{N+1} = +\infty \quad \text{and} \quad y_i(t) = \frac{\tilde{x}_{i-1}(t) + \tilde{x}_i(t)}{2},\ i = 2, \cdots, N. \tag{3.19}$$

Obviously, $\sum_{i=1}^N \Phi_i(t, x) = 1$, for $t \in [0, t^*]$. Taking $L > 0$ and $L/K > 0$ large enough, we have

$$|1 - \Phi_i| \le 4e^{-\frac{L}{4K}}, \quad \text{on } \left[\tilde{x}_i - \frac{L}{4}, \tilde{x}_i + \frac{L}{4}\right], \tag{3.20}$$

and

$$|\Phi_i| \le 4e^{-\frac{L}{4K}} \quad \text{on } \left[\tilde{x}_j - \frac{L}{4}, \tilde{x}_j + \frac{L}{4}\right], \quad \text{for } j \ne i. \tag{3.21}$$

Then, we introduce the localized conservation laws of E and F in terms of weight functions, for $i = 1, \cdots, N$:

$$E_i(\boldsymbol{u}) = E_i(u, \eta) = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + u_x^2 + \eta^2\right) \Phi_i \, dx, \tag{3.22}$$

$$F_i(\boldsymbol{u}) = F_i(u, \eta) = \frac{1}{2}\int_{\mathbb{R}} \left(u^3 + \sigma u u_x^2 + 2u\eta + u\eta^2 - Au^2\right) \Phi_i \, dx. \tag{3.23}$$

Moreover, for simplicity, we set

$$\sigma_0 = \frac{1}{4}\min(c_1, c_2 - c_1, \cdots, c_N - c_{N-1}). \tag{3.24}$$

LEMMA 3.3. Let $\boldsymbol{u}$ be the solution of the system (2.3) such that $\boldsymbol{u} \in U(\alpha, \frac{L}{2})$ on $[0, t^*]$ where $\{\tilde{x}_i(t)\}_{i=1}^N$ are defined in Lemma 3.2. There exist $\alpha_0 > 0$ and $L_0 > 0$ depending on $\{c_i\}_{i=1}^N$ and σ such that if $0 < \alpha < \alpha_0$ and $L > L_0$, for $\max\{4, \sigma^2\} \le K = O(L^{1/2})$, it follows that

$$I_{j,K}(t) - I_{j,K}(0) \le O(e^{-\frac{L}{4K}}), \tag{3.25}$$

for all $j = 2, \cdots, N$ and $t \in [0, t^*]$, where $I_{j,K} = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + u_x^2 + \eta^2\right) \Psi_{j,K} \, dx$.

Proof. We know system (2.3) can be written in abstract Hamiltonian form as (2.6), that is,

$$\begin{pmatrix} u \\ \eta \end{pmatrix}_t = \begin{pmatrix} -\partial_x(1 - \partial_x^2)^{-1} & 0 \\ 0 & -\partial_x \end{pmatrix} \begin{pmatrix} F_u' \\ F_\eta' \end{pmatrix}, \tag{3.26}$$

where

$$F'_u = \frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \sigma u u_{xx} - Au + \eta + \frac{1}{2}\eta^2, \qquad F'_\eta = u + u\eta. \tag{3.27}$$

Fixing j, differentiating $I_{j,K}$ with respect to t and using (3.26), we get

$$\begin{aligned}
\frac{d}{dt}I_{j,K}(t) =& -\frac{\dot{y}_j}{2}\int_{\mathbb{R}}(u^2+u_x^2+\eta^2)\partial_x\Psi_{j,K}\,dx + \int_{\mathbb{R}}u(u_t-u_{txx})\Psi_{j,K}\,dx \\
&- \int_{\mathbb{R}}uu_{tx}\partial_x\Psi_{j,K}\,dx + \int_{\mathbb{R}}\eta\eta_t\Psi_{j,K}\,dx \\
=& -\frac{\dot{y}_j}{2}\int_{\mathbb{R}}(u^2+u_x^2+\eta^2)\partial_x\Psi_{j,K}\,dx - \int_{\mathbb{R}}u\Psi_{j,K}\partial_x F'_u\,dx \\
&+ \int_{\mathbb{R}}u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}\partial_x^2 F'_u\,dx - \int_{\mathbb{R}}\eta\Psi_{j,K}\partial_x F'_\eta\,dx \\
=& -\frac{\dot{y}_j}{2}\int_{\mathbb{R}}(u^2+u_x^2+\eta^2)\partial_x\Psi_{j,K}\,dx + \int_{\mathbb{R}}u_x\Psi_{j,K}F'_u\,dx \\
&+ \int_{\mathbb{R}}u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}F'_u\,dx + \int_{\mathbb{R}}\eta_x\Psi_{j,K}F'_\eta\,dx + \int_{\mathbb{R}}\eta\partial_x\Psi_{j,K}F'_\eta\,dx \\
:=& -\frac{\dot{y}_j}{2}\int_{\mathbb{R}}(u^2+u_x^2+\eta^2)\partial_x\Psi_{j,K}\,dx + J_1(t)+J_2(t)+J_3(t)+J_4(t).
\end{aligned} \tag{3.28}$$

Substituting (3.27) into (3.28) and using integration by parts, $J_1(t)$, $J_2(t)$, $J_3(t)$ and $J_4(t)$ become

$$\begin{aligned}
J_1(t) =& \int_{\mathbb{R}}u_x\Psi_{j,K}\left(\frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \sigma u u_{xx} - Au + \eta + \frac{1}{2}\eta^2\right)dx \\
=& -\frac{1}{2}\int_{\mathbb{R}}\partial_x\Psi_{j,K}u^3\,dx + \frac{\sigma}{2}\int_{\mathbb{R}}\partial_x\Psi_{j,K}uu_x^2\,dx + \frac{A}{2}\int_{\mathbb{R}}\partial_x\Psi_{j,K}u^2 + \int_{\mathbb{R}}\Psi_{j,K}u_x\eta\,dx \\
&+ \frac{1}{2}\int_{\mathbb{R}}\Psi_{j,K}u_x\eta^2\,dx,
\end{aligned}$$

$$\begin{aligned}
J_2(t) =& \int_{\mathbb{R}}u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}\left(\frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \sigma u u_{xx} - Au + \eta + \frac{1}{2}\eta^2\right)dx \\
=& \int_{\mathbb{R}}u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}\left(\frac{3}{2}u^2 + \frac{\sigma}{2}u_x^2 - \frac{\sigma}{2}(u^2)_{xx} - Au + \eta + \frac{1}{2}\eta^2\right)dx \\
=& \frac{\sigma}{2}\int_{\mathbb{R}}\partial_x\Psi_{j,K}u^3\,dx + \int_{\mathbb{R}}u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}\left(\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 - Au + \eta + \frac{1}{2}\eta^2\right)dx,
\end{aligned}$$

$$\begin{aligned}
J_3(t) =& \int_{\mathbb{R}}\eta_x\Psi_{j,K}\,(u+u\eta)\,dx \\
=& -\int_{\mathbb{R}}\Psi_{j,K}u_x\eta\,dx - \int_{\mathbb{R}}\partial_x\Psi_{j,K}u\eta\,dx - \frac{1}{2}\int_{\mathbb{R}}\Psi_{j,K}u_x\eta^2\,dx - \frac{1}{2}\int_{\mathbb{R}}\partial_x\Psi_{j,K}u\eta^2\,dx, \\
J_4(t) =& \int_{\mathbb{R}}\eta\partial_x\Psi_{j,K}\,(u+u\eta)\,dx = \int_{\mathbb{R}}\partial_x\Psi_{j,K}u\eta\,dx + \int_{\mathbb{R}}\partial_x\Psi_{j,K}u\eta^2\,dx.
\end{aligned}$$

Combining $J_1(t)$, $J_2(t)$, $J_3(t)$ and $J_4(t)$, we can simplify $J(t) = J_1(t)+J_2(t)+J_3(t)+J_4(t)$ as follows:

$$\begin{aligned} J(t) =& \frac{A}{2}\int_{\mathbb{R}} u^2\partial_x\Psi_{j,K}\,dx - A\int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}u\,dx \\ &+ \frac{1}{2}\int_{\mathbb{R}} u\partial_x\Psi_{j,K}\Big((\sigma-1)u^2+\sigma u_x^2+\eta^2\Big)\,dx + \int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}\eta\,dx \\ &+ \int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}\left(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2+\frac{1}{2}\eta^2\right)\,dx \\ =& \frac{A}{2}\int_{\mathbb{R}} u^2\partial_x\Psi_{j,K}\,dx - A\int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}u\,dx + Q_1(t)+Q_2(t)+Q_3(t). \end{aligned} \tag{3.29}$$

