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# IDEAL CHARACTERIZATIONS OF MULTIPLE IMPACTS: A FRAME–INDEPENDENT APPROACH BY MEANS OF JET–BUNDLE GEOMETRY

By

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**Abstract.** We present, in the geometric setup given by the space–time bundle $\mathcal{M}$ and its first jet–extension $J_1(\mathcal{M})$, an ideal constitutive characterization based on the conservation of kinetic energy for a general mechanical system with a finite number of degrees of freedom in contact/impact with a multiple unilateral constraint $\mathcal{C}$ comprising a finite number of regular constraints of codimension 1.

We prove that the geometric structures associated to the elements of $\mathcal{C}$ determine a natural criterion to choose the simplest non-trivial constitutive characterization among the various possibilities preserving the kinetic energy in multiple impacts.

We put this specific choice at the core of an algorithm that determines the right–velocity of the system once the massive properties of the system, the elements of the multiple constraint and the left–velocity of the system are known, in cases of both single and multiple contact/impact.

We show the application of the algorithm in three significant examples: the Newton Cradle, the simultaneous impact of a disk with two disks at rest and in contact, the impact of a disk with a disk at rest and in contact with two other disks.

**Introduction.** Classical Contact/Impact Mechanics of systems subject to unilateral constraints has in the last few decades enjoyed renewed interest due to the wide variety of applications to physical and engineering problems such as the study of granular materials, kinematic chains and robotics.

The great breadth of possible contact/impact (from now on C/I) situations can be subdivided according to various criteria. One criterion is the nature of the massive parts

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comprising the mechanical system (single or multi–body); another criterion is the type of the constraints acting on the massive parts (ideal or non–ideal, with or without friction, positional and/or kinetics); a third criterion is the nature of the C/I between massive parts and constraints (single or multi–point).

The analysis of the system can be tackled by looking for a solution of the motion as a whole, or by splitting the motion into periods of differentiable motion separated by the instants of impulsive motion (the so-called "event driven" approach).

The main different methodologies that can be chosen to base the analysis of the C/I laws governing the interaction between different massive parts of the system or between massive parts and constraints include algebraic laws, local deformation laws and full deformation laws. Algebraic laws describe the C/I interaction by algebraic equations that relate pre–C/I and post–C/I quantities, possibly without the analysis of mechanical quantities such as forces and deformations in the period of C/I. Local deformation laws analyze and model deformation and slip only in a restricted region, usually referred to as a "contact" region, close to the points where the contacts between massive parts and/or constraints happen, and assume that the motion of the bulk of the massive parts of the system is governed by rigid–body Mechanics models and equations during the C/I. Full deformation laws establish that all the massive parts of the system are subject to the Continuum Mechanics models and equations.

Obviously, for each of these possible methodologies, extended bibliographies and lists of references could be exhibited. We will go back to the bibliography at the end of this introduction.

This paper concerns the motion of a general (single or multi–body) system subject to ideal frictionless positional constraints and involved in a general (single or multi–point) impact. We use the event–driven approach, using the model given by algebraic impact laws, in a frame–independent context.

We present a frame–independent kinetic–energy–preserving constitutive characterization of single or multiple C/I of general systems with a finite number of degrees of freedom with multiple positional unilateral constraints.

The great importance of frame independence for any result pertaining to velocity and kinetic energy in Classical Mechanics should be universally known and understood. We refer to the Appendix for a discussion about the importance of a frame–independent and coordinate–free description of C/I Mechanics. Since the pre–C/I and post–C/I quantities taken into account by the algebraic impact laws will simply be velocities and kinetic energy, due to the obvious fact that velocity and kinetic energy are frame–dependent quantities, a frame–independent context is mandatory to give physical meaning to the impact laws. Let us promptly recall that naive differential geometric setups such as the configuration space $\mathcal{Q}$, the finite dimensional Riemannian manifold whose points represent the possible configurations of the mechanical system, with its tangent space $T(\mathcal{Q})$, or the trivial product bundles $\mathbb{R} \times \mathcal{Q}$ and $\mathbb{R} \times T(\mathcal{Q})$, although ensuring a coordinate–free context, do not provide a frame–independent context. To achieve frame independence (and of course a coordinate–free setup), we will work in the differential geometric environment given by the so-called space–time configuration bundle $\mathcal{M}$ and its first jet–extension

$J_1(\mathcal{M})$, endowed with a vertical positive definite metric: this will be more fitting for introducing the concept of frame of reference and the related concepts of relative velocity and relative kinetic energy.

The general properties of the constitutive characterizations preserving the kinetic energy for an impulsive positional constraint were described in [1], also for a constraint of codimension greater than 1. In particular, it was shown that, for impacts with constraints of codimension greater than 1, requiring just the preservation of kinetic energy in the impact is not sufficient to uniquely determine the constitutive characterization, but a specific choice relying on geometric arguments and a maximality principle was suggested. Nevertheless, additional information about the nature of the constraint, for instance if the constraint is the intersection of two or more unilateral constraints of codimension 1, could (possibly) suggest a different choice.

In this paper we investigate this possibility. With this aim, we extend some concepts of the geometric description of a single unilateral constraint $\mathcal{S}$, viewed as the fibred subbundle of the space–time bundle $\mathcal{M}$, to the case of a finite set $\mathcal{C}$ of unilateral constraints of codimension 1. In particular we generalize the concept of a set of rest frames of a constraint to the case of a multiple constraint $\mathcal{C}$. This generalization allows one to obtain a meaningful concept of orthogonal components of velocity with respect to each constraint or subset of constraints of $\mathcal{C}$, introducing then new geometric tools that suggest how to select a meaningful frame–independent requirement of preservation of kinetic energy in case of multiple impact. This preservation condition, when employed in a suitable algorithm selecting those elements of $\mathcal{C}$ actually acting on the system in the analyzed impact, determines a constitutive characterization of $\mathcal{C}$ that turns out in general different from that obtained by considering $\mathcal{C}$ as a single constraint of codimension greater than 1. Furthermore, it better describes the impulsive behavior of the system. The applicability and the usefulness of the algorithm implementing this constitutive characterization, and a critical comparison between the two characterizations, are exhibited and discussed in some examples.

Although the results presented in the paper are grounded on the very basic approach given by an algebraic law involving only kinetic energy (then without local analysis of forces or other mechanical quantities pertaining to the contact), the strengthening given by the frame invariance requirement turns them out powerful enough to apply to non-trivial multiple ideal impacts of multi-body systems, such as the opening break shots in billiard games. Moreover, it will be clear that the results presented in the paper constitute the first step in order to analyze with the same methods more complicated situations, such as non–ideal (frictionless or not) multiple impacts, possibly with positional and kinetic multiple constrains, also in the presence of anisotropy.

In Section 1 we present the notation for and give a summary of the bundle foundations suitable for geometrizing the frame–independent approach to Impulsive Mechanics. Starting with the notion of space–time configuration bundle $\mathcal{M}$, we recall the affine and vector structures of its first jet-extension $J_1(\mathcal{M})$ and its vertical bundle $V(\mathcal{M})$ respectively, together with the concepts of vertical metric and frame of reference. Moreover we describe how the presence of a unilateral constraint $\mathcal{S}$, viewed as a fibred subbundle of $\mathcal{M}$, enriches the geometric setup with the subbundles $J_1(\mathcal{S}) \subset J_1(\mathcal{M})$ and $V(\mathcal{S}) \subset V(\mathcal{M})$, with the set of rest frames of the constraint and with the projection operators determined by the vertical metric. We end the section by recalling the ideal constitutive characterization of a single unilateral constraint, possibly of codimension greater than 1, based on the requirement of conservation of kinetic energy for all the rest frames of the constraint.

In Section 2, we simply generalize the main geometric concepts recalled in Section 1 to the case of a finite set $\mathcal{C} = \{\mathcal{S}_\xi \subset \mathcal{M},\, \xi = 1, \ldots, r\}$ of unilateral constraints acting on the system. In order to ensure a clear physical meaning for $\mathcal{C}$, we also require three conditions about codimension, regularity and independence of the elements of $\mathcal{C}$.

In Section 3, after some general remarks and premises, we propose a kinetic energy preserving constitutive characterization of the multiple constraint $\mathcal{C}$: to do so, we present a detailed algorithm that describes how to calculate a right–velocity $\mathbf{p}_R$ of the system for a given left–velocity $\mathbf{p}_L$. In accordance with our premises, the algorithm is devised in such a way that it takes into account the nature of the contact (absence of contact, single or multiple contacts with or without impact), it gives the same result as the usual ideal constitutive characterization for a single C/I and it handles the elements of $\mathcal{C}$ involved in case of multiple contact without distinguishing between them.

In Section 4, we present three meaningful examples, mainly inspired by simplified billiard strokes, of the behavior of the algorithm of Section 3 in situations involving multiple C/I: a simplified version of the classical example of the Newton Cradle; a simplified splitting stroke of the cue ball on two target balls; a simplified break shot of a cue ball on three target balls. Moreover, with the limitations imposed by the computational complexity, we compare the results of the algorithm with the results of the characterization presented in [1] for constraints of codimension greater than 1.

In Section 5, we conclude the paper with final remarks about the termination analysis of the algorithm and mention some possible future developments and generalizations of the techniques presented in the paper: for instance, the presence of permanent or impulsive kinetic constraints, the non-ideality of the C/I phenomenon and the possible anisotropy of $\mathcal{C}$.

In the Appendix we recall some basic notions about the geometric setup for a frame–independent approach to Classical Mechanics of systems with a finite number of degrees of freedom, briefly underlining the intrinsic limitations of the usual frame–dependent approach based on the geometry of the configuration space $\mathcal{Q}$.

The list of references is based on a minimality criterion: to make the paper self–consistent. Different choices, even if restricted to the works pertaining to multiple impacts, would oblige us to draw up a wide list of citations that, although important for a better comprehension of the various problems, methods and techniques of multi–impact Mechanics, are not focused on the specific approach presented in the paper and would then draw away the attention of the reader from the peculiarities (especially about frame independence) of this paper. Anyway, for large but not recent or exhaustive lists of general references, see for example the books [2–5].

**1. Preliminaries.** In this section we briefly present, also in order to fix notation, the bundle environment characterizing Geometric Impulsive Mechanics. For a very detailed treatise about the geometry of jet bundles, see [6,7]. For a concise overview of the use of jet-bundle theory in a frame–independent approach to Classical Mechanics, see [1,8–11].

1.1. *Geometry, kinematics and general impulsive aspects of free mechanical systems.* The usual geometric setup suitable to study frame–independent and time dependent Classical Mechanics of a system with a finite number $n$ of degrees of freedom consists of:

- a bundle $\pi_t : \mathcal{M} \to \mathbb{E}$, being $\mathcal{M}$ an $(n+1)$–dimensional differentiable manifold and $\mathbb{E}$ the affine time line. The elements of $\mathcal{M}$ are called space–time configurations of the system. We require, as a regularity condition, that all the fibres of $\mathcal{M}$ are diffeomorphic to the same manifold $\mathcal{Q}$, usually known as the configuration space of the system. The manifold $\mathcal{Q}$ takes intrinsically into account all the ideal positional bilateral constraints acting on the system. The bundle $\mathcal{M}$ is then non-canonically diffeomorphic to the product bundle $\mathbb{R} \times \mathcal{Q}$ and it can be described by fibred coordinates $(t, x^i)$, $i = 1, \ldots, n$, where $t$ is the time coordinate expressing the absolute time axiom.
- The first jet–extension $\pi : J_1(\mathcal{M}) \to \mathcal{M}$ of the bundle $\mathcal{M}$, representing the space of absolute velocities of the system. It is a $(2n+1)$–dimensional affine subbundle of the tangent bundle $T(\mathcal{M})$ of $\mathcal{M}$. The bundle $J_1(\mathcal{M})$ can be referred to as admissible jet–coordinates $(t, x^i, \dot{x}^i)$, $i = 1, \ldots, n$. Using local coordinates, the elements of $J_1(\mathcal{M})$ have the form $\mathbf{p} = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i}$ (the sum over repeated indexes is implicitly understood).
- The vertical vector bundle $\pi : V(\mathcal{M}) \to \mathcal{M}$ of the vectors of $T(\mathcal{M})$ that are vertical with respect to $\pi_t$, that is, that are tangent to the fibres of $\mathcal{M}$. The bundle $V(\mathcal{M})$ is the vector bundle modelling the affine bundle $J_1(\mathcal{M})$. The bundle $V(\mathcal{M})$ too can be referred to as admissible local coordinates $(t, x^i, \dot{x}^i)$ or, as usual, $(t, x^i, u^i)$. Then, using local coordinates, the elements of $V(\mathcal{M})$ have the form $\vec{U} = u^i \frac{\partial}{\partial x^i}$.
- A positive definite scalar product $\Phi : V(\mathcal{M}) \times_{\mathcal{M}} V(\mathcal{M}) \to \mathbb{R}$, acting on the fibres of $V(\mathcal{M})$. It is usually called the vertical metric, it takes intrinsically into account the mass properties of the system and it expresses in a wide sense the absolute space axiom (see [8] for a more detailed analysis of this aspect). Using local coordinates, the vertical metric is expressed by the positive definite matrix $g_{ij} = \Phi(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.
- The class $\mathcal{H}_{\mathcal{M}}$ of the frames of reference for the system (without any assumption of rigidity), that is, the set of global sections $\mathbf{h}_{\mathcal{M}} : \mathcal{M} \to J_1(\mathcal{M})$. Using local coordinates, the elements of $J_1(\mathcal{M})$ have the form $\mathbf{h}_{\mathcal{M}} = \frac{\partial}{\partial t} + h^i(t, x^j) \frac{\partial}{\partial x^i}$.

Some remarks about the geometric framework described above are in order:

1) every section $\mathbf{h}_{\mathcal{M}} \in \mathcal{H}_{\mathcal{M}}$ determines a fibred diffeomorphism $\psi_{\mathbf{h}} : \mathcal{M} \to \mathbb{R} \times \mathcal{Q}$ identifying the elements lying on the same integral line of $\mathbf{h}_{\mathcal{M}}$. Vice versa every global fibred diffeomorphism $\psi : \mathcal{M} \to \mathbb{R} \times \mathcal{Q}$ determines a frame of reference $\mathbf{h}_{\mathcal{M}} = (\psi^{-1})_*(\frac{\partial}{\partial t}) \in \mathcal{H}_{\mathcal{M}}$.
2) For every frame $\mathbf{h}_{\mathcal{M}} \in \mathcal{H}_{\mathcal{M}}$ and absolute velocity $\mathbf{p} \in J_1(\mathcal{M})$, the vertical vector $\vec{\Delta}_{\mathbf{h}_{\mathcal{M}}}(\mathbf{p}) = \mathbf{p} - \mathbf{h}_{\mathcal{M}}(\pi(\mathbf{p}))$ represents the relative velocity of $\mathbf{p}$ with respect

to $\mathbf{h}_{\mathcal{M}}$. Moreover, the function $\mathcal{K}_{\mathbf{h}_{\mathcal{M}}} : J_1(\mathcal{M}) \to \mathbb{R}$ such that $\mathcal{K}_{\mathbf{h}_{\mathcal{M}}}(\mathbf{p}) = \frac{1}{2}\,\Phi\,(\vec{\Delta}_{\mathbf{h}_{\mathcal{M}}}(\mathbf{p}), \vec{\Delta}_{\mathbf{h}_{\mathcal{M}}}(\mathbf{p}))$ represents the kinetic energy function relative to the frame $\mathbf{h}_{\mathcal{M}}$.

3) The vertical bundle $V(\mathcal{M})$ represents the space of possible impulses acting on the system (see [1]). In fact, the relation between the space $J_1(\mathcal{M})$ of absolute velocities and the space $V(\mathcal{M})$ of impulses is made clear by the very nature of impulsive problems: given an input velocity $\mathbf{p}_L$ (the so-called "left–velocity") of the system and an impulse $\vec{I}$, the sum

$$\mathbf{p}_R = \mathbf{p}_L + \vec{I} \tag{1}$$

must be an output velocity ("right–velocity") of the system. A similar situation is perfectly framed in the relations between affine spaces and modelling vector spaces.

4) Since forces and accelerations are not analyzed in the paper, the introduction of the second jet–extension $\pi : J_2(\mathcal{M}) \to J_1(\mathcal{M})$ is not necessary.

The description of an impulsive dynamic problem in this geometric context is very simple: it consists of determining an element $\mathbf{p}_R \in J_1(\mathcal{M})$, the right–velocity, once an element $\mathbf{p}_L \in J_1(\mathcal{M})$, the left–velocity, is known. Impulsive dynamics of free (in the sense of not subject to the action of unilateral constraints) systems is then assigned once an active impulse, that is, a global section $\vec{I} : J_1(\mathcal{M}) \to V(\mathcal{M})$, is assigned. Equation (1), viewed as the evolution equation for impulsive systems, determines the behavior of the system.

1.2. *Additional positional constraints.* The geometric description of a mechanical system subject to unilateral constraints requires additional structures. An additional positional constraint acting on the system is a globally $\mathbb{E}$–fibred subbundle $i : \mathcal{S} \to \mathcal{M}$ of $\mathcal{M}$. Once again we require that all the fibres of $\mathcal{S}$ with respect to the restriction of the fibre map $\pi_t$ are diffeomorphic to the same submanifold of the configuration space $\mathcal{Q}$. The subbundle $\mathcal{S}$ can be described by fibred coordinates $(t, q^\alpha)$, $\alpha = 1, \dots, s < n$. The immersion $i : \mathcal{S} \to \mathcal{M}$ can be described in a parametric way $x^i = x^i(t, q^\alpha)$, $i = 1, \dots, n; \alpha = 1, \dots, s$ or in a cartesian way $F_\rho(t, x^i) = 0$, $\rho = 1, \dots, n - s$.

We say that a space–time configuration $m \in \mathcal{M}$ of the system is a *contact position with* $\mathcal{S}$ (briefly, the system is *in contact with* $\mathcal{S}$) if $m \in i(\mathcal{S})$ (briefly, $m \in \mathcal{S}$).

We are not interested in $\mathcal{S}$ as additional bilateral constraint, since we suppose that all bilateral constraints are intrinsically taken into account in the construction of the space–time bundle $\mathcal{M}$. We need to introduce some geometric structures determined by the immersion $i : \mathcal{S} \to \mathcal{M}$ in order to analyze $\mathcal{S}$ viewed as unilateral constraint.

- The (restriction of the) bundle $\pi_t : \mathcal{S} \to \mathbb{E}$ determines its own first jet–extension $\pi : J_1(\mathcal{S}) \to \mathcal{S}$, its corresponding vertical bundle $\pi : V(\mathcal{S}) \to \mathcal{S}$ and the natural immersions $i_* : J_1(\mathcal{S}) \to J_1(\mathcal{M})$ and $i_* : V(\mathcal{S}) \to V(\mathcal{M})$. The vertical metric $\Phi$ of $V(\mathcal{M})$ can be restricted to a vertical metric $\varphi : V(\mathcal{S}) \times_{\mathcal{S}} V(\mathcal{S}) \to \mathbb{R}$.
- The immersion $i : \mathcal{S} \to \mathcal{M}$ also determines the pullback bundles $i^*(J_1(\mathcal{M})) \to \mathcal{S}$ of the possible velocities of the system when the system is in contact with $\mathcal{S}$ and $i^*(V(\mathcal{M})) \to \mathcal{S}$ of the possible impulses acting on the system when the system is in contact with $\mathcal{S}$. The pullback bundles $i^*(J_1(\mathcal{M}))$ and $i^*(V(\mathcal{M}))$

can be described by admissible local coordinates $(t, q^\alpha, \dot{x}^i)$ and $(t, q^\alpha, u^i)$, $\alpha = 1, \dots, s; i = 1, \dots, n$ respectively.

- Together with the set $\mathcal{H}_\mathcal{S}$ of global sections $\mathbf{h}_\mathcal{S} : \mathcal{S} \to J_1(\mathcal{S})$, the constraint $\mathcal{S}$ selects the class $\mathcal{H}_{\rfloor_{J_1(\mathcal{S})}} \subset \mathcal{H}_\mathcal{M}$ of those frames of reference of $\mathcal{M}$ tangent to $\mathcal{S}$, that is, whose restriction $\mathbf{h}_{\rfloor_\mathcal{S}} : \mathcal{S} \to J_1(\mathcal{M})$ has image in $i_*(J_1(\mathcal{S}))$. The class $\mathcal{H}_{\rfloor_{J_1(\mathcal{S})}}$ represents the set of all those frames of reference of $\mathcal{M}$ for which the constraint $\mathcal{S}$ could be considered at rest.

The whole geometric construction can be summarized by the following diagram:

$$
\begin{tikzcd}
J_1(\mathcal{S}) \arrow[r, "i_*"] \arrow[d, "\pi"] & i^*(J_1(\mathcal{M})) \arrow[r, "i_*"] \arrow[d, "\pi"] & J_1(\mathcal{M}) \arrow[d, "\pi"] \arrow[dd, dotted, bend right, "\vec{\Delta}_{\mathbf{h}_\mathcal{M}}"'] \\
\mathcal{S} \arrow[r, equal] & \mathcal{S} \arrow[r, "i"] & \mathcal{M} \arrow[u, dotted, bend right, "\mathbf{h}_\mathcal{M}"'] \\
V(\mathcal{S}) \arrow[r, "i_*"] \arrow[u, "\pi"] & i^*(V(\mathcal{M})) \arrow[r, "i_*"] \arrow[u, "\pi"] & V(\mathcal{M}) \arrow[u, "\pi"]
\end{tikzcd}
$$

The part of this geometric construction singled out by the subdiagram:

$$
\begin{tikzcd}
J_1(\mathcal{S}) \arrow[r, "i_*"] \arrow[d, "\pi"] & i^*(J_1(\mathcal{M})) \arrow[d, "\pi"] \arrow[dd, dotted, bend right, "\vec{\Delta}_{\mathbf{h}_\mathcal{M}}"'] \\
\mathcal{S} \arrow[r, equal] \arrow[u, dotted, bend left, "\mathbf{h}_{\rfloor_\mathcal{S}}"] & \mathcal{S} \arrow[u, dotted, bend right, "\mathbf{h}_\mathcal{M}"'] \\
V(\mathcal{S}) \arrow[r, "i_*"] \arrow[u, "\pi"] & i^*(V(\mathcal{M})) \arrow[u, "\pi"]
\end{tikzcd}
$$

is fundamental to describe in a frame–independent and event-driven way the impulsive behavior of the mechanical system. In fact

1) since all the bundles have the same base space $\mathcal{S}$, all their elements have space–time configurations belonging to $\mathcal{S}$, and then with the system in contact with $\mathcal{S}$.
2) The elements of $J_1(\mathcal{S})$ are absolute velocities of the system tangent to $\mathcal{S}$, while the elements of $i^*(J_1(\mathcal{M}))$ are absolute velocities of the system when the system is in contact with $\mathcal{S}$ but not necessarily velocities tangent to $\mathcal{S}$. The elements of $i^*(J_1(\mathcal{M}))$ are then all the possible “left” and “right” velocities of the system before and after the contact.