To deal with the first two terms in (3.29), we adopt the same trick in [21]. Setting $h = (1-\partial_x^2)^{-1}u$ infers

$$\begin{aligned} -A\int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}u\,dx =& -A\int_{\mathbb{R}} h\partial_x\Psi_{j,K}(1-\partial_x^2)h\,dx \\ =& -A\int_{\mathbb{R}}(h^2+h_x^2)\partial_x\Psi_{j,K}\,dx + \frac{A}{2}\int_{\mathbb{R}} h^2\partial_x^3\Psi_{j,K}\,dx. \end{aligned}$$

According to the definition of h, the following identity holds:

$$\int_{\mathbb{R}} u^2\Psi_{j,K}\,dx = \int_{\mathbb{R}}(h^2+h_{xx}^2+2h_x^2)\partial_x\Psi_{j,K}\,dx - \int_{\mathbb{R}} h^2\partial_x^3\Psi_{j,K}\,dx.$$

Then for $m > \frac{A}{2}$, we have

$$\begin{aligned} -A\int_{\mathbb{R}}\left(h^2+h_x^2\right)\partial_x\Psi_{j,K}\,dx =& \left(-\frac{A}{2}+m\right)\int_{\mathbb{R}}\left(h^2+h_x^2\right)\partial_x\Psi_{j,K}\,dx \\ &- \left(m+\frac{A}{2}\right)\int_{\mathbb{R}}\left(h^2+h_x^2\right)\partial_x\Psi_{j,K}\,dx \\ \le& \left(-\frac{A}{2}+m\right)\int_{\mathbb{R}} u^2\partial_x\Psi_{j,K}\,dx + \left(-\frac{A}{2}+m\right)\int_{\mathbb{R}} h^2\partial_x^3\Psi_{j,K}\,dx \\ &- \left(m+\frac{A}{2}\right)\int_{\mathbb{R}}\left(h^2+h_x^2\right)\partial_x\Psi_{j,K}\,dx, \end{aligned}$$

which implies

$$\begin{aligned} -A\int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}u\,dx \le& (-\frac{A}{2}+m)\int_{\mathbb{R}} u^2\partial_x\Psi_{j,K}\,dx + m\int_{\mathbb{R}} h^2\partial_x^3\Psi_{j,K}\,dx \\ &- (m+\frac{A}{2})\int_{\mathbb{R}} h^2\partial_x\Psi_{j,K}\,dx. \end{aligned}$$

It follows from (3.16) that for $K>0$, if

$$\frac{10m}{K^2} \le (m+\frac{A}{2}) \Leftrightarrow K \ge \sqrt{\frac{10m}{m+\frac{A}{2}}},$$

then

$$-A\int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}u\,dx \le (-\frac{A}{2}+m)\int_{\mathbb{R}} u^2\partial_x\Psi_{j,K}\,dx. \tag{3.30}$$

To consider the upper bound for Q_1, Q_2 and Q_3, we divide $\mathbb{R}$ into two parts: $D_j \cup D_j^c$, where $D_j = [\tilde{x}_{j-1} + \frac{L}{4}, \tilde{x}_j - \frac{L}{4}]$. The two crucial estimates related to D_j and D_j^c are listed in the following. For $x \in D_j$,

$$\begin{aligned} \|u(t)\|_{L^\infty(D_j)} &\le \sum_{i=1}^N \|\varphi_{c_i}(\cdot - \tilde{x}_i(t))\|_{L^\infty(D_j)} + \|u - \sum_{i=1}^N \varphi_{c_i}(\cdot - \tilde{x}_i(t))\|_{L^\infty(D_j)} \\ &\le O(e^{-\frac{L}{8}}) + O(\sqrt{\alpha}). \end{aligned} \tag{3.31}$$

For $x \in D_j^c$, taking advantage of (3.5), (3.19) and the definition of $\Psi_{j,K}$, we know

$$|x - y_j(t)| \ge \frac{\tilde{x}_j(t) - \tilde{x}_{j-1}(t)}{2} - \frac{L}{4} \ge \frac{(c_j - c_{j-1})t}{4} + \frac{L}{8} \ge \sigma_0 t + \frac{L}{8},$$

which implies, for $K = O(\sqrt{L})$ and sufficiently large L_0,

$$\left| \frac{x - y_j(t)}{K} \right| \ge \frac{\sigma_0 t + \frac{L}{8}}{K} > 1.$$

Hence, there holds

$$\partial_x \Psi_{j,K}(t,x) = \frac{1}{K} \Psi'_{j,K}(\frac{x - y_j(t)}{K}) \le \frac{1}{K} e^{-\frac{1}{K}(\sigma_0 t + \frac{L}{8})}, \quad x \in D_j^c. \tag{3.32}$$

For $\alpha_0 > 0$ small enough and $L_0 > \sigma^2$ large enough, since $\partial_x \Psi_{j,K}, u^2, u_x^2, \eta^2$ are all positive, then

$$\begin{aligned} Q_1(t) &= \frac{1}{2} \int_{D_j} u \partial_x \Psi_{j,K} \Big((\sigma - 1)u^2 + \sigma u_x^2 + \eta^2 \Big) \, dx \\ &\quad + \frac{1}{2} \int_{D_j^c} u \partial_x \Psi_{j,K} \Big((\sigma - 1)u^2 + \sigma u_x^2 + \eta^2 \Big) \, dx \\ &\le \frac{1}{2} \max\{|\sigma - 1|, |\sigma|, 1\} \, \|u(t)\|_{L^\infty(D_j)} \int_{D_j} \partial_x \Psi_{j,K} \Big(u^2 + u_x^2 + \eta^2 \Big) \, dx \\ &\quad + \frac{1}{2} \max\{|\sigma - 1|, |\sigma|, 1\} \, \|u(t)\|_{L^\infty(D_j^c)} \\ &\quad \times \sup_{x \in D_j^c} |\partial_x \Psi_{j,K}(x - y_j(t))| \int_{D_j^c} \Big(u^2 + u_x^2 + \eta^2 \Big) \, dx \\ &\le \frac{C}{K} \|\boldsymbol{u}_0\|_X^3 \, e^{-\frac{\sigma_0 t + \frac{L}{8}}{K}} + \frac{\sigma_0}{8} \int_{\mathbb{R}} \partial_x \Psi_{j,K} \Big(u^2 + u_x^2 + \eta^2 \Big) \, dx. \end{aligned}$$

For $Q_2(t)$, by the Cauchy inequality and Hölder's inequality we know that

$$\begin{aligned} \int_{\mathbb{R}} u \partial_x \Psi_{j,K} (1 - \partial_x^2)^{-1} \eta \, dx &= \int_{\mathbb{R}} u \partial_x \Psi_{j,K} p * \eta \, dx \\ &\le \frac{1}{2a} \int_{\mathbb{R}} u^2 \partial_x \Psi_{j,K} \, dx + \frac{a}{2} \int_{\mathbb{R}} \partial_x \Psi_{j,K} (p * \eta)^2 \, dx \\ &\le \frac{1}{2a} \int_{\mathbb{R}} u^2 \partial_x \Psi_{j,K} \, dx + \frac{a}{4} \int_{\mathbb{R}} \partial_x \Psi_{j,K} (p * \eta^2) \, dx, \end{aligned} \tag{3.33}$$

where

$$\begin{aligned}(p*\eta)^2(x) &= \left[\frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\eta(y)\,dy\right]^2\\ &\le \frac{1}{4}\left(\int_{\mathbb{R}} e^{-|x-y|}\,dy\right)\left(\int_{\mathbb{R}} e^{-|x-y|}\eta^2(y)\,dy\right)\\ &= \frac{1}{2}\left(p*\eta^2\right)(x).\end{aligned}$$

For $x \in D_j^c$, by the property of convolution and (3.32), the following inequality holds:

$$\begin{aligned}\int_{D_j^c} \partial_x\Psi_{j,K} p*\eta^2\,dx &\le \sup_{x\in D_j^c}|\partial_x\Psi_{j,K}|\int_{D_j^c} p*\left(u^2+u_x^2+\eta^2\right)\,dx\\ &\le \frac{1}{2}\sup_{x\in D_j^c}|\partial_x\Psi_{j,K}|\int_{\mathbb{R}} e^{-|x|}dx\int_{\mathbb{R}}\left(u^2+u_x^2+\eta^2\right)\,dx \qquad (3.34)\\ &\le \frac{C}{K}\left\|\boldsymbol{u}_0\right\|_X^2 e^{-\frac{\sigma_0 t+\frac{L}{8}}{K}}.\end{aligned}$$

By the given condition (3.16), we have

$$\left(1-\partial_x^2\right)\partial_x\Psi_{j,K} \ge \left(1-\frac{10}{K^2}\right)\partial_x\Psi_{j,K},$$

which is equivalent to

$$\left(1-\partial_x^2\right)^{-1}\partial_x\Psi_{j,K} \le \left(1-\frac{10}{K^2}\right)^{-1}\partial_x\Psi_{j,K}, \quad \text{for} \quad K\ge 4. \qquad (3.35)$$

Hence, for $x \in D_j$,

$$\begin{aligned}\int_{D_j}\partial_x\Psi_{j,K}(p*\eta^2)\,dx &\le \int_{D_j}\eta^2\left(1-\partial_x^2\right)^{-1}\partial_x\Psi_{j,K}\,dy\\ &\le \left(1-\frac{10}{K^2}\right)^{-1}\int_{\mathbb{R}}\eta^2\partial_x\Psi_{j,K}\,dx.\end{aligned} \qquad (3.36)$$

Combining (3.33), (3.34) and (3.36), we obtain

$$Q_2(t) \le \frac{1}{2a}\int_{\mathbb{R}} u^2\partial_x\Psi_{j,K}\,dx + \frac{a}{4}\left(1-\frac{10}{K^2}\right)^{-1}\int_{\mathbb{R}}\eta^2\partial_x\Psi_{j,K}\,dx + \frac{C}{K}\left\|\boldsymbol{u}_0\right\|_X^2 e^{-\frac{\sigma_0 t+\frac{L}{8}}{K}}.$$

$Q_3(t)$ can be treated with the same process. For $x \in D_j^c$, we deduce from (3.32) that

$$\begin{aligned}&\int_{D_j^c} u\partial_x\Psi_{j,K}\left(1-\partial_x^2\right)^{-1}\left(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2+\frac{1}{2}\eta^2\right)\,dx\\ &\le \frac{1}{2}\max\{|3-\sigma|,|\sigma|,1\}\left\|u\right\|_{L^\infty}\sup_{x\in D_j^c}|\partial_x\Psi_{j,K}(x-y_j(t))|\int_{\mathbb{R}} p*\left(u^2+u_x^2+\eta^2\right)\,dx\\ &= \frac{1}{4}\max\{|3-\sigma|,|\sigma|,1\}\left\|u\right\|_{L^\infty}\sup_{x\in D_j^c}|\partial_x\Psi_{j,K}(x-y_j(t))|\int_{\mathbb{R}} e^{-|x|}*\left(u^2+u_x^2+\eta^2\right)\,dx\\ &\le \frac{C}{K}\left\|\boldsymbol{u}_0(t)\right\|_X^3 e^{-\frac{\sigma_0 t+\frac{L}{8}}{K}}.\end{aligned}$$

For $x \in D_j$, using equation (3.31) and taking $L_0 > \sigma^2$ large enough, we infer

$$\begin{aligned}
&\int_{D_j} u\partial_x \Psi_{j,K} \left(1-\partial_x^2\right)^{-1} \left(\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\eta^2\right) dx \\
&\le \frac{1}{2}\max\{|3-\sigma|, |\sigma|, 1\} \|u\|_{L^\infty} \int_{D_j} \partial_x \Psi_{j,K} \left(1-\partial_x^2\right)^{-1} \left(u^2+u_x^2+\eta^2\right) dx \\
&\le \frac{1}{4}\max\{|3-\sigma|, |\sigma|, 1\} \|u\|_{L^\infty} \int_{\mathbb{R}} \partial_x \Psi_{j,K} e^{-|x|} * \left(u^2+u_x^2+\eta^2\right) dx \\
&\le \frac{\sigma_0}{8} \int_{\mathbb{R}} \partial_x \Psi_{j,K} \Big(u^2+u_x^2+\eta^2\Big) dx.
\end{aligned}$$

Thus, the estimates on D_j^c and D_j guarantee that

$$Q_3(t) \le \frac{C}{K}\|\boldsymbol{u}_0\|_X^3 e^{-\frac{\sigma_0 t + \frac{L}{8}}{K}} + \frac{\sigma_0}{8}\int_{\mathbb{R}} \partial_x \Psi_{j,K} \Big(u^2+u_x^2+\eta^2\Big) dx.$$

Substituting the estimates of $Q_1(t)$, $Q_2(t)$, and $Q_3(t)$ and (3.30) into (3.29), we derive that

$$\begin{aligned}
J(t) \le &\frac{C}{K}\left(\|\boldsymbol{u}_0(t)\|_X^3 + \|\boldsymbol{u}_0(t)\|_X^2\right) e^{-\frac{\sigma_0 t + \frac{L}{8}}{K}} + \frac{\sigma_0}{4}\int_{\mathbb{R}} \partial_x \Psi_{j,K} \Big(u^2+u_x^2+\eta^2\Big) dx \\
&+ \left(m + \frac{1}{2a}\right)\int_{\mathbb{R}} u^2 \partial_x \Psi_{j,K}\, dx + \frac{a}{4}\left(1-\frac{10}{K^2}\right)^{-1} \int_{\mathbb{R}} \eta^2 \partial_x \Psi_{j,K}\, dx.
\end{aligned}$$

For $\alpha_0 > 0$ small enough and $L_0 > \sigma^2$ large enough, by the definition of σ_0 in (3.24) and equation (3.5), we have

$$-\frac{\dot{y}_j(t)}{2} = -\frac{\dot{\bar{x}}_{j-1}(t) - c_{j-1}}{4} - \frac{\dot{\bar{x}}_j(t) - c_j}{4} - \frac{c_{j-1}+c_j}{4} \le -\frac{c_1+\sigma_0}{2}.$$

Associating the above estimate with $J(t)$ and (3.28), there obtains the inequality

$$\begin{aligned}
\frac{d}{dt} I_{j,K}(t) \le &\frac{C}{K}\left(\|\boldsymbol{u}_0(t)\|_X^3 + \|\boldsymbol{u}_0(t)\|_X^2\right) e^{-\frac{\sigma_0 t + \frac{L}{8}}{K}} - \frac{\sigma_0}{4}\int_{\mathbb{R}} \partial_x \Psi_{j,K} \Big(u^2+u_x^2+\eta^2\Big) dx \\
&+ \left(m - \frac{c_1}{2} + \frac{1}{2a}\right)\int_{\mathbb{R}} u^2 \partial_x \Psi_{j,K}\, dx \\
&+ \left(-\frac{c_1}{2} + \frac{a}{4}\left(1-\frac{10}{K^2}\right)^{-1}\right)\int_{\mathbb{R}} \eta^2 \partial_x \Psi_{j,K}\, dx.
\end{aligned}$$

Let m and a satisfy the following conditions:

$$m - \frac{c_1}{2} + \frac{1}{2a} \le 0, \quad -\frac{c_1}{2} + \frac{a}{4}\left(1-\frac{10}{K^2}\right)^{-1} \le 0, \quad m > \frac{A}{2}. \tag{3.37}$$

Then

$$\frac{d}{dt} I_{j,K}(t) \le \frac{C}{K} e^{-\frac{\sigma_0 t + \frac{L}{8}}{K}} - \frac{\sigma_0}{4}\int_{\mathbb{R}} \partial_x \Psi_{j,K} \Big(u^2+u_x^2+\eta^2\Big) dx.$$

The desired results (3.25) now follow by applying Gronwall's inequality. □

REMARK 3.4. We can find such m and a to meet the conditions in (3.37) as follows. Let

$$f(x) = \frac{A + x + \sqrt{(A+x)^2+2}}{2}.$$

Since $c_1 > \frac{A+\sqrt{A^2+2}}{2} = f(0)$ and $f(x)$ is increasing for $x > 0$, then there exists $\epsilon_0 > 0$ such that

$$f(0) < f(\epsilon_0) < c_1, \quad 0 < \epsilon_0 < \frac{c_1 - A}{2}, \quad \epsilon_0 \to 0.$$

Choose $m = \frac{A}{2} + \epsilon_0$. Then from the first two conditions in (3.37), to find such a, we need to show that the following inequality is true:

$$\frac{1}{c_1 - A - 2\epsilon_0} \leq 2c_1(1 - \frac{10}{K^2}).$$

Actually, since $c_1 > \frac{A+\sqrt{A^2+2}}{2}$, taking $K \geq \sqrt{\frac{20c_1(c_1-A-2\epsilon_0)}{2c_1(c_1-A-2\epsilon_0)-1}}$, we have that the inequality mentioned above holds, which implies that such m and a exist.

In the next lemma, we present another version of Lemma 3.7 in [9], where the assumption is not strictly orthogonal. This is helpful while proving the local coercivity of the linear operator $H_c : X \to X^*$ at $\boldsymbol{\varphi}_c$, $\forall c \in \{c_i\}_{i=1}^N$.