3) The elements of $i^*(V(\mathcal{V}))$ are both the relative velocities of the system in contact with $\mathcal{S}$ once a frame of reference $\mathbf{h}_{\mathcal{M}}$ (not necessarily in $\mathcal{H}\rfloor_{J_1(\mathcal{S})}$) is assigned and the absolute impulses possibly acting on the system in contact with $\mathcal{S}$ (in the sense expressed by equation (1)).

1.3. *Orthogonal component of an absolute velocity.* Thanks to the vertical metric $\Phi$, the vector bundle $i^*(V(\mathcal{M}))$ can be split (with a mild abuse of notation) into the direct sum

$$i^*(V(\mathcal{M})) = V(\mathcal{S}) \oplus (V(\mathcal{S}))^{\perp}. \tag{2}$$

For every absolute velocity $\mathbf{p} \in i^*(J_1(\mathcal{M}))$ and every frame of reference $\mathbf{h}_{\mathcal{M}} \in \mathcal{H}_{\mathcal{M}}$ the relative velocity $\vec{\Delta}_{\mathbf{h}_{\mathcal{M}}}(\mathbf{p}) \in i^*(V(\mathcal{M}))$ can then be split using (2) into the sum $\vec{\Delta}_{\mathbf{h}_{\mathcal{M}}}(\mathbf{p}) = \vec{v}_{\mathcal{S}}^{\parallel}(\mathbf{p}) + \vec{v}_{\mathcal{S}}^{\perp}(\mathbf{p})$ of the tangential and orthogonal components of $\vec{\Delta}_{\mathbf{h}_{\mathcal{M}}}(\mathbf{p})$ with respect to $\mathcal{S}$. In the general case, both these components depend on the frame $\mathbf{h}_{\mathcal{M}}$. However it is known (see [1]) that the orthogonal component $\vec{v}_{\mathcal{S}}^{\perp}(\mathbf{p})$ does not change when the frames are chosen in the class $\mathcal{H}\rfloor_{J_1(\mathcal{S})}$ for which the constraint could be considered at rest, while the tangential part $\vec{v}_{\mathcal{S}}^{\parallel}(\mathbf{p})$ remains dependent on the frame even when the choice is restricted to $\mathcal{H}\rfloor_{J_1(\mathcal{S})}$. This (partially) invariant orthogonal component is then the sole vertical vector giving a frame invariant meaning to the concept of orthogonal component $\vec{v}_{\mathcal{S}}^{\perp}(\mathbf{p})$ of the absolute velocity $\mathbf{p}$ with respect to the constraint $\mathcal{S}$.

Recalling that both $J_1(\mathcal{S})$ and $i^*(J_1(\mathcal{M}))$ have an affine nature, the affine bundle $i^*(J_1(\mathcal{M}))$ can be split (once again with a mild abuse of notation) into the sum

$$i^*(J_1(\mathcal{M})) = J_1(\mathcal{S}) \oplus (V(\mathcal{S}))^{\perp}. \tag{3}$$

However in this case the meaning of the symbol $\oplus$ is related to the action of the modelling bundle $i^*(V(\mathcal{M}))$ on the affine bundle $i^*(J_1(\mathcal{M}))$. The splitting (3) determines the projection operators

$$\mathcal{P}_{\mathcal{S}}: \quad i^*(J_1(\mathcal{M})) \to \quad J_1(\mathcal{S}); \qquad \mathcal{P}_{\mathcal{S}}^{\perp}: \quad i^*(J_1(\mathcal{M})) \to \quad (V(\mathcal{S}))^{\perp}. \tag{4}$$

The vertical vector $\mathcal{P}_{\mathcal{S}}^{\perp}(\mathbf{p}) \in (V(\mathcal{S}))^{\perp}$ associated to a velocity $\mathbf{p} \in i^*(J_1(\mathcal{M}))$ is (see [1]) the orthogonal component $\vec{v}_{\mathcal{S}}^{\perp}(\mathbf{p})$ of the absolute velocity $\mathbf{p}$ with respect to the constraint $\mathcal{S}$.

Let us stress however that, as a consequence of the properties of the splittings (2), (3), any statement regarding the constraint $\mathcal{S}$ and involving "the orthogonal or tangent component of the velocity with respect to $\mathcal{S}$":

i) can have an invariant nature only if we restrict our attention to the frames of reference of the subclass $\mathcal{H}\rfloor_{J_1(\mathcal{S})}$ of the possible rest frames of $\mathcal{S}$;

ii) cannot in any way have an invariant nature if the tangent component $\vec{v}_{\mathcal{S}}^{\parallel}(\mathbf{p})$ is involved.

If the unilateral constraint $\mathcal{S}$ is of codimension 1, the bundle $(V(\mathcal{S}))^{\perp}$ has fibres of dimension 1 and we can choose a (possibly unit) vector $\vec{u}_{\mathcal{S}}^{\perp}$ such that $(V(\mathcal{S}))^{\perp}$ is generated by $\vec{u}_{\mathcal{S}}^{\perp}$, that is, $(V(\mathcal{S}))^{\perp} = Lin(\vec{u}_{\mathcal{S}}^{\perp})$. If so, every orthogonal velocity $\vec{v}_{\mathcal{S}}^{\perp}(\mathbf{p})$ is a multiple of $\vec{u}_{\mathcal{S}}^{\perp}$.

It is a straightforward matter to show that, if $F(t, x^i) = 0$ is the local cartesian representation of $\mathcal{S}$ in fibred coordinates, the vertical vector

$$\vec{u}^{\perp}_{\mathcal{S}} = g^{ij} \frac{\partial F}{\partial x^j} \frac{\partial}{\partial x^i}\,, \tag{5}$$

where $g^{ij}$ is the inverse of the matrix $g_{ij}$ giving the local representation of the vertical metric $\Phi$, is a possible choice for the generator of $(V(\mathcal{S}))^{\perp}$.

1.4. *Incoming or outgoing nature of an absolute velocity.* Determining the "incoming" (in the sense of "causing a collision") or "outgoing" (in the sense of "result of a collision") nature of a velocity $\mathbf{p} \in i^*(J_1(\mathcal{M}))$ is crucial to establish if the system in the contact position $\pi(\mathbf{p}) \in \mathcal{S}$ does or does not have an impact with the constraint. However, this can depend on the nature of the constraint $\mathcal{S}$, as illustrated by the following examples.

Example 1.1. A massive point particle of mass $M$ moves in a three–dimensional euclidean halfspace. The space–time $\mathcal{M}$ can be described by global euclidean coordinates $(t, x, y, z)$, the vertical metric is $g_{ij} = diag(M, M, M)$ and the constraint $\mathcal{S}$ can be described by the immersion $(t, x, y) \hookrightarrow (t, x, y, 0)$ or by the condition $z = 0$. Given an absolute velocity $\mathbf{p} \in i^*(J_1(\mathcal{M}))$, we have:

$$\mathbf{p} = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z},$$

and

$$\begin{cases} \mathcal{P}_{\mathcal{S}}(\mathbf{p}) = \dfrac{\partial}{\partial t} + \dot{x}\dfrac{\partial}{\partial x} + \dot{y}\dfrac{\partial}{\partial y}, \\ \mathcal{P}^{\perp}_{\mathcal{S}}(\mathbf{p}) = \vec{v}^{\perp}_{\mathcal{S}}(\mathbf{p}) = \dot{z}\dfrac{\partial}{\partial z}. \end{cases}$$

The "incoming" or "outgoing" nature of $\mathbf{p}$ with respect to $\mathcal{S}$ depends on what halfspace $z \leq 0$ or $z \geq 0$ is admissible for the material point. For instance, if $z \geq 0$ is the admissible halfspace, choosing $\vec{u}^{\perp}_{\mathcal{S}} = \dfrac{\partial}{\partial z}$, the incoming nature of $\mathbf{p}$ is equivalent to the condition $\Phi\left(\vec{v}^{\perp}_{\mathcal{S}}(\mathbf{p}), \vec{u}^{\perp}_{\mathcal{S}}\right) = \Phi\left(\dot{z}\dfrac{\partial}{\partial z}, \dfrac{\partial}{\partial z}\right) = M\,\dot{z} < 0$ that ensures the presence of an impact.

Example 1.2. A massive point particle moves in a three–dimensional euclidean space crossed by a straight line. The space–time $\mathcal{M}$ can be described by global euclidean coordinates $(t, x, y, z)$ and the constraint $\mathcal{S}$ can be described by the immersion $(t, x) \hookrightarrow (t, x, 0, 0)$ or by the conditions $y = z = 0$. Given an absolute velocity $\mathbf{p} \in i^*(J_1(\mathcal{M}))$, we have:

$$\mathbf{p} = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z},$$

and

$$\begin{cases} \mathcal{P}_{\mathcal{S}}(\mathbf{p}) = \dfrac{\partial}{\partial t} + \dot{x}\dfrac{\partial}{\partial x}, \\ \mathcal{P}^{\perp}_{\mathcal{S}}(\mathbf{p}) = \vec{v}^{\perp}_{\mathcal{S}}(\mathbf{p}) = \dot{y}\dfrac{\partial}{\partial y} + \dot{z}\dfrac{\partial}{\partial z}. \end{cases}$$

Due to the codimension of $\mathcal{S}$, in this case it is not clear how to determine the "incoming" or "outgoing" nature of $\mathbf{p}$ with respect to $\mathcal{S}$, since all the velocities $\mathbf{p} \in i^*(J_1(\mathcal{M})), \mathbf{p} \notin J_1(\mathcal{S})$ can be either velocities of the point before the impact with $\mathcal{S}$ or velocities of the point after the impact with $\mathcal{S}$.

The examples above illustrate how the codimension of the unilateral constraint plays an important role in the analysis of the impulsive phenomenon, even in order to determine if an impact happens. The following example points out the difference between a constraint that is intrinsically of codimension greater than 1 (such as the constraint of Example 1.2: a straight line in the three–dimensional space) and a constraint that is actually of codimension greater than 1 but it is obtained by the simultaneous action of two or more constraints of codimension 1.