LEMMA 3.5. Assume $\delta > 0, \sigma \leq 1$ and $\boldsymbol{\varphi}_c = (\varphi_c, \psi_c)$ is a smooth solitary wave of (2.3). There exists a constant $C_\delta > 0$ such that

$$\langle H_c \boldsymbol{\zeta}, \boldsymbol{\zeta} \rangle \geq C_\delta \left\| \boldsymbol{\zeta} \right\|_X^2,$$

for all $\boldsymbol{\zeta} \in X$ satisfying $|\langle \boldsymbol{\zeta}, \boldsymbol{\nu} \rangle| + |\langle \boldsymbol{\zeta}, \partial_x \boldsymbol{\nu} \rangle| \leq \delta \left\| \boldsymbol{\zeta} \right\|_X$, where $\boldsymbol{\nu} = \left(\left(1 - \partial_x^2\right) \varphi_c, \psi_c \right)$.

Proof. According to the spectrum analysis of H_c, let $\partial_c \boldsymbol{\varphi}$ be decomposed into $a_0 \boldsymbol{\chi} + b_0 \boldsymbol{\varphi}' + a_0 \boldsymbol{p}_0$ with $a_0 > 0$, where $\boldsymbol{\chi}$ is the eigenvector of negative eigenvalue, i.e. $H_c \boldsymbol{\chi} = -|\lambda_1| \boldsymbol{\chi}$, $\boldsymbol{\varphi}'$ is the eigenvector of eigenvalue 0, i.e. $H_c \boldsymbol{\varphi}' = 0$, $\boldsymbol{p}_0$ is the positive subspace of H_c. Without loss of generality, we assume $\|\boldsymbol{\chi}\| = 1$. Therefore,

$$\begin{aligned}
\langle H_c \partial_c \boldsymbol{\varphi}, \partial_c \boldsymbol{\varphi} \rangle =& \langle H_c \left(a_0 \boldsymbol{\chi} + b_0 \boldsymbol{\varphi}' + a_0 \boldsymbol{p}_0 \right), a_0 \boldsymbol{\chi} + b_0 \boldsymbol{\varphi}' + a_0 \boldsymbol{p}_0 \rangle \\
=& a_0^2 \langle H_c \boldsymbol{\chi}, \boldsymbol{\chi} \rangle + a_0^2 \langle H_c \boldsymbol{p}_0, \boldsymbol{p}_0 \rangle \\
=& -a_0^2 |\lambda_1| + a_0^2 \langle H_c \boldsymbol{p}_0, \boldsymbol{p}_0 \rangle.
\end{aligned}$$

Since $\langle H_c \partial_c \boldsymbol{\varphi}, \partial_c \boldsymbol{\varphi} \rangle = -d''(c) < 0$, then $\langle H_c \boldsymbol{p}_0, \boldsymbol{p}_0 \rangle < |\lambda_1|$. Let $\varepsilon > 0$, such that

$$\langle H_c \boldsymbol{p}_0, \boldsymbol{p}_0 \rangle \leq \frac{1}{1+\varepsilon} |\lambda_1|.$$

By the assumption we know $|(\boldsymbol{\zeta}, \boldsymbol{\varphi})_{H^1 \times L^2}| + |(\boldsymbol{\zeta}, \boldsymbol{\varphi}')_{H^1 \times L^2}| = |\langle \boldsymbol{\zeta}, \boldsymbol{\nu} \rangle| + |\langle \boldsymbol{\zeta}, \partial_x \boldsymbol{\nu} \rangle| \leq \delta \left\| \boldsymbol{\zeta} \right\|_X$. Then there exists $\varepsilon = \varepsilon(\delta)$ such that $\boldsymbol{\zeta}$ has the decomposition $a\boldsymbol{\chi} + \varepsilon \boldsymbol{\varphi}' + \boldsymbol{p_1}$ with $\boldsymbol{p}_1$ in the positive subspace of H_c. If $a = 0$, then $\boldsymbol{\zeta} = \varepsilon \boldsymbol{\varphi}' + \boldsymbol{p}_1$. It is easy to check that $\langle H_c \boldsymbol{\zeta}, \boldsymbol{\zeta} \rangle \geq C_\delta \left\| \boldsymbol{\zeta} \right\|_X^2$. If $a \neq 0$, then $\boldsymbol{\zeta} = a\boldsymbol{\chi} + \varepsilon \boldsymbol{\varphi}' + a\boldsymbol{p}$ with $\boldsymbol{p}$ in the positive subspace of H_c. Differentiating $\langle cE'(\boldsymbol{\varphi}_c) - f'(\boldsymbol{\varphi}_c), \boldsymbol{\zeta} \rangle = 0$ with respect to c, we have $\langle H_c \partial_c \boldsymbol{\varphi}, \boldsymbol{\zeta} \rangle + \langle E'(\boldsymbol{\varphi}_c), \boldsymbol{\zeta} \rangle = 0$, i.e.

$$\langle H_c \partial_c \boldsymbol{\varphi}, \boldsymbol{\zeta} \rangle = \langle -a_0 |\lambda_1| \boldsymbol{\chi} + a_0 H_c \boldsymbol{p}_0, a\boldsymbol{\chi} + \varepsilon \boldsymbol{\varphi}' + a\boldsymbol{p} \rangle,$$

which implies

$$|\langle H_c \boldsymbol{p}_0, \boldsymbol{p} \rangle| \leq |\lambda_1| + \frac{\delta \left\| \boldsymbol{\zeta} \right\|_X}{|aa_0|}.$$

Consequently,

$$\begin{aligned}\langle H_c\boldsymbol{\zeta},\boldsymbol{\zeta}\rangle &= \langle H_c\left(a\boldsymbol{\chi}+\varepsilon\boldsymbol{\varphi}'+a\boldsymbol{p}\right), a\boldsymbol{\chi}+\varepsilon\boldsymbol{\varphi}'+a\boldsymbol{p}\rangle\\ &= -a^2\left|\lambda_1\right|+a^2\langle H_c\boldsymbol{p},\boldsymbol{p}\rangle\\ &\geq -a^2\left|\lambda_1\right|+a^2\frac{\langle H_c\boldsymbol{p}_0,\boldsymbol{p}\rangle^2}{\langle H_c\boldsymbol{p}_0,\boldsymbol{p}_0\rangle}\\ &\geq \varepsilon a^2\left|\lambda_1\right|=\varepsilon\left|\lambda_1\right|\left\|a\boldsymbol{\chi}\right\|_X^2.\end{aligned}$$

In fact, the function $\Gamma(f,g)=\langle H_cf,g\rangle$ is a nonnegative sesquilinear form on a positive subspace of H_c. Thus, we have the Cauchy-Schwarz inequality

$$\left|\Gamma\left(f,g\right)\right|^2\leq\langle H_cf,f\rangle\langle H_cg,g\rangle.$$

On the other hand, in view of $\boldsymbol{p}$ in the positive subspace of H_c, there exists $\lambda_3>0$, such that $\langle H_c\boldsymbol{p},\boldsymbol{p}\rangle\geq\lambda_3\left\|\boldsymbol{p}\right\|_X^2$ for all $\boldsymbol{p}$ orthogonal to $\boldsymbol{\chi}$ and $\boldsymbol{\varphi}'$, which implies

$$\langle H_c\boldsymbol{\zeta},\boldsymbol{\zeta}\rangle\geq-\left|\lambda_1\right|\left\|a\boldsymbol{\chi}\right\|_X^2+\lambda_3\left\|a\boldsymbol{p}\right\|_X^2.$$

Then, it follows that for $k=\frac{\varepsilon}{\varepsilon+2}\min\{\left|\lambda_1\right|,\lambda_3\}$,

$$\langle H_c\boldsymbol{\zeta},\boldsymbol{\zeta}\rangle\geq k\left(\left\|a\boldsymbol{\chi}\right\|_X^2+\left\|a\boldsymbol{p}\right\|_X^2\right).$$

Since ε and δ are small enough compared with k, we obtain

$$\langle H_c\boldsymbol{\zeta},\boldsymbol{\zeta}\rangle\geq C_\delta\left\|\boldsymbol{\zeta}\right\|_X^2,$$

which completes the proof of this lemma. □

Now, it remains to show the last property in this subsection. Taking advantage of Lemma 3.5, we will prove that H_{c_i} is a local coercive operator as follows.

LEMMA 3.6. There exist $\sigma\leq1$, $\delta>0$, $C_\delta>0$ and $C>0$ depending only on c_1, such that for all $c\geq c_1$, $\Theta\in C^2(\mathbb{R})>0$ and $\boldsymbol{\zeta}=(\zeta,\xi)\in X$ satisfying

$$\left|\langle\sqrt{\Theta}\boldsymbol{\zeta},\boldsymbol{\nu}\rangle\right|+\left|\langle\sqrt{\Theta}\boldsymbol{\zeta},\partial_x\boldsymbol{\nu}\rangle\right|\leq\delta\left\|\boldsymbol{\zeta}\right\|_X$$

and

$$\left|\frac{(\Theta')^2}{4\Theta}\right|+c\left|\sigma\Theta'\right|+\left|\frac{\Theta''}{2}\right|\leq\min\left\{\frac{1}{4},\frac{C_\delta}{4(c+c|\sigma|)}\right\}\Theta,$$

where $\boldsymbol{\nu}=\left(\left(1-\partial_x^2\right)\varphi_c,\psi_c\right)$, it holds that

$$\begin{aligned}\int_{\mathbb{R}}\Big[&\Theta\left(\left(c-\sigma\varphi_c\right)\left(\partial_x\zeta\right)^2+\left(-3\varphi_c+\sigma\varphi_{c,xx}+c+A\right)\zeta^2\right)\\ &-2\Theta\frac{c}{c-\varphi}\zeta\xi+\Theta\left(c-\varphi\right)\xi^2+\sigma\Theta'\zeta^2\varphi_{c,x}\Big]\,dx\geq C\int_{\mathbb{R}}\Theta\left(\zeta^2+\zeta_x^2+\xi^2\right)\,dx.\end{aligned}\tag{3.38}$$

Proof. Following with the strategy in [21], we have

$$\begin{aligned}
&\langle H_c\sqrt{\Theta}\boldsymbol{\zeta},\sqrt{\Theta}\boldsymbol{\zeta}\rangle_{L^2\times L^2}=\langle H_c\sqrt{\Theta}\begin{pmatrix}\zeta\\ \xi\end{pmatrix},\sqrt{\Theta}\begin{pmatrix}\zeta\\ \xi\end{pmatrix}\rangle_{L^2\times L^2}\\
&=\langle\begin{pmatrix}\left(-\partial_x\left(\left(c-\sigma\varphi_c\right)\partial_x\right)-3\varphi_c+\sigma\varphi_{c,xx}+c+A\right)\sqrt{\Theta}\zeta-\frac{c}{c-\varphi}\sqrt{\Theta}\xi\\ -\frac{c}{c-\varphi}\sqrt{\Theta}\zeta+\left(c-\varphi\right)\sqrt{\Theta}\xi\end{pmatrix},\begin{pmatrix}\sqrt{\Theta}\zeta\\ \sqrt{\Theta}\xi\end{pmatrix}\rangle_{L^2\times L^2}\\
&=\int_{\mathbb{R}}\Theta\left[\left(c-\sigma\varphi_c\right)\left(\partial_x\zeta\right)^2+\left(-3\varphi_c+\sigma\varphi_{c,xx}+c+A\right)\zeta^2\right]\,dx+\int_{\mathbb{R}}\frac{c-\sigma\varphi_c}{4}\frac{\left(\Theta'\right)^2}{\Theta}\zeta^2\,dx\\
&\quad+\int_{\mathbb{R}}\left(c-\sigma\varphi_c\right)\Theta'\zeta\partial_x\zeta\,dx-2\int_{\mathbb{R}}\Theta\frac{c}{c-\varphi_c}\zeta\xi\,dx+\int_{\mathbb{R}}\Theta\left(c-\varphi_c\right)\xi^2\,dx\\
&=\int_{\mathbb{R}}\Bigg[\Theta\left(\left(c-\sigma\varphi_c\right)\left(\partial_x\zeta\right)^2+\left(-3\varphi_c+\sigma\varphi_{c,xx}+c+A\right)\zeta^2\right)-2\Theta\frac{c}{c-\varphi_c}\zeta\xi+\Theta\left(c-\varphi_c\right)\xi^2\\
&\quad+\sigma\Theta'\zeta^2\varphi_{c,x}\Bigg]\,dx+\int_{\mathbb{R}}\left[\left(c-\sigma\varphi_c\right)\left(\frac{\left(\Theta'\right)^2}{4\Theta}-\frac{\Theta''}{2}\right)-\frac{\sigma}{2}\varphi_{c,x}\Theta'\right]\zeta^2\,dx\\
&:=\Lambda+\int_{\mathbb{R}}\left[\left(c-\sigma\varphi_c\right)\left(\frac{\left(\Theta'\right)^2}{4\Theta}-\frac{\Theta''}{2}\right)-\frac{\sigma}{2}\varphi_{c,x}\Theta'\right]\zeta^2\,dx
\end{aligned}$$

and

$$\begin{aligned}
\left\|\sqrt{\Theta}\begin{pmatrix}\zeta\\ \xi\end{pmatrix}\right\|_X^2&=\int_{\mathbb{R}}\left(\sqrt{\Theta}\zeta\right)^2+\left(\partial_x\left(\sqrt{\Theta}\zeta\right)\right)^2+\left(\sqrt{\Theta}\xi\right)^2\,dx\\
&=\int_{\mathbb{R}}\Theta\left(\zeta^2+\left(\partial_x\zeta\right)^2+\xi^2\right)\,dx+\int_{\mathbb{R}}\left(\frac{\left(\Theta'\right)^2}{4\Theta}-\frac{\Theta''}{2}\right)\zeta^2\,dx.
\end{aligned}$$

Due to Lemma 3.5, there exist $\delta>0$ and $C_\delta>0$, such that if for $c\geq c_1$

$$\left|\langle\boldsymbol{\zeta},\boldsymbol{\nu}\rangle_{L^2\times L^2}\right|+\left|\langle\boldsymbol{\zeta},\partial_x\boldsymbol{\nu}\rangle_{L^2\times L^2}\right|\leq\delta\left\|\boldsymbol{\zeta}\right\|_X,\ \text{where}\ \boldsymbol{\nu}=\left(\left(1-\partial_x^2\right)\varphi_c,\psi_c\right),\tag{3.39}$$

then

$$\langle H_c\boldsymbol{\zeta},\boldsymbol{\zeta}\rangle_{L^2\times L^2}\geq C_\delta\left\|\boldsymbol{\zeta}\right\|_X^2.\tag{3.40}$$

Hence, taking $\boldsymbol{\zeta}=\sqrt{\Theta}\boldsymbol{\zeta}$, integrating with the conditions that φ_c is bounded by 0 and $c-A_1$, $|\varphi_c'|$ is bounded by $c-A_1$ and

$$\left|\frac{\left(\Theta'\right)^2}{4\Theta}\right|+c\left|\sigma\Theta'\right|+\left|\frac{\Theta''}{2}\right|\leq\min\left\{\frac{1}{4},\frac{C_\delta}{4(c+c\left\|\sigma\right\|)}\right\}\Theta,$$

we deduce the inequality

$$\Lambda+\frac{C_\delta}{4}\int_{\mathbb{R}}\Theta\zeta^2\,dx\geq C_\delta\int_{\mathbb{R}}\Theta\left(\zeta^2+\left(\partial_x\zeta\right)^2+\xi^2\right)\,dx-\min\left\{\frac{1}{4},\frac{C_\delta}{4(c+c|\sigma|)}\right\}\int_{\mathbb{R}}\Theta\zeta^2\,dx,$$

which implies there exists C such that

$$\Lambda\geq C\int_{\mathbb{R}}\Theta\left(\zeta^2+\left(\partial_x\zeta\right)^2+\xi^2\right)\,dx.$$

This completes the proof of Lemma 3.6. □

3.2. *Stability of the sum of an ordered train of solitary waves.* According to Proposition 3.1, to prove the main theorem is to prove inequality (3.2), where $x_i(t)$ are taken as $\tilde{x}_i(t)$ in Lemma 3.2 and M will be determined later. Indeed, the following three lemmas contribute to the estimate of $\boldsymbol{v}$ at time $t = t^*$.