EXAMPLE 1.3. A massive point particle moves in a three–dimensional euclidean part of space delimited by two non–parallel planes. The space–time $\mathcal{M}$ is once again described by global euclidean coordinates $(t, x, y, z)$. Let the constraint $\mathcal{S}_1$ be described by the immersion $(t, x, y) \hookrightarrow (t, x, y, 2y)$, equivalent to the condition $z - 2y = 0$, and the constraint $\mathcal{S}_2$ be described by the immersion $(t, x, z) \hookrightarrow (t, x, 2z, z)$, equivalent to $y - 2z = 0$. We consider the point in contact with the intersection $\mathcal{S}_1 \cap \mathcal{S}_2$, that is, a constraint of codimension 2. Once again, given an absolute velocity $\mathbf{p} \in i^*(J_1(\mathcal{M}))$ we have:

$$\mathbf{p} = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z},$$

and we have

$$\begin{cases} \mathcal{P}_{\mathcal{S}_1 \cap \mathcal{S}_2}(\mathbf{p}) = \dfrac{\partial}{\partial t} + \dot{x}\dfrac{\partial}{\partial x}, \\[2ex] \vec{v}^{\perp}_{\mathcal{S}_1 \cap \mathcal{S}_2}(\mathbf{p}) = \dot{y}\dfrac{\partial}{\partial y} + \dot{z}\dfrac{\partial}{\partial z}, \end{cases}$$

that are the same projection and orthogonal velocity of Example 1.2. Nevertheless, the situations described by Examples 1.2 and 1.3 are different, for at least three main reasons. The first regards the part of $\mathcal{M}$ that is admissible for the material point: the whole $\mathcal{M}$ for the Example 1.2, only one of the four parts in which $\mathcal{M}$ is divided by $\mathcal{S}_1$ and $\mathcal{S}_2$ for the Example 1.3. The second is that in Example 1.3 we have three orthogonal velocities:

$$\vec{v}^{\perp}_{\mathcal{S}_1}(\mathbf{p}) = \frac{2}{5}\,(2\dot{y} - \dot{z})\,\frac{\partial}{\partial y} - \frac{1}{5}\,(2\dot{y} - \dot{z})\,\frac{\partial}{\partial z},$$

$$\vec{v}^{\perp}_{\mathcal{S}_2}(\mathbf{p}) = \frac{1}{5}\,(\dot{y} - 2\dot{z})\,\frac{\partial}{\partial y} - \frac{2}{5}\,(\dot{y} - 2\dot{z})\,\frac{\partial}{\partial z},$$

and $\vec{v}^{\perp}_{\mathcal{S}_1 \cap \mathcal{S}_2}(\mathbf{p})$ instead of a single orthogonal velocity $\vec{v}^{\perp}_{\mathcal{S}}(\mathbf{p})$ of Example 1.2. Note that a straightforward calculation shows that, in the general case, $\vec{v}^{\perp}_{\mathcal{S}_1 \cap \mathcal{S}_2}(\mathbf{p}) \neq \vec{v}^{\perp}_{\mathcal{S}_1}(\mathbf{p}) + \vec{v}^{\perp}_{\mathcal{S}_2}(\mathbf{p})$. The third is that in cases such as Example 1.3 we can check if the left–velocity $\mathbf{p}_L$ is an incoming velocity for $\mathcal{S}_1$ and/or $\mathcal{S}_2$, calculating the scalar products $\Phi\left(\vec{v}^{\perp}_{\mathcal{S}_1}(\mathbf{p}_L), \vec{u}^{\perp}_{\mathcal{S}_1}\right)$ and $\Phi\left(\vec{v}^{\perp}_{\mathcal{S}_2}(\mathbf{p}_L), \vec{u}^{\perp}_{\mathcal{S}_2}\right)$.

The following result facilitates the check of the “incoming” or “outgoing” nature of a velocity $\mathbf{p}$ with respect to a unilateral constraint of codimension 1.

LEMMA 1.1. Let $\mathcal{S}$ be a constraint of codimension 1, let $F_{\mathcal{S}}(t, x^i) = 0$ be the cartesian representation of $\mathcal{S}$ and $\mathbf{p} \in i^*(J_1(\mathcal{M}))$. Then, for a suitable choice of $\vec{u}^{\perp}_{\mathcal{S}}$, we have:

$$\Phi\left(\vec{v}^{\perp}_{\mathcal{S}}(\mathbf{p}), \vec{u}^{\perp}_{\mathcal{S}}\right) = \mathbf{p}(F_{\mathcal{S}})\,.$$

*Proof.* Due to the splitting (3), we have:

$$\mathbf{p}(F_{\mathcal{S}}) = \mathcal{P}_{\mathcal{S}}(\mathbf{p})(F_{\mathcal{S}}) + \vec{v}_{\mathcal{S}}^{\perp}(\mathbf{p})(F_{\mathcal{S}}) = \vec{v}_{\mathcal{S}}^{\perp}(\mathbf{p})(F_{\mathcal{S}})$$

because $\mathcal{P}_{\mathcal{S}}(\mathbf{p})(F_{\mathcal{S}}) = 0$ since $\mathcal{P}_{\mathcal{S}}(\mathbf{p})$ is tangent to $\mathcal{S}$ by construction. Then using local coordinates and recalling (5), with obvious notation we have:

$$\begin{aligned}\Phi\left(\vec{v}_{\mathcal{S}}^{\perp}(\mathbf{p}), \vec{u}_{\mathcal{S}}^{\perp}\right) &= \Phi\left((\vec{v}_{\mathcal{S}}^{\perp})^h \frac{\partial}{\partial x^h}, g^{ij}\frac{\partial F_{\mathcal{S}}}{\partial x^j}\frac{\partial}{\partial x^i}\right) = g_{hi}g^{ij}(\vec{v}_{\mathcal{S}}^{\perp})^h \frac{\partial F_{\mathcal{S}}}{\partial x^j} \\ &= (\vec{v}_{\mathcal{S}}^{\perp})^h \frac{\partial F_{\mathcal{S}}}{\partial x^h} = \vec{v}_{\mathcal{S}}^{\perp}(\mathbf{p})(F_{\mathcal{S}}) = \mathbf{p}(F_{\mathcal{S}}).\end{aligned}$$

□

Later on, given $\mathcal{S}$ of codimension 1, we preferably but not compulsorily choose the generator $\vec{u}_{\mathcal{S}}^{\perp}$ and the cartesian representation $F_{\mathcal{S}}(t, x^i) = 0$ such that a velocity $\mathbf{p}$ is incoming for $\mathcal{S}$ if $\Phi\left(\vec{v}_{\mathcal{S}}^{\perp}(\mathbf{p}), \vec{u}_{\mathcal{S}}^{\perp}\right) = \mathbf{p}(F_{\mathcal{S}}) < 0$, and $\mathbf{p}$ is outgoing for $\mathcal{S}$ if $\Phi\left(\vec{v}_{\mathcal{S}}^{\perp}(\mathbf{p}), \vec{u}_{\mathcal{S}}^{\perp}\right) = \mathbf{p}(F_{\mathcal{S}}) \geq 0$.

1.5. *Impulsive constitutive characterization of a constraint.* The impulsive evolution problem for a system subject to a unilateral constraint, although apparently very similar to that of a free system, is conceptually very different. Both the problems are based on equation (1) and we saw that for free systems the evolution is known once an active impulse is a priori assigned. In the constrained case instead, the reactive impulse due to the constraint is in general unknown. Only the assignment of an impulsive constitutive characterization of the constraint fit to uniquely determine the reactive impulse makes possible the knowledge of the evolution of the system.

A constitutive characterization of an impulsive constraint is then a map

$$\begin{array}{rccc}\vec{I} & : & i^*(J_1(\mathcal{M})) & \to & i^*(V(\mathcal{M})) \\ & & \mathbf{p}_L & \rightsquigarrow & \vec{I}(\mathbf{p}_L)\end{array} \tag{6}$$

assigning to each possible left–velocity $\mathbf{p}_L \in i^*(J_1(\mathcal{M}))$ the reactive impulse $\vec{I}(\mathbf{p}_L) \in i^*(V(\mathcal{M}))$. The right–velocity $\mathbf{p}_R \in i^*(J_1(\mathcal{M}))$ giving the evolution of the system is then uniquely determined by equation (1) written as $\mathbf{p}_R = \mathbf{p}_L + \vec{I}(\mathbf{p}_L)$.

In the case of constraint $\mathcal{S}$ of codimension 1, the requirement of conservation of kinetic energy of the colliding system for every frame of reference in the class $\mathcal{H}\rfloor_{J_1(\mathcal{S})}$ (that, similarly to the case of the orthogonal component $\vec{v}_{\mathcal{S}}^{\perp}(\mathbf{p})$ of an absolute velocity $\mathbf{p}$, is the only class for which the conservation of kinetic energy has a clear meaning) uniquely determines (see [1]) the ideal constitutive characterization (6) of the constraint in the form:

$$\begin{array}{rccc}\vec{I}_{ideal} & : & i^*(J_1(\mathcal{M})) & \to & i^*(V(\mathcal{M})) \\ & & \mathbf{p}_L & \rightsquigarrow & -2\,\vec{v}_{\mathcal{S}}^{\perp}(\mathbf{p}_L).\end{array} \tag{7}$$

More precisely, since $codim(\mathcal{S}) = 1$, a preliminary check to verify if the left–velocity $\mathbf{p}_L$ is of incoming nature is mandatory. This can be done by adding the condition $\Phi\left(\vec{v}_{\mathcal{S}}^{\perp}(\mathbf{p}_L), \vec{u}_{\mathcal{S}}^{\perp}\right) < 0$ for a suitable choice of the generator $\vec{u}_{\mathcal{S}}^{\perp}$ of $(V(\mathcal{S}))^{\perp}$. Then the kinetic–energy–preserving ideal constitutive characterization for constraints $\mathcal{S}$ with

$codim(\mathcal{S}) = 1$ is the rule:

$$\vec{I}_{ideal}(\mathbf{p}_L) = \begin{cases} -2\,\vec{v}^{\perp}_{\mathcal{S}}(\mathbf{p}_L) & \text{if } \Phi\left(\vec{v}^{\perp}_{\mathcal{S}}(\mathbf{p}_L), \vec{u}^{\perp}_{\mathcal{S}}\right) < 0; \\ \\ 0 & \text{if } \Phi\left(\vec{v}^{\perp}_{\mathcal{S}}(\mathbf{p}_L), \vec{u}^{\perp}_{\mathcal{S}}\right) \geq 0. \end{cases} \tag{8}$$

In case of constraints of codimension greater than 1, such as the constraint of Example 1.2, the conservation of kinetic energy does not uniquely determine the ideal constitutive characterization (see once again [1]). Nevertheless, in the absence of additional information about the constraint, and recalling that in the general case there is no need to check the incoming nature of $\mathbf{p}_L$, the constitutive characterization (7) is still a possible choice that, because of geometric reasons and maximality principles, turns out to be the most natural among those preserving kinetic energy.

However, when the constraint of codimension greater than 1 is given by intersection of two (or more) constraints of codimension 1, such as the constraint of Example 1.3, the situation becomes different. In fact in this case the geometric structure of the system is enriched by different orthogonal velocities $\vec{v}^{\perp}_{\mathcal{S}_1}(\mathbf{p})$, $\vec{v}^{\perp}_{\mathcal{S}_2}(\mathbf{p})$ and $\vec{v}^{\perp}_{\mathcal{S}_1\cap\mathcal{S}_2}(\mathbf{p})$, and moreover by the fact that in general $\vec{v}^{\perp}_{\mathcal{S}_1\cap\mathcal{S}_2}(\mathbf{p}) \neq \vec{v}^{\perp}_{\mathcal{S}_1}(\mathbf{p}) + \vec{v}^{\perp}_{\mathcal{S}_2}(\mathbf{p})$. These velocities will play a crucial role in our choice of ideal impulsive constitutive characterization of multiple constraints.

**2. Geometry and kinematics of systems with multiple contacts.** In this section we analyze in detail the geometric structure of a constraint $\mathcal{C}$ of codimension greater than 1 given by intersection of constraints of codimension 1 (for brevity, a multiple constraint $\mathcal{C}$). Moreover, we define the notions of incoming and outgoing velocity for $\mathcal{C}$ and we introduce the several different orthogonal velocities determined by $\mathcal{C}$.

2.1. *Geometry of systems with multiple contacts.* The geometric setup described in the previous section can be generalized to mechanical systems subject to (possible) multiple constraints. A multiple constraint is a set $\mathcal{C} = \{\mathcal{S}_\xi \subset \mathcal{M},\, \xi = 1, \ldots, r\}$ of unilateral constraints acting on the system, where $r$ is finite and $\pi_t : \mathcal{M} \to \mathbb{E}$ is the usual space–time manifold of the system.