When there is no ambiguity, we write $\boldsymbol{u}=\boldsymbol{u}(t^*), X=(\tilde{x}_1,\cdots,\tilde{x}_N)=(\tilde{x}_1(t^*),\cdots,\tilde{x}_N(t^*))$, $E_i(\boldsymbol{u}) = E_i(\boldsymbol{u}(t^*))$, $F_i(\boldsymbol{u}) = E_i(\boldsymbol{u}(t^*))$, and so forth. Define $\boldsymbol{\mu}_i = (\mu_i, \tau_i) \in X$, for $i = 1, 2, \cdots, N$, by

$$\boldsymbol{u} = (1 + a_i)\boldsymbol{R}_X + \boldsymbol{\mu}_i, \text{ satisfying } \langle E_i'(\boldsymbol{R}_X), \boldsymbol{\mu}_i\rangle_{L^2\times L^2} = 0, \tag{3.41}$$

where

$$\begin{aligned}\boldsymbol{R}_X =&(R_X, P_X) = \sum_{i=1}^{N} \boldsymbol{R}_i = (\sum_{i=1}^{N} R_i(\cdot), \sum_{i=1}^{N} P_i(\cdot))\\ =& \left(\sum_{i=1}^{N} \varphi_{c_i}(\cdot - \tilde{x}_i(t^*)), \sum_{i=1}^{N} \psi_{c_i}(\cdot - \tilde{x}_i(t^*))\right).\end{aligned}$$

By (3.20) and (3.21), and the exponential decay of $\boldsymbol{\varphi}_{c_i}$, we have

$$\begin{aligned}\langle E_i'(\boldsymbol{R}_X), \boldsymbol{R}_X\rangle =&\langle E_i'(\boldsymbol{\varphi}_{c_i}), \boldsymbol{\varphi}_{c_i}\rangle + O(e^{-\frac{L}{4}})\\ =& \|\boldsymbol{\varphi}_{c_i}\|_X^2 + O(e^{-\frac{L}{4}})\\ >& \frac{1}{2}\|\boldsymbol{\varphi}_{c_i}\|_X^2,\end{aligned} \tag{3.42}$$

which implies $\boldsymbol{\mu}_i$ $(i = 1, \ldots, N)$ are well defined.

We also set $\boldsymbol{v} = (v, w) = \boldsymbol{u} - \boldsymbol{R}_X$. The goal of this subsection is to show $\|\boldsymbol{v}\|_X \le O(\sqrt{\varepsilon} + e^{-\frac{L}{8}})$. If $\|\boldsymbol{v}\|_X \le \sqrt{\varepsilon} + e^{-\frac{L}{8}}$, then we can set $M = 2$ to reach the conclusion (3.2). Hence, we subsequently assume $\|\boldsymbol{v}\|_X \ge \sqrt{\varepsilon} + e^{-\frac{L}{8}}$.

By computing the variational derivatives of localized conservation laws E_i and F_i, we know

$$\begin{cases}(E_i)''_{uu} = \Phi_i\,(1 - \partial_{xx}) - \Phi_i'\partial_x,\\ (E_i)''_{u\eta} = (E_i)''_{\eta u} = 0,\\ (E_i)''_{\eta\eta} = \Phi_i\end{cases}$$

and

$$\begin{cases}(F_i)''_{uu} = \left(3u - \sigma u_x\partial_x - \sigma u_{xx} - \sigma u\partial_x^2 - A\right)\Phi_i - \sigma\left(u_x\Phi_i' + u\Phi_i' + \partial_x\right),\\ (F_i)''_{u\eta} = (F_i)''_{\eta u} = (\eta + 1)\,\Phi_i,\\ (F_i)''_{\eta\eta} = u\Phi_i.\end{cases}$$

Then the localized Hessian operator H_{c_i} has following form:

$$H_{c_i} = c_iE''(\boldsymbol{\varphi}_{c_i}) - F''\left(\boldsymbol{\varphi}_{c_i}\right) = \begin{pmatrix} L_{c_i} & -\Phi_i(1 + \psi_{c_i})\\ -\Phi_i(1 + \psi_{c_i}) & \Phi_i\left(c_i - \varphi_{c_i}\right)\end{pmatrix}, \tag{3.43}$$

where

$$L_{c_i} = -\partial_x\left(\Phi_i(c_i - \sigma\varphi_{c_i})\partial_x\right) + \Phi_i\left(-3\varphi_{c_i} + \sigma\varphi_{c_i,xx} + c_i + A\right) + \sigma\partial_x\varphi_{c_i}.$$

For abbreviation, we use H_i instead of H_{c_i} and L_i instead of L_{c_i}.

The first lemma in this subsection gives the estimate of a_i and $\boldsymbol{\mu}_i$ through the almost monotonicity property.

LEMMA 3.7. Let $\boldsymbol{u} = \boldsymbol{R}_X + \boldsymbol{v} = (1+a_i)\boldsymbol{R}_X + \boldsymbol{\mu}_i$ satisfy $\langle E_i'(\boldsymbol{R}_X), \boldsymbol{\mu}_i\rangle = 0$. Assume $\|\boldsymbol{v}\|_X \geq \sqrt{\varepsilon} + e^{-\frac{L}{8}}$. Then for $i = 1, \cdots, N$,

$$|a_i| \leq O(\|\boldsymbol{v}\|_X^2) \tag{3.44}$$

and

$$\|\boldsymbol{\mu}_i\|_X \sim \|\boldsymbol{v}\|_X . \tag{3.45}$$

Proof. From the definition of localized conservation laws (3.22) and (3.23) and the property of Φ_i (3.20) and (3.21), we know

$$I_{j,K}\left(\boldsymbol{u}(t^*)\right) = \sum_{j=i}^{N} E_j\left(\boldsymbol{u}(t^*)\right), \text{ for } i = 2, \cdots, N, \tag{3.46}$$

$$E\left(\boldsymbol{u}(t^*)\right) = \sum_{j=1}^{N} E_j\left(\boldsymbol{u}(t^*)\right). \tag{3.47}$$

In light of the Taylor formula and equation (3.41), we infer that for $t = t^*$,

$$\begin{aligned} \sum_{j=1}^{N} E_j\left(\boldsymbol{u}\right) =& \sum_{j=1}^{N} E_j\left(\boldsymbol{R}_X\right) + \sum_{j=1}^{N} \langle E_j'\left(\boldsymbol{R}_X\right), \boldsymbol{v}\rangle + O(\|\boldsymbol{v}\|_X^2) \\ =& \sum_{j=1}^{N} E_j\left(\boldsymbol{\varphi}_{c_j}\right) + \sum_{j=1}^{N} a_j \langle E_j'\left(\boldsymbol{R}_X\right), \boldsymbol{R}_X\rangle + O(\|\boldsymbol{v}\|_X^2). \end{aligned} \tag{3.48}$$

Moreover, the way we construct $\Psi_{j,K}$ implies

$$E_j\left(\boldsymbol{R}_k\right) \leq O(e^{-\frac{L}{4}}), \quad \text{for} \quad j \neq k, \quad \text{and} \quad E_j\left(\boldsymbol{R}_j\right) = E\left(\boldsymbol{\varphi}_{c_j}\right) + O(e^{-\frac{L}{4}}). \tag{3.49}$$

Note that the conservation law $E(\boldsymbol{u})$ and almost monotonicity of $I_{j,K}$ provide

$$\begin{aligned} I_{j,K}(\boldsymbol{u}(t^*)) \leq& I_{j,K}(\boldsymbol{u}_0) + O(e^{-\frac{L}{4}}) \leq \sum_{j=i}^{N} E\left(\boldsymbol{\varphi}_{c_j}\right) + O(e^{-\frac{L}{4}}) + O(\varepsilon), \\ E\left(\boldsymbol{u}(t^*)\right) =& E\left(\boldsymbol{u}_0\right) = \sum_{j=1}^{N} E\left(\boldsymbol{\varphi}_{c_j}\right) + O(e^{-\frac{L}{4}}) + O(\varepsilon). \end{aligned} \tag{3.50}$$

Hence, from (3.46)-(3.50), we deduce

$$\sum_{j=i}^{N} a_j \langle E_j'\left(\boldsymbol{R}_X\right), \boldsymbol{R}_X\rangle \leq O(\|\boldsymbol{v}\|_X^2), \quad i = 1, \cdots, N. \tag{3.51}$$

Similarly, in view of the conservation law $F(\boldsymbol{u})$ and Taylor's formula, it follows that for $t = t^*$,

$$\begin{aligned} F\left(\boldsymbol{u}(t^*)\right) &= \sum_{i=1}^{N} F_i\left(\boldsymbol{u}(t^*)\right) = \sum_{i=1}^{N} F_i\left(\boldsymbol{R}_X\right) + \sum_{i=1}^{N} \langle F_i'\left(\boldsymbol{R}_X\right), \boldsymbol{v}\rangle + O(\|\boldsymbol{v}\|_X^2) \\ &= \sum_{i=1}^{N} F_i\left(\boldsymbol{\varphi}_{c_i}\right) + \sum_{i=1}^{N} \langle F_i'\left(\boldsymbol{R}_X\right), \boldsymbol{v}\rangle + O(\|\boldsymbol{v}\|_X^2) + O(e^{-\frac{L}{4}}) \end{aligned} \tag{3.52}$$

and

$$F\left(\boldsymbol{u}(t^*)\right) = F\left(\boldsymbol{u}_0\right) = \sum_{j=1}^{N} F\left(\boldsymbol{\varphi}_{c_j}\right) + O(e^{-\frac{L}{4}}) + O(\varepsilon). \tag{3.53}$$