For every constraint $\mathcal{S}_\xi$ we can introduce the geometric structures described in Subsections 1.2 and 1.3: the jet bundle $J_1(\mathcal{S}_\xi)$, the vertical bundle $V(\mathcal{S}_\xi)$, the pullback bundles $i^*_\xi(J_1(\mathcal{M}))$ and $i^*_\xi(V(\mathcal{M}))$, the class of frames $\mathcal{H}_{\rfloor_{J_1(\mathcal{S}_\xi)}}$, the splittings $V(\mathcal{S}_\xi) \oplus (V(\mathcal{S}_\xi))^\perp$ and $J_1(\mathcal{S}_\xi) \oplus (V(\mathcal{S}_\xi))^\perp$ of $i^*_\xi(V(\mathcal{M}))$ and $i^*_\xi(J_1(\mathcal{M}))$ and the projectors $\mathcal{P}_{\mathcal{S}_\xi}$ and $\mathcal{P}^{\perp}_{\mathcal{S}_\xi}$.

Although the following analysis can be performed in very general cases, we require three conditions on the elements of $\mathcal{C}$ in order to avoid very singular situations with bare physical meaning (such as the one of Example 1.2 or even worse).

CONDITION 1 (Dimensional). *$codim(\mathcal{S}_\xi) = 1 \quad \forall\, \xi = 1, \ldots, r$.*

Of course, the system can be subject to multiple contacts only if there exists at least one intersection $\mathcal{S}_{\xi_1} \cap \ldots \cap \mathcal{S}_{\xi_k} \neq \emptyset$ for some $k$–uple of indexes $\{\xi_1, \ldots, \xi_k,\, 2 \leq k \leq r\}$. The possible simultaneous action of more than one constraint on the system is then pointed out by the map

$$\begin{array}{rccl} Cont: & \mathcal{M} & \to & Parts\,(\{1, \ldots, r\}) \\ & m & \rightsquigarrow & \{\xi \,\text{s.t.}\, m \in \mathcal{S}_\xi\}\,. \end{array} \tag{9}$$

If $Cont(m) = \emptyset$, the system in the space–time configuration $m$ is not in contact with any constraint, while if $Cont(m) = \{\xi_1, \ldots, \xi_k, \, k \leq r\} \neq \emptyset$ the system in the space–time configuration $m$ is simultaneously in contact with the constraints $\mathcal{S}_{\xi_1}, \ldots, \mathcal{S}_{\xi_k}$.

Due to the generality of the approach, the intersection $\mathcal{S}_{\xi_1} \cap \ldots \cap \mathcal{S}_{\xi_k}$, even when not empty, might not be a subbundle of $\mathcal{M}$ (or even a regular manifold at all): constraints $\mathcal{S}_{\xi_1}, \ldots, \mathcal{S}_{\xi_k}$ with an effective time dependence easily provide examples of non regular intersections. Then we require the following condition:

CONDITION 2 (Regularity). If $\mathcal{S}_{\xi_1} \cap \ldots \cap \mathcal{S}_{\xi_k} \neq \emptyset$, then it is (a regular manifold and) a subbundle of $\mathcal{M}$.

If $\mathcal{S}_{\xi_1} \cap \ldots \cap \mathcal{S}_{\xi_k} \neq \emptyset$, we can consider the class $\mathcal{H}_{\rfloor_{\xi_1 \ldots \xi_k}}$ of the frames of $\mathcal{H}_{\mathcal{M}}$ that are simultaneously tangent to all the constraints $\mathcal{S}_{\xi_1}, \ldots, \mathcal{S}_{\xi_k}$. Of course, if $\mathbf{h} \in \mathcal{H}_{\rfloor_{\xi_1 \ldots \xi_k}}$, then $\mathbf{h} \in \mathcal{H}_{\rfloor_{\xi_l}}$ for all $l = 1, \ldots, k$. Moreover Condition 2 implies that $dim(\mathcal{S}_{\xi_1} \cap \ldots \cap \mathcal{S}_{\xi_k}) \geq 1$ and then the set $\mathcal{H}_{\rfloor_{\xi_1 \ldots \xi_k}}$ is non-empty.

Recalling that, due to Condition 1, we can introduce the generators $\vec{u}_\xi^\perp$ for each vertical subspace $(V(\mathcal{S}_\xi))^\perp$, we finally require the following condition:

CONDITION 3 (Independence). If $m$ is a multiple contact configuration with $Cont(m) = \{\xi_1, \ldots, \xi_k\}$, then the vectors $\vec{u}_{\xi_1}^\perp(m), \ldots, \vec{u}_{\xi_k}^\perp(m)$ are linearly independent.

Note that, due to (5), Condition 3 implies that for every $m \in \mathcal{S}_{\xi_1} \cap \ldots \cap \mathcal{S}_{\xi_k}$, we have $Codim(\mathcal{S}_{\xi_1} \cap \ldots \cap \mathcal{S}_{\xi_k}) = k$.

2.2. *Kinematics of systems with multiple contacts.* For every $\mathbf{p}$ such that $\pi(\mathbf{p}) \in \mathcal{S}_{\xi_1} \cap \ldots \cap \mathcal{S}_{\xi_k}$ we have in particular that $\pi(\mathbf{p}) \in \mathcal{S}_{\xi_l}$ for all $l = 1, \ldots, k$. By using the projectors $\mathcal{P}_{\mathcal{S}_\xi}^\perp$, we can introduce the $k$ orthogonal velocities $\vec{v}_{\xi_l}^\perp(\mathbf{p}) \in (V(\mathcal{S}_{\xi_l}))^\perp$ for all $l = 1, \ldots, k$.

DEFINITION 2.1. Let $\mathcal{C} = \{\mathcal{S}_\xi \subset \mathcal{M}, \, \xi = 1, \ldots, r\}$ be a multiple constraint satisfying Conditions 1, 2, 3 and let $\mathbf{p} \in i^*(J_1(\mathcal{S}))$ such that $Cont(\pi(\mathbf{p})) = \{\xi_1, \ldots, \xi_k\}$ (or, that is the same, $\pi(\mathbf{p}) \in \mathcal{S}_{\xi_1} \cap \ldots \cap \mathcal{S}_{\xi_k}$). Then:

- $\mathbf{p}$ is an incoming velocity for $\mathcal{C}$ if it is an incoming velocity for at least one $\mathcal{S}_{\xi_l} \in \{\mathcal{S}_{\xi_1}, \ldots, \mathcal{S}_{\xi_k}\} \quad \Leftrightarrow \quad \exists \; l \in \{1, \ldots, k\}$ s.t. $\Phi\left(\vec{v}_{\mathcal{S}_{\xi_l}}^\perp(\mathbf{p}), \vec{u}_{\mathcal{S}_{\xi_l}}^\perp\right) < 0$;
- $\mathbf{p}$ is an outgoing velocity for $\mathcal{C}$ if it is an outgoing velocity for all the elements $\mathcal{S}_{\xi_1}, \ldots, \mathcal{S}_{\xi_k} \quad \Leftrightarrow \quad \Phi\left(\vec{v}_{\mathcal{S}_{\xi_l}}^\perp(\mathbf{p}), \vec{u}_{\mathcal{S}_{\xi_l}}^\perp\right) \geq 0 \;\; \forall \, l = 1, \ldots, k$.

Of course the system in the contact position $\pi(\mathbf{p})$ is subject to an impact if and only if $\mathbf{p}$ is an incoming velocity for $\mathcal{C}$.

Note moreover that, if $\mathbf{p}$ is such that $\pi(\mathbf{p}) \in \mathcal{S}_{\xi_1} \cap \ldots \cap \mathcal{S}_{\xi_k}$, we can also introduce the orthogonal component of the velocity $\vec{v}_{\eta_1 \ldots \eta_z}^\perp(\mathbf{p}) \in (V(\mathcal{S}_{\xi_{\eta_1}} \cap \ldots \cap \mathcal{S}_{\xi_{\eta_z}}))^\perp$ for every possible choice of intersection $\mathcal{S}_{\xi_{\eta_1}} \cap \ldots \cap \mathcal{S}_{\xi_{\eta_z}}$ of constraints chosen among $\mathcal{S}_{\xi_1}, \ldots, \mathcal{S}_{\xi_k}$. However, all these vertical vectors will not be used in the following, with the exceptions of the "single" orthogonal velocities $\vec{v}_{\xi_l}^\perp(\mathbf{p}) \in (V(\mathcal{S}_{\xi_l}))^\perp \; l = 1, \ldots, k$ and the "global" orthogonal velocity $\vec{v}_{1 \ldots k}^\perp(\mathbf{p}) \in (V(\mathcal{S}_{\xi_1} \cap \ldots \cap \mathcal{S}_{\xi_k}))^\perp$.

**3. Constitutive characterization of ideal multiple impulsive constraints.** In this section, we list the essential requirements for a general constitutive characterization of an ideal multiple constraint $\mathcal{C}$. Stressing once again that the only requirement of preservation of kinetic energy is insufficient to uniquely determine the constitutive characterization for multiple constraints, we briefly discuss the leeway in the choice of a constitutive characterization satisfying these requirements. Then we present our choice. The map assigning to each left velocity $\mathbf{p}_L$ the corresponding reactive impulse $\vec{I}_{ideal}(\mathbf{p}_L)$ is defined in the form of an algorithm. Finally we discuss the main properties of this constitutive characterization and of the algorithm.

3.1. *General aspects.* The geometric construction described in the previous section gives us all the instruments to present ideal constitutive characterizations for a multiple constraint $\mathcal{C} = \{\mathcal{S}_\xi \subset \mathcal{M},\, \xi = 1, \ldots, r\}$ in a very general form. The main fixed points that we must take into account in the assignment of the map (6) are:

a) we require the preservation of kinetic energy of the system before and after the contact with the constraints for every frame of reference for which the requirement has a clear meaning. This is an ideality requirement.
b) In the absence of additional information about the constraints, we must not distinguish the elements of $\mathcal{C}$ involved in a multiple contact. This is an isotropy requirement.
c) In case of single contact with only one of the elements of $\mathcal{C}$, the constitutive characterization must coincide with (8). This is a coherence requirement.

Let $\pi(\mathbf{p}_L) \in \mathcal{S}_{\xi_1} \cap \ldots \cap \mathcal{S}_{\xi_k}$ and suppose, for simplicity, that $\mathbf{p}_L$ is an incoming velocity for all the constraints $\mathcal{S}_{\xi_l}$, $l = 1, \ldots, k$. The constitutive characterization for the multiple contact can in general involve all the possible orthogonal velocities $\vec{v}^{\perp}_{\eta_1 \ldots \eta_z}(\mathbf{p}_L) \in (V(\mathcal{S}_{\xi_{\eta_1}} \cap \ldots \cap \mathcal{S}_{\xi_{\eta_z}}))^{\perp}$ for every possible choice of intersection $\mathcal{S}_{\xi_{\eta_1}} \cap \ldots \cap \mathcal{S}_{\xi_{\eta_z}}$ of the constraints $\mathcal{S}_{\xi_1}, \ldots, \mathcal{S}_{\xi_k}$. However, condition c) implies that no distinction can be done among the $\mathcal{S}_{\xi_1}, \ldots, \mathcal{S}_{\xi_k}$.

A very simple choice of constitutive characterization satisfying conditions a), b), c) involves the sole global orthogonal velocity $\vec{v}^{\perp}_{1\ldots k}(\mathbf{p}_L) \in (V(\mathcal{S}_{\xi_1} \cap \ldots \cap \mathcal{S}_{\xi_k}))^{\perp}$, and in this case the most reasonable choice is analogous to (8). Nevertheless this choice does not take into account the nature of the constraint $\mathcal{C}$ as formed by constraints of codimension 1.