Thus, from equation (3.52) and (3.53), there holds

$$\sum_{i=1}^{N} \langle F_i'\left(\boldsymbol{R}_X\right), \boldsymbol{v}\rangle = O(\|\boldsymbol{v}\|_X^2). \tag{3.54}$$

On the other hand, by the identity $F'\left(\boldsymbol{\varphi}_c\right) = cE'\left(\boldsymbol{\varphi}_c\right)$, (3.20) and (3.21), we know

$$\|F_i'\left(\boldsymbol{R}_X\right) - c_i E_i'\left(\boldsymbol{R}_X\right)\|_{X^*} \leq \|F'\left(\boldsymbol{\varphi}_{c_i}\right) - c_i E'\left(\boldsymbol{\varphi}_{c_i}\right)\|_{X^*} + O(e^{-\frac{L}{4}}) \leq O(e^{-\frac{L}{4}}),$$

which implies

$$\begin{aligned} \sum_{i=1}^{N} \langle F_i'\left(\boldsymbol{R}_X\right), \boldsymbol{v}\rangle &= \sum_{i=1}^{N} \langle F_i'\left(\boldsymbol{R}_X\right) - c_i E_i'\left(\boldsymbol{R}_X\right), \boldsymbol{v}\rangle + \sum_{i=1}^{N} \langle c_i E_i'\left(\boldsymbol{R}_X\right), \boldsymbol{v}\rangle \\ &= \sum_{i=1}^{N} \langle F_i'\left(\boldsymbol{R}_X\right) - c_i E_i'\left(\boldsymbol{R}_X\right), \boldsymbol{v}\rangle + \sum_{i=1}^{N} c_i a_i \langle E_i'\left(\boldsymbol{R}_X\right), \boldsymbol{R}_X\rangle \\ &= \sum_{i=1}^{N} c_i a_i \langle E_i'\left(\boldsymbol{R}_X\right), \boldsymbol{R}_X\rangle + O(\|\boldsymbol{v}\|_X^2). \end{aligned} \tag{3.55}$$

Hence, combining (3.54) and (3.55), we obtain

$$\sum_{i=1}^{N} c_i a_i \langle E_i'\left(\boldsymbol{R}_X\right), \boldsymbol{R}_X\rangle = O(\|\boldsymbol{v}\|_X^2).$$

Using the Abel transformation, the above estimate yields

$$\sum_{i=1}^{N} (c_i - c_{i-1}) \sum_{j=i}^{N} a_j \langle E_j'\left(\boldsymbol{R}_X\right), \boldsymbol{R}_X\rangle + c_1 \sum_{i=1}^{N} a_i \langle E_i'\left(\boldsymbol{R}_X\right), \boldsymbol{R}_X\rangle = O(\|\boldsymbol{v}\|_X^2). \tag{3.56}$$

Furthermore, associating (3.51) with (3.56), we infer

$$|a_i \langle E_i'\left(\boldsymbol{R}_X\right), \boldsymbol{R}_X\rangle| \leq O(\|\boldsymbol{v}\|_X^2), \forall i = 1, \cdots, N,$$

which implies

$$|a_i| \leq O(\|\boldsymbol{v}\|_X^2).$$

Consequently, by (3.41), we have

$$\|\boldsymbol{\mu}_i\|_X \sim \|\boldsymbol{v}\|_X.$$

This completes the proof of Lemma 3.7. □

The second lemma in this subsection is to show that $E_i(\boldsymbol{\mu}_i)$ is bounded above by the property of local coercivity of H_i.

LEMMA 3.8. Let a_i and $\boldsymbol{\mu}_i$ satisfy Lemma 3.7. We have

$$\langle H_i(\boldsymbol{R}_X)\boldsymbol{\mu}_i, \boldsymbol{\mu}_i\rangle \geq C_\delta E_i\left(\boldsymbol{\mu}_i\right), \quad i = 1, \cdots, N, \tag{3.57}$$

where E_i and H_i are defined in (3.22) and (3.43), respectively.

Proof. Notice that (3.57) is equivalent to (3.38) while letting $\Theta = \Phi_i$, $\boldsymbol{\zeta} = \boldsymbol{\mu}_i$, $\boldsymbol{\nu} = \big((1-\partial_x^2)R_i, P_i\big)$ and $H_c = H_i$ in Lemma 3.6. Hence, in order to apply this lemma, we claim that for $\boldsymbol{\nu} = \big((1-\partial_x^2)R_i, P_i\big)$,

$$\left|\langle\sqrt{\Phi_i}\boldsymbol{\mu}_i, \partial_x\boldsymbol{\nu}\rangle\right| + \left|\langle\sqrt{\Phi_i}\boldsymbol{\mu}_i, \boldsymbol{\nu}\rangle\right| \leq \left(O(\sqrt{\varepsilon}) + O(e^{-\frac{L}{8}})\right)\|\boldsymbol{\mu}_i\|_X.$$

In fact,

$$\begin{aligned}
\left\langle\sqrt{\Phi_i}\boldsymbol{\mu}_i, \partial_x\boldsymbol{\nu}\right\rangle =& \left\langle\boldsymbol{\mu}_i, \partial_x\boldsymbol{\nu}\right\rangle + \left\langle\left(\sqrt{\Phi_i}-1\right)\boldsymbol{\mu}_i, \partial_x\boldsymbol{\nu}\right\rangle \\
=& \left\langle\mu_i, (1-\partial_x^2)\partial_x R_i\right\rangle + \left\langle\tau_i, \partial_x P_i\right\rangle + \left\langle\left(\sqrt{\Phi_i}-1\right)\mu_i, (1-\partial_x^2)\partial_x R_i\right\rangle \\
&+ \left\langle\left(\sqrt{\Phi_i}-1\right)\tau_i, \partial_x P_i\right\rangle \\
=& -a_i\left\langle R_i, (1-\partial_x^2)\partial_x R_i\right\rangle - a_i\left\langle P_i, \partial_x P_i\right\rangle \\
&+ \left\langle\left(\sqrt{\Phi_i}-1\right)\mu_i, (1-\partial_x^2)\partial_x R_i\right\rangle \\
&+ \left\langle\left(\sqrt{\Phi_i}-1\right)\tau_i, \partial_x P_i\right\rangle.
\end{aligned}$$

Then, by (3.20) and (3.21), we deduce

$$\left|\left\langle\sqrt{\Phi_i}\boldsymbol{\mu}_i, \partial_x\boldsymbol{\nu}\right\rangle\right| \leq \left(O(\sqrt{\varepsilon}) + O(e^{-\frac{L}{8}})\right)\|\boldsymbol{\mu}_i\|_X.$$

On the other hand, adopting the same method, we have

$$\begin{aligned}
\left\langle\sqrt{\Phi_i}\boldsymbol{\mu}_i, \boldsymbol{\nu}\right\rangle =& \left\langle\boldsymbol{\mu}_i, \boldsymbol{\nu}\right\rangle + \left\langle\left(\sqrt{\Phi_i}-1\right)\boldsymbol{\mu}_i, \boldsymbol{\nu}\right\rangle \\
=& \left\langle\mu_i, (1-\partial_x^2)R_i\right\rangle + \left\langle\tau_i, P_i\right\rangle + \left\langle\left(\sqrt{\Phi_i}-1\right)\mu_i, (1-\partial_x^2)R_i\right\rangle \\
&+ \left\langle\left(\sqrt{\Phi_i}-1\right)\tau_i, P_i\right\rangle \\
=& \langle E_i'(R_X), \boldsymbol{\mu}_i\rangle - \int_{\mathbb{R}}(1-\Phi_i)\left(\mu_i R_i + \partial_x\mu_i\partial_x R_i + \tau_i P_i\right)\,dx \\
&- \sum_{j=1, j\neq i}^{N}\int_{\mathbb{R}}(1-\Phi_i)\left(\mu_i R_j + \partial_x\mu_i\partial_x R_j + \tau_i P_j\right)\,dx \\
&+ \left\langle\left(\sqrt{\Phi_i}-1\right)\mu_i, (1-\partial_x^2)R_i\right\rangle + \left\langle\left(\sqrt{\Phi_i}-1\right)\tau_i, P_i\right\rangle,
\end{aligned}$$

which infers

$$\left|\left\langle\sqrt{\Phi_i}\boldsymbol{\mu}_i, \boldsymbol{\nu}\right\rangle\right| \leq \left(O(\sqrt{\varepsilon}) + O(e^{-\frac{L}{8}})\right)\|\boldsymbol{\mu}_i\|_X.$$

Hence, according to Lemma 3.6, we know that (3.57) holds, which implies that the proof of Lemma 3.8 is compete. □

In the last lemma, we prove that $\sum_{i=1}^{N}\langle H_i(\boldsymbol{R}_X)\boldsymbol{\mu}_i,\boldsymbol{\mu}_i\rangle$, which is the upper bound of $E_i(\boldsymbol{\mu}_i)$, can be controlled by an infinitesimal with the application of conservation laws and the Taylor formula.