To do so, we can involve the orthogonal velocities $\vec{v}^{\perp}_{\xi_l}(\mathbf{p}_L) \in (V(\mathcal{S}_{\xi_\eta}))^{\perp}$, $l = 1, \ldots, k$. In this case, the map (6) can be assigned in the form

$$\begin{array}{rccl} \vec{I}_{ideal} & : & i^*(J_1(\mathcal{M})) & \rightarrow & i^*(V(\mathcal{M})) \\ & & \mathbf{p}_L & \rightsquigarrow & \lambda_1 \vec{v}^{\perp}_{\xi_1}(\mathbf{p}_L) + \ldots + \lambda_k \vec{v}^{\perp}_{\xi_k}(\mathbf{p}_L), \end{array} \tag{10}$$

but different coefficients $\lambda_j$ would distinguish the elements of the multiple contact. Then the simplest constitutive characterization involving the velocities $\vec{v}^{\perp}_{\xi_l}(\mathbf{p})$ and satisfying condition b) is

$$\begin{array}{rccl} \vec{I}_{ideal} & : & i^*(J_1(\mathcal{M})) & \rightarrow & i^*(V(\mathcal{M})) \\ & & \mathbf{p}_L & \rightsquigarrow & \lambda \left( \vec{v}^{\perp}_{\xi_1}(\mathbf{p}_L) + \ldots + \vec{v}^{\perp}_{\xi_k}(\mathbf{p}_L) \right) \end{array} \tag{11}$$

with the coefficient $\lambda$ suitable to ensure condition a). Then we prove the following

LEMMA 3.1. The coefficient $\lambda$ of the definition (11) satisfying the ideality requirement is

$$\lambda = -2 \frac{\Phi\left(\vec{v}^{\perp}_{1\ldots k}(\mathbf{p}_L), \vec{v}^{\perp}_{\xi_1}(\mathbf{p}_L) + \ldots + \vec{v}^{\perp}_{\xi_k}(\mathbf{p}_L)\right)}{\Phi\left(\vec{v}^{\perp}_{\xi_1}(\mathbf{p}_L) + \ldots + \vec{v}^{\perp}_{\xi_k}(\mathbf{p}_L), \vec{v}^{\perp}_{\xi_1}(\mathbf{p}_L) + \ldots + \vec{v}^{\perp}_{\xi_k}(\mathbf{p}_L)\right)} \ . \tag{12}$$

*Proof.* For every left–velocity $\mathbf{p}_L$ and frame of reference $\mathbf{h} \in \mathcal{H}_{\rfloor \xi_1 \ldots \xi_k}$, we have to impose the preservation of kinetic energy of the system with respect to the frame $\mathbf{h} \in \mathcal{H}_{\rfloor \xi_1 \ldots \xi_k}$, that is,

$$\mathcal{K}_{\mathbf{h}}(\mathbf{p}_L) = \mathcal{K}_{\mathbf{h}}(\mathbf{p}_L + \vec{I}_{ideal}(\mathbf{p}_L)) \qquad \forall\, \mathbf{p}_L \in i^*(J_1(\mathcal{M})),\ \forall\, \mathbf{h} \in \mathcal{H}_{\rfloor \xi_1 \ldots \xi_k}.$$

This means

$$\frac{1}{2}\,\Phi\,(\mathbf{p}_L - \mathbf{h}, \mathbf{p}_L - \mathbf{h}) = \frac{1}{2}\,\Phi\,\left(\mathbf{p}_L + \vec{I}_{ideal}(\mathbf{p}_L) - \mathbf{h}, \mathbf{p}_L + \vec{I}_{ideal}(\mathbf{p}_L) - \mathbf{h}\right)$$

or, that is the same,

$$2\,\Phi\,\left(\mathbf{p}_L - \mathbf{h}, \vec{I}_{ideal}(\mathbf{p}_L)\right) + \Phi\,\left(\vec{I}_{ideal}(\mathbf{p}_L), \vec{I}_{ideal}(\mathbf{p}_L)\right) = 0.$$

Using (11) and the projection operators $\mathcal{P}_{\rfloor \xi_1 \ldots \xi_k}$ and $\mathcal{P}^{\perp}_{\rfloor \xi_1 \ldots \xi_k}$, we can split $\mathbf{p}_L - \mathbf{h} = [P_{\rfloor \xi_1 \ldots \xi_k}(\mathbf{p}_L) - \mathbf{h}] + \vec{v}^{\perp}_{1\ldots k}(\mathbf{p}_L)$ so that we obtain

$$2\,\Phi\,\left([P_{\rfloor \xi_1 \ldots \xi_k}(\mathbf{p}_L) - \mathbf{h}] + \vec{v}^{\perp}_{1\ldots k}(\mathbf{p}_L), \lambda\left(\vec{v}^{\perp}_{\xi_1}(\mathbf{p}_L) + \ldots + \vec{v}^{\perp}_{\xi_k}(\mathbf{p}_L)\right)\right)$$

$$+\Phi\,\left(\lambda\left(\vec{v}^{\perp}_{\xi_1}(\mathbf{p}_L) + \ldots + \vec{v}^{\perp}_{\xi_k}(\mathbf{p}_L)\right), \lambda\left(\vec{v}^{\perp}_{\xi_1}(\mathbf{p}_L) + \ldots + \vec{v}^{\perp}_{\xi_k}(\mathbf{p}_L)\right)\right) = 0.$$

Since $\lambda$ must be different from zero, the thesis follows from the orthogonality between the vectors $P_{\rfloor \xi_1 \ldots \xi_k}(\mathbf{p}_L) - \mathbf{h}$ and $\vec{v}^{\perp}_{\xi_l}(\mathbf{p}_L)$ for all $l = 1, \ldots, k$. □

3.2. *The algorithm.* The effective application of the constitutive characterization described above to significant physical situations is not immediate, since a detailed analysis of the impact is necessary, mainly in order to determine which of the one–dimensional constraints forming the multiple constraint $\mathcal{C}$ is really involved in the impact, and if the resulting "right–velocity" is an outgoing velocity for $\mathcal{C}$. Then we describe the procedure in form of an algorithm generalizing the constitutive characterization of a single constraint (8). By the very nature of the method, the approach is of event–driven type.

Given a left–velocity $\mathbf{p}_L \in J_1(\mathcal{M})$ and a set $\mathcal{C} = \{\mathcal{S}_\xi \subset \mathcal{M},\ \xi = 1, \ldots, r\}$ of constraints satisfying the dimensional, regularity and independence conditions of Section 2, then:

**Step 1.** Calculate $cont(\pi(\mathbf{p}_L))$:

i) if $cont(\pi(\mathbf{p}_L)) = \emptyset$, then the system is not in contact with any constraints. Then go to **Step FINAL**.

ii) If $cont(\pi(\mathbf{p}_L)) = \{\xi_{\eta_1}, \ldots, \xi_{\eta_z}\} \neq \emptyset$, then the system is in (possibly multiple) contact with the constraints $\mathcal{S}_{\xi_{\eta_1}}, \ldots, \mathcal{S}_{\xi_{\eta_z}}$. Then go to the next step.

**Step 2.** Determine the generators $\vec{u}^{\perp}_{\xi_{\eta_a}}$ of $(V(\mathcal{S}_{\xi_{\eta_a}}))^{\perp}$ in the space–time configuration $\pi(\mathbf{p}_L)$ for all $a = 1, \ldots, z$. Then go to the next step.

**Step 3.** Determine the orthogonal velocities $\vec{v}^{\perp}_{\xi_{\eta_a}}(\mathbf{p}_L) \in (V(\mathcal{S}_{\xi_{\eta_a}}))^{\perp}$ for all $a = 1, \ldots, z$. Then go to the next step.

**Step 4.** For all $a = 1, \ldots, z$, calculate the scalar product $\Phi\left(\vec{v}^{\perp}_{\xi_{\eta_a}}(\mathbf{p}_L), \vec{u}^{\perp}_{\xi_{\eta_a}}\right)$:

i) if $a$ is such that $\Phi\left(\vec{v}^{\perp}_{\xi_{\eta_a}}(\mathbf{p}_L), \vec{u}^{\perp}_{\xi_{\eta_a}}\right) \geq 0$, then the system is in contact with $\mathcal{S}_{\xi_{\eta_a}}$ but the system does not impact with $\mathcal{S}_{\xi_{\eta_a}}$, since $\vec{v}^{\perp}_{\xi_{\eta_a}}(\mathbf{p}_L)$ is an outgoing velocity for $\mathcal{S}_{\xi_{\eta_a}}$.

ii) if $a$ is such that $\Phi\left(\vec{v}^{\perp}_{\xi_{\eta_a}}(\mathbf{p}_L), \vec{u}^{\perp}_{\xi_{\eta_a}}\right) < 0$, then the system is in contact with $\mathcal{S}_{\xi_{\eta_a}}$ and the system impacts with $\mathcal{S}_{\xi_{\eta_a}}$, since $\vec{v}^{\perp}_{\xi_{\eta_a}}(\mathbf{p}_L)$ is an incoming velocity for $\mathcal{S}_{\xi_{\eta_a}}$.

Determine the set $\{\beta_1, \ldots, \beta_l\} \subset \{\eta_1, \ldots, \eta_z\}$ of indexes such that $\Phi\left(\vec{v}^{\perp}_{\xi_{\beta_b}}(\mathbf{p}_L), \vec{u}^{\perp}_{\xi_{\beta_b}}\right) < 0$ for all $b = 1, \ldots, l$. Then go to the next step.

**Step 5.** i) If $\{\beta_1, \ldots, \beta_l\} = \emptyset$, then go to **Step FINAL**;

ii) if $\{\beta_1, \ldots, \beta_l\}$ is formed by a single index $\{\chi\}$, then go to **Step 6**;

iii) if $\{\beta_1, \ldots, \beta_l\} \neq \emptyset$ and it is not formed by a single index $\{\chi\}$, then go to **Step 7**.

**Step 6.** Set

$$\begin{aligned} \vec{I}_{ideal}(\mathbf{p}_L) &= -2\, \vec{v}^{\perp}_{\chi}(\mathbf{p}_L), \\ \mathbf{p}_L &:= \mathbf{p}_L + \vec{I}_{ideal}(\mathbf{p}_L). \end{aligned}$$

Then go back to **Step 3**.

**Step 7.** Determine the orthogonal velocity $\vec{v}^{\perp}_{\beta_1 \ldots \beta_l}(\mathbf{p}_L)$, then go to the next step.

**Step 8.** Calculate the coefficient

$$\lambda = -2\, \frac{\Phi\left(\vec{v}^{\perp}_{\beta_1 \ldots \beta_l}(\mathbf{p}_L), \vec{v}^{\perp}_{\xi_{\beta_1}}(\mathbf{p}_L) + \ldots + \vec{v}^{\perp}_{\xi_{\beta_l}}(\mathbf{p}_L)\right)}{\Phi\left(\vec{v}^{\perp}_{\xi_{\beta_1}}(\mathbf{p}_L) + \ldots + \vec{v}^{\perp}_{\xi_{\beta_l}}(\mathbf{p}_L), \vec{v}^{\perp}_{\xi_{\beta_1}}(\mathbf{p}_L) + \ldots + \vec{v}^{\perp}_{\xi_{\beta_l}}(\mathbf{p}_L)\right)}.$$

Then go to the next step.

**Step 9.** Set

$$\begin{aligned} \vec{I}_{ideal}(\mathbf{p}_L) &= \lambda \left(\vec{v}^{\perp}_{\xi_{\beta_1}}(\mathbf{p}_L) + \ldots + \vec{v}^{\perp}_{\xi_{\beta_l}}(\mathbf{p}_L)\right), \\ \mathbf{p}_L &:= \mathbf{p}_L + \vec{I}_{ideal}(\mathbf{p}_L). \end{aligned}$$

Then go back to **Step 3**.

**Step FINAL.** Set $\mathbf{p}_R := \mathbf{p}_L$.

3.3. *Remarks about the algorithm.* Some remarks are in order to clarify the characteristics of the iterative method described above.

1) The algorithm does not change the absolute velocity of the system when the system is not in contact with any constraint (Step 1, pt. i)), and also when the system is in contact with one or more constraints but its left–velocity $\mathbf{p}_L$ is not an incoming velocity for $\mathcal{C}$ (Step 5, pt.1)). Of course in this case the algorithm stops after the first iteration.
2) The algorithm generalizes the constitutive characterization (8) when the set $\mathcal{C}$ of constraints is formed by a single constraint $\mathcal{S}$. In fact, once the contact is assured (by Step 1, pt. ii)), if $\mathbf{p}_L$ is not an incoming velocity (Step 4, pt. i)), then we

are in the case of Step 5, pt. i) and the velocity does not change (and $\vec{I}(\mathbf{p}_L) = 0$ such as in the second row of (8)). Otherwise, if $\mathbf{p}_L$ is an incoming velocity (Step 4, pt. ii)), then we have to apply Step 5, pt. ii) and Step 6, obtaining a result analogous to the first row of (8). It is a straightforward matter to show that the new $\mathbf{p}_L$ determines an outgoing orthogonal velocity. Then Step 4, pt. i) works and the algorithm stops after the first iteration giving the same result of (8). Then the constitutive characterization complies with condition c) of Subsection 3.1.

3) For effective multiple impact, the algorithm handles the elements of $\mathcal{C}$ involved in the impact in a uniform manner. This is a clear consequence of the structure of the coefficient (12) and of the discussion before Lemma 3.1. Then the constitutive characterization complies with condition b) of Subsection 3.1.
4) The algorithm preserves the kinetic energy of the system with respect to all the frames of reference $\mathbf{h} \in \mathcal{H}_{\rfloor_{1\dots r}} = \{\mathbf{h} \in \mathcal{H}_{\mathcal{M}} | \mathbf{h} \in \mathcal{H}_{\rfloor_{J_1(\mathcal{S}_\xi)}}\ \xi = 1, \dots, r\}$. More precisely, the algorithm preserves the kinetic energy of the system with respect to all the frames of reference $\mathbf{h} \in \mathcal{H}_{\rfloor_{\beta_1\dots\beta_r}} \supset \mathcal{H}_{\rfloor_{1\dots r}}$ for which the subset $\mathcal{C}' = \{\mathcal{S}_{\beta_b}, b = 1, \dots, z\} \subset \mathcal{C}$ formed by all the constraints $\mathcal{S}_{\beta_b}$ effectively involved in the impact could be considered at rest.

   The conservation of kinetic energy is obvious when the system does not have collision or have a collision with a single constraint. For effective multiple impact, the conservation of kinetic energy follows from the property of the coefficient (12) of Lemma 3.1. Then the constitutive characterization complies with condition a) of Subsection 3.1.
5) The need to go back to Step 3 at the end of Step 6 and Step 9 is due to the possibility that the new $\mathbf{p}_L$ given by Step 6 or Step 9 is an incoming velocity for constraints previously excluded by the impact by Step 4, i). This can occur only in case of multiple contact, but it can happen even for very simple systems such as the Newton Cradle (whose behavior will be analyzed in the next section).
6) The algorithm stops when the conditions $\Phi\left(\vec{u}^{\perp}_{\xi_{\eta_a}}, \vec{v}^{\perp}_{\xi_{\eta_a}}(\mathbf{p}_L)\right) \geq 0$ hold for all the orthogonal velocities $\vec{v}^{\perp}_{\xi_{\eta_a}}(\mathbf{p}_L), a = 1, \dots, z$, that is, when $\mathbf{p}_R$ is an outgoing velocity for $\mathcal{C}$. A detailed analysis of the conditions to ensure that the algorithm stops, the so–called termination analysis, is not in line with the aims of this paper but we sketch this argument in Section 5.
7) When the constraints $\mathcal{S}_\xi \in \mathcal{C}$ are known in their cartesian form $F_{\mathcal{S}_\xi}(t, x^i) = 0$, taking into account Lemma 1.1, the application of the algorithm can be shortened by replacing Steps 2 and 4 with a single step: we can determine the incoming or outgoing nature of $\mathbf{p}_L$ with respect to $\mathcal{S}_{\xi_{\eta_a}}$ just after Step 1 by calculating the sign of $\mathbf{p}_L(F_{\mathcal{S}_{\xi_{\eta_a}}})$ in the contact position of the system. Then we can proceed determining the orthogonal velocities $\vec{v}^{\perp}_{\xi_{\beta_b}}(\mathbf{p}_L)$.

Let us stress the fact that the constitutive characterization determined by the algorithm is not the only one respecting the conditions a), b), c) of Subsection 3.1, since other characterizations can be constructed by using all the possible orthogonal velocities $\vec{v}^{\perp}_{\eta_1\dots\eta_z}(\mathbf{p}_L) \in (V(\mathcal{S}_{\xi_{\eta_1}} \cap \ldots \cap \mathcal{S}_{\xi_{\eta_z}}))^{\perp}$ for every possible choice of intersection

$\mathcal{S}_{\xi_{\eta_1}} \cap \ldots \cap \mathcal{S}_{\xi_{\eta_z}}$ of the constraints $\mathcal{S}_{\xi_1}, \ldots, \mathcal{S}_{\xi_k}$ involved in the impact. However, the characterization determined by the algorithm is the simplest one taking into account the geometric structure of the multiple constraint.

**4. Examples.** In this section we present the application of the algorithm to simple mechanical systems inspired by simplified billiard situations. Moreover, when the complexity of the calculation will allow, we compare the different behaviors of the system thought of as subject to an intersection of constraints of codimension 1 or simply as subject to an unspecified constraint of codimension greater than 1.

*Example 1. The Newton Cradle.* Five equal disks of mass $M$ and radius $R$ can move in a plane. Labelling the disks with the numbers $1, 2, 3, 4, 5$, the space–time configuration is described by 16 coordinates $(t, x_i, y_i, \vartheta_i), i = 1, \ldots, 5$ where $(x_i, y_i)$ are the coordinates of the center of the $i$–th disk and $\vartheta_i$ is its orientation. The set of constraints is given (with obvious notation) by the functions

$$S_{(i,j)} : \ (x_j - x_i)^2 + (y_j - y_i)^2 - 4R^2 \ = \ 0 \quad i, j = 1, \ldots, 5, \ \ i < j.$$

We consider the particular case when the object disks $2, 3, 4$ and $5$ are at rest and in contact with their centers aligned on a straight line, while the cue disk 1 moves in the straight line direction and collides with disk 2 (see the upper and middle parts of Fig. 1).

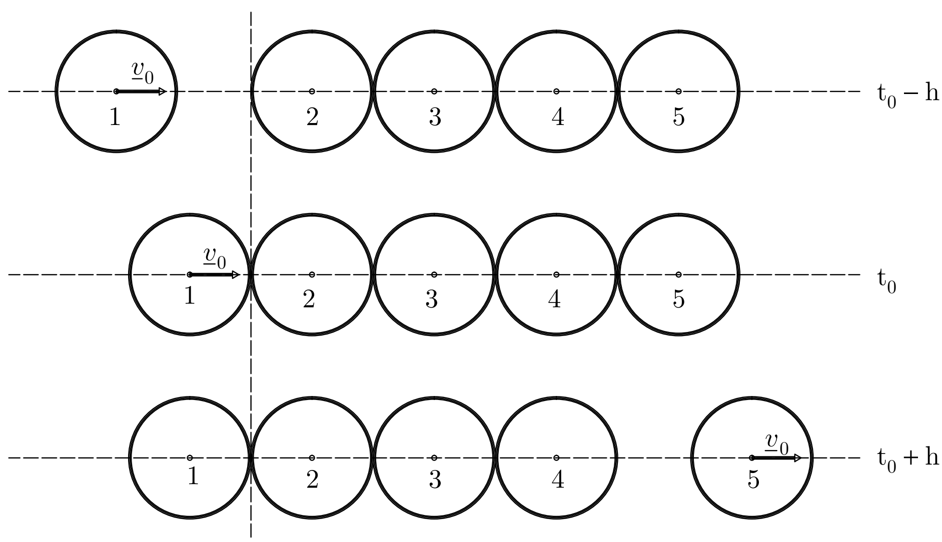


FIG. 1. Newton Cradle with one disk moving

In this case we can consider $\mathbf{p}_L = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x_1}$ with $v_0 > 0$, $cont(\pi(\mathbf{p}_L)) = \{(1,2), (2,3), (3,4), (4,5)\}$, so that Step 1, pt. ii) applies.

We determine the nature of the left–velocity $\mathbf{p}_L$ with respect to the contact constraint $\mathcal{C} = \{S_{(1,2)}, S_{(2,3)}, S_{(3,4)}, S_{(4,5)}\}$ by using the shortened version of the algorithm described in Remark 7) of Subsection 3.3. We have

$$\begin{aligned}\mathbf{p}_L(S_{(1,2)}) &= -4Rv_0 < 0,\\ \mathbf{p}_L(S_{(2,3)}) &= \mathbf{p}_L(S_{(3,4)}) = \mathbf{p}_L(S_{(4,5)}) = 0,\end{aligned}$$

so that the velocity $\mathbf{p}_L$ is an incoming velocity only for the constraints $\mathcal{S}_{(1,2)}$ (and it is tangent to the other constraints). The standard calculation of $\vec{v}^{\perp}_{(1,2)}(\mathbf{p}_L)$ gives

$$\vec{v}^{\perp}_{(1,2)}(\mathbf{p}_L) = +\frac{1}{2}\,v_0\frac{\partial}{\partial x_1} - \frac{1}{2}\,v_0\frac{\partial}{\partial x_2}\,.$$

We apply Steps 5.ii) and 6 obtaining the "new" left–velocity

$$\mathbf{p}_L := \mathbf{p}_L - 2\,\vec{v}^{\perp}_{(1,2)}(\mathbf{p}_L) = \frac{\partial}{\partial t} + v_0\,\frac{\partial}{\partial x_2}$$

and we have to restart determining the nature of the new left-velocity with respect to $\mathcal{C}$. We have:

$$\begin{aligned}\mathbf{p}_L(S_{(1,2)}) &= 4Rv_0 > 0,\\ \mathbf{p}_L(S_{(2,3)}) &= -4Rv_0 < 0,\\ \mathbf{p}_L(S_{(3,4)}) &= \mathbf{p}_L(S_{(4,5)}) = 0,\end{aligned}$$

so that the new left–velocity is an outgoing velocity for $S_{(1,2)}$ and an incoming velocity for $S_{(2,3)}$ (and it is tangent to the other contact constraints). The standard calculation of $\vec{v}^{\perp}_{(2,3)}(\mathbf{p}_L)$ gives

$$\vec{v}^{\perp}_{(2,3)}(\mathbf{p}_L) = +\frac{1}{2}\,v_0\frac{\partial}{\partial x_2} - \frac{1}{2}\,v_0\frac{\partial}{\partial x_3}.$$

We apply Steps 5.ii) and 6 obtaining the "new" left–velocity

$$\mathbf{p}_L := \mathbf{p}_L - 2\,\vec{v}^{\perp}_{(2,3)}(\mathbf{p}_L) = \frac{\partial}{\partial t} + v_0\,\frac{\partial}{\partial x_3}$$

and we have to restart. We have:

$$\begin{aligned}\mathbf{p}_L(S_{(1,2)}) &= 0,\\ \mathbf{p}_L(S_{(2,3)}) &= 4Rv_0 > 0,\\ \mathbf{p}_L(S_{(3,4)}) &= -4Rv_0 < 0,\\ \mathbf{p}_L(S_{(4,5)}) &= 0,\end{aligned}$$

so that the new left–velocity is an outgoing velocity for $S_{(2,3)}$ and an incoming velocity for $S_{(3,4)}$ (and it is tangent to the other contact constraints).