LEMMA 3.9. Let a_i and $\boldsymbol{\mu}_i$ satisfy Lemma 3.7. If E_i, F_i are the two localized conservation laws of system (2.3) and H_i is defined in (3.43), then

$$\sum_{i=1}^{N}\langle H_i(\boldsymbol{R}_X)\boldsymbol{\mu}_i,\boldsymbol{\mu}_i\rangle = O(\|\boldsymbol{v}\|_X^3)+O(e^{-\frac{L}{4}})+O(\varepsilon). \tag{3.58}$$

Proof. By the Taylor formula, we know

$$\begin{aligned}\sum_{i=1}^{N}(c_iE_i(\boldsymbol{u})-F_i(\boldsymbol{u})) =& \sum_{i=1}^{N}(c_iE_i(\boldsymbol{R}_X)-F_i(\boldsymbol{R}_X))+\sum_{i=1}^{N}\langle c_iE_i'(\boldsymbol{R}_X)-F_i'(\boldsymbol{R}_X),\boldsymbol{v}_i\rangle\\ &+\frac{1}{2}\sum_{i=1}^{N}\langle c_iE_i''(\boldsymbol{R}_X)-F_i''(\boldsymbol{R}_X),\boldsymbol{v}_i^2\rangle\\ &+O(\|\boldsymbol{v}\|_X^3)+O(e^{-\frac{L}{4}})\\ =&\sum_{i=1}^{N}(c_iE_i(\boldsymbol{R}_X)-F_i(\boldsymbol{R}_X))+\frac{1}{2}\sum_{i=1}^{N}\langle c_iE_i''(\boldsymbol{R}_X)-F_i''(\boldsymbol{R}_X),\boldsymbol{v}_i^2\rangle\\ &+O(\|\boldsymbol{v}\|_X^3)+O(e^{-\frac{L}{4}}).\end{aligned}$$

More precisely, the decomposition of $\boldsymbol{v}$ and (3.41) ensure

$$\begin{aligned}\sum_{i=1}^{N}(c_iE_i(\boldsymbol{u})-F_i(\boldsymbol{u})) =& \sum_{i=1}^{N}(c_iE_i(\boldsymbol{R}_X)-F_i(\boldsymbol{R}_X))+\frac{1}{2}\sum_{i=1}^{N}\langle H_i(\boldsymbol{R}_X)\boldsymbol{\mu}_i,\boldsymbol{\mu}_i\rangle\\ &+\sum_{i=1}^{N}a_i\langle H_i(\boldsymbol{R}_X)\boldsymbol{R}_X,\boldsymbol{\mu}_i\rangle+\sum_{i=1}^{N}\frac{a_i^2}{2}\langle H_i(\boldsymbol{R}_X)\boldsymbol{R}_X,\boldsymbol{R}_X\rangle\\ &+O(\|\boldsymbol{v}\|_X^3)+O(e^{-\frac{L}{4}}),\end{aligned}$$

which implies

$$\begin{aligned}\sum_{i=1}^{N}\langle H_i(\boldsymbol{R}_X)\boldsymbol{\mu}_i,\boldsymbol{\mu}_i\rangle =& \sum_{i=1}^{N}2\left[c_iE_i(\boldsymbol{u})-c_iE_i(\boldsymbol{R}_X)\right]-\sum_{i=1}^{N}2\left[F_i(\boldsymbol{u})-F_i(\boldsymbol{R}_X)\right]\\ &-\sum_{i=1}^{N}2a_i\langle H_i(\boldsymbol{R}_X)\boldsymbol{R}_X,\boldsymbol{\mu}_i\rangle-\sum_{i=1}^{N}a_i^2\langle H_i(\boldsymbol{R}_X)\boldsymbol{R}_X,\boldsymbol{R}_X\rangle\\ &+O(\|\boldsymbol{v}\|_X^3)+O(e^{-\frac{L}{4}}).\end{aligned} \tag{3.59}$$

For the first term on the left side of (3.59), Abel's transformation infers

$$\begin{aligned}\sum_{i=1}^{N}c_iE_i(\boldsymbol{u}) &= \sum_{i=2}^{N}(c_i-c_{i-1})\sum_{j=i}^{N}E_j(\boldsymbol{u})+c_1\sum_{j=1}^{N}E_j(\boldsymbol{u})\\ &=\sum_{i=2}^{N}(c_i-c_{i-1})\,I_i(t^*,\boldsymbol{u}(t^*))+c_1E(\boldsymbol{u}(t^*))\end{aligned}$$

$$\leq \sum_{i=2}^{N} (c_i - c_{i-1})\, I_i(0, \boldsymbol{u}_0) + c_1 E(\boldsymbol{u}_0) + O(e^{-\frac{L}{4}})$$
$$\leq \sum_{i=2}^{N} (c_i - c_{i-1}) \sum_{j=i}^{N} E(\boldsymbol{\varphi}_{c_j}) + c_1 \sum_{j=1}^{N} E(\boldsymbol{\varphi}_{c_j}) + O(e^{-\frac{L}{4}}) + O(\varepsilon)$$
$$\leq \sum_{i=1}^{N} c_i E(\boldsymbol{\varphi}_{c_i}) + O(e^{-\frac{L}{4}}) + O(\varepsilon)$$
$$\leq \sum_{i=1}^{N} c_i E_i(\boldsymbol{R}_X) + O(e^{-\frac{L}{4}}) + O(\varepsilon). \tag{3.60}$$

For the second term on the left side of (3.59), by the conservation law we obtain

$$\sum_{i=1}^{N} F_i(\boldsymbol{u}) = \sum_{i=1}^{N} F_i(\boldsymbol{u}_0) = \sum_{i=1}^{N} F_i(\boldsymbol{R}_X) + O(e^{-\frac{L}{4}}) + O(\varepsilon). \tag{3.61}$$

The estimates for the third term and fourth term on the left side of (3.59) follow with Lemma 3.7:

$$\sum_{i=1}^{N} a_i \langle H_i(\boldsymbol{R}_X)\boldsymbol{R}_X, \boldsymbol{\mu}_i\rangle = O(\|\boldsymbol{v}\|_X^3) + O(e^{-\frac{L}{4}}) + O(\varepsilon) \tag{3.62}$$

and

$$\sum_{i=1}^{N} a_i^2 \langle H_i(\boldsymbol{R}_X)\boldsymbol{R}_X, \boldsymbol{R}_X\rangle = O(\|\boldsymbol{v}\|_X^3) + O(e^{-\frac{L}{4}}) + O(\varepsilon). \tag{3.63}$$

In consequence, substituting (3.60)-(3.63) into (3.59), we complete the proof of Lemma 3.9. □

Proof of Theorem 1.1. According to equations (3.57) and (3.58), we have

$$\sum_{i=1}^{N} E_i(\boldsymbol{\mu}_i) \leq \sum_{i=1}^{N} \langle H_i(\boldsymbol{R}_X)\boldsymbol{\mu}_i, \boldsymbol{\mu}_i\rangle = O(\|\boldsymbol{v}\|_X^3) + O(e^{-\frac{L}{4}}) + O(\varepsilon). \tag{3.64}$$

Meanwhile, by the definition of $\boldsymbol{\mu}_i$, (3.41), and the property of (3.44) the following estimate holds:

$$\sum_{i=1}^{N} E_i(\boldsymbol{\mu}_i) = \sum_{i=1}^{N} E_i(\boldsymbol{v}) + O(\|\boldsymbol{v}\|_X^3). \tag{3.65}$$

Hence, in light of

$$\sum_{i=1}^{N} E_i(\boldsymbol{v}) = E(\boldsymbol{v}) = \|\boldsymbol{v}\|_X^2\,, \tag{3.66}$$

there exists a constant C, such that

$$\|\boldsymbol{v}\|_X \leq O(e^{-\frac{L}{8}}) + O(\sqrt{\varepsilon}) \leq C(e^{-\frac{L}{8}} + \sqrt{\varepsilon}). \tag{3.67}$$

Let M be $2C$ in (3.1). This then concludes the result of Proposition 3.1. Consequently, the proof of Theorem 1.1 is complete. □

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