Two further iterations of the algorithm give the left–velocity

$$\mathbf{p}_L = \frac{\partial}{\partial t} + v_0\,\frac{\partial}{\partial x_5}$$

such that

$$\begin{aligned}\mathbf{p}_L(S_{(1,2)}) &= \mathbf{p}_L(S_{(2,3)}) = \mathbf{p}_L(S_{(3,4)}) = 0,\\ \mathbf{p}_L(S_{(4,5)}) &= 4Rv_0 > 0\,.\end{aligned}$$

Then, at last, we have that $\mathbf{p}_L$ is a globally outgoing velocity for all the elements of $\mathcal{C}$. Therefore we can apply Step FINAL and we obtain the right–velocity

$$\mathbf{p}_R = \frac{\partial}{\partial t} + v_0\,\frac{\partial}{\partial x_5}.$$

The lower part of Fig. 1 illustrates the (well-known) ideal behavior of the system after the impact: the cue disk together with disks 2, 3 and 4 remain at rest and disk 5 starts moving with the velocity that the cue disk had before the impact.

It can be easily shown that an initial spin of the cue disk does not have significant effect on the impact and it will be found unchanged in the final right–velocity. Then, if the initial left–velocity is $\mathbf{p}_L = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x_1} + \omega_0 \frac{\partial}{\partial \vartheta_1}$, we obtain a final right–velocity $\mathbf{p}_L = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x_5} + \omega_0 \frac{\partial}{\partial \vartheta_1}$ with the additional spin component still pertaining to the cue disk.

Note that, if we consider the set $\mathcal{C}$ of constraints as a single constraint of codimension 4 and we apply the constitutive characterization based on the global orthogonal velocity $\vec{v}^{\perp}_{(1,2)\ldots(4,5)}(\mathbf{p}_L)$, we obtain the outgoing velocity

$$\mathbf{p}_R = \frac{\partial}{\partial t} - \frac{3}{5} v_0 \frac{\partial}{\partial x_1} + \frac{2}{5} v_0 \frac{\partial}{\partial x_2} + \frac{2}{5} v_0 \frac{\partial}{\partial x_3} + \frac{2}{5} v_0 \frac{\partial}{\partial x_4} + \frac{2}{5} v_0 \frac{\partial}{\partial x_5}\,.$$

This means that disks 2,3,4 and 5 start moving in the $x$ direction with velocity $\frac{2}{5} v_0$ and the cue disk goes back in the $x$ direction with velocity $-\frac{3}{5} v_0$ (see Fig. 2). It is known that this behavior, although formally correct, does not have an experimental validation. Moreover, it is the same result that can be obtained for the impact of the cue disk with the rigid body formed by the four disks thought of as fixed together. This was easily predicted, since in this case the method does not barely take into account the nature of multiple contact, but only the "collective" impact with the entire constraint neglecting the incoming or outgoing character of the single velocities involved in the impact.

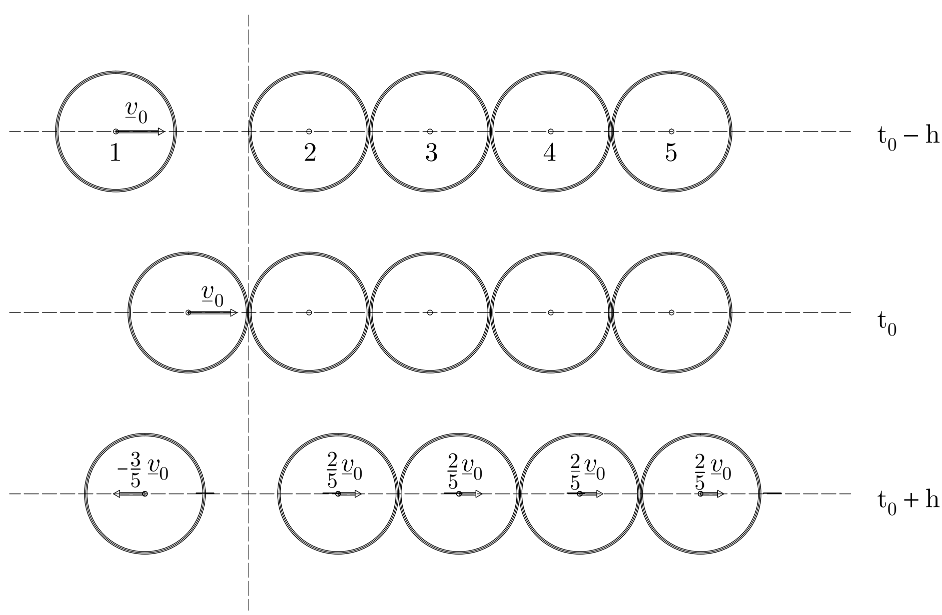


FIG. 2. Alternative Newton Cradle with one disk moving

It is well known that the mechanical system describing the Newton Cradle can model different impacts, for instance the one with cue disks 1 and 2. In this case, starting with the left-velocity $\mathbf{p}_L = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x_1} + v_0 \frac{\partial}{\partial x_2}$ with $v_0 > 0$ the algorithm gives $\mathbf{p}_R = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x_4} + v_0 \frac{\partial}{\partial x_5}$, in agreement with the experimental results.

We remark that the Newton Cradle shows in a very simple situation the necessity of the return to Step 3 of the algorithm.

*Example 2. Split n.1.* Three equal disks of mass $M$ and radius $R$ can move in a plane. Labelling the disks with the numbers $1, 2, 3$, the space–time configuration is described by 10 coordinates $(t, x_i, y_i, \vartheta_i), i = 1, \ldots, 3$ where $(x_i, y_i)$ are the coordinates of the center of the $i$–th disk and $\vartheta_i$ is its orientation. The set of constraints is given by the functions

$$\begin{aligned} S_{(1,2)} :\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - 4R^2 &= 0\,, \\ S_{(1,3)} :\ (x_3 - x_1)^2 + (y_3 - y_1)^2 - 4R^2 &= 0\,, \\ S_{(2,3)} :\ (x_3 - x_2)^2 + (y_3 - y_2)^2 - 4R^2 &= 0\,. \end{aligned}$$

We consider the particular case when the object disks 2 and 3 are at rest and in contact, while the cue disk 1 moves and simultaneously impacts with both the disks 2 and 3 (see Fig. 3).

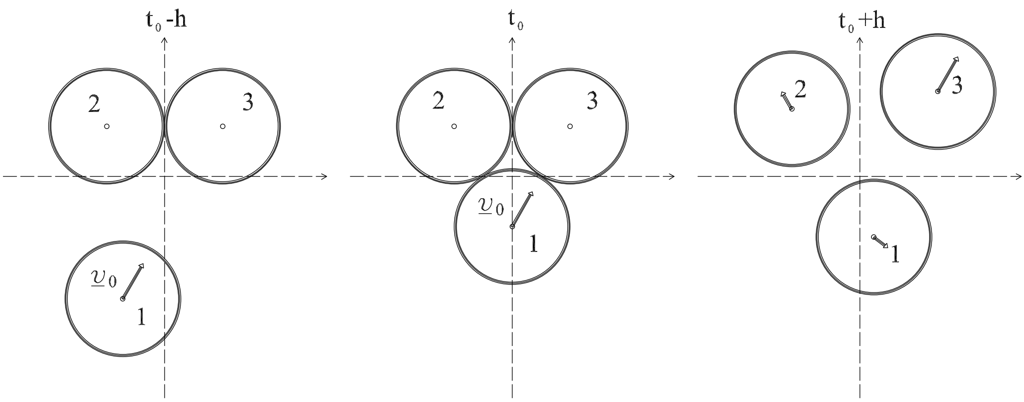


FIG. 3. Simultaneous impact of a disk with two disks at rest and in contact

In this case we can consider $\mathbf{p}_L = \frac{\partial}{\partial t} + \mu v_0 \frac{\partial}{\partial x_1} + v_0 \frac{\partial}{\partial y_1}$ with $v_0 > 0$ and $cont(\pi(\mathbf{p}_L)) = \{(1,2), (1,3), (2,3)\}$, so Step 1, pt. ii) applies.

In the contact space–time configuration we have then:

$$\begin{array}{rcll} \mathbf{p}_L(S_{(1,2)}) & = & 2(\mu - \sqrt{3})Rv_0 & < 0 \Leftrightarrow \mu < \sqrt{3}, \\ \mathbf{p}_L(S_{(1,3)}) & = & -2(\mu + \sqrt{3})Rv_0 & < 0 \Leftrightarrow \mu > -\sqrt{3}, \\ \mathbf{p}_L(S_{(2,3)}) & = & 0. & \end{array}$$

The left–velocity is then an incoming velocity for both $S_{(1,2)}$ and $S_{(1,3)}$ if and only if $-\sqrt{3} < \mu < \sqrt{3}$ (and it is tangent to the other contact $S_{(2,3)}$). In this case we have then a true multiple impact. The standard calculation of $\vec{v}^{\perp}_{(1,2)}(\mathbf{p}_L), \vec{v}^{\perp}_{(1,3)}(\mathbf{p}_L)$ gives

$$\begin{aligned}\vec{v}^{\perp}_{(1,2)}(\mathbf{p}_L) &= \left(\frac{\mu-\sqrt{3}}{8}\,v_0\right)\frac{\partial}{\partial x_1} - \sqrt{3}\left(\frac{\mu-\sqrt{3}}{8}\,v_0\right)\frac{\partial}{\partial y_1}\\ &\quad -\left(\frac{\mu-\sqrt{3}}{8}\,v_0\right)\frac{\partial}{\partial x_2} + \sqrt{3}\left(\frac{\mu-\sqrt{3}}{8}\,v_0\right)\frac{\partial}{\partial y_2},\end{aligned}$$

$$\begin{aligned}\vec{v}^{\perp}_{(1,3)}(\mathbf{p}_L) &= \left(\frac{\mu+\sqrt{3}}{8}\,v_0\right)\frac{\partial}{\partial x_1} + \sqrt{3}\left(\frac{\mu+\sqrt{3}}{8}\,v_0\right)\frac{\partial}{\partial y_1}\\ &\quad -\left(\frac{\mu+\sqrt{3}}{8}\,v_0\right)\frac{\partial}{\partial x_3} - \sqrt{3}\left(\frac{\mu+\sqrt{3}}{8}\,v_0\right)\frac{\partial}{\partial y_3}.\end{aligned}$$

Applying Step 7, we obtain

$$\begin{aligned}\vec{v}^{\perp}_{(1,2)/(1,3)}(\mathbf{p}_L) &= \frac{1}{3}\,\mu v_0\frac{\partial}{\partial x_1} + \frac{3}{5}\,v_0\frac{\partial}{\partial y_1}\\ &\quad -\left(\frac{\mu}{6}-\frac{\sqrt{3}}{10}\right)v_0\frac{\partial}{\partial x_2} + \sqrt{3}\left(\frac{\mu}{6}-\frac{\sqrt{3}}{10}\right)v_0\frac{\partial}{\partial y_2}\\ &\quad -\left(\frac{\mu}{6}+\frac{\sqrt{3}}{10}\right)v_0\frac{\partial}{\partial x_3} - \sqrt{3}\left(\frac{\mu}{6}+\frac{\sqrt{3}}{10}\right)v_0\frac{\partial}{\partial y_3}\,.\end{aligned}$$

Step 8 gives

$$\lambda = -2\,\frac{\Phi\left(\vec{v}^{\perp}_{(1,2)/(1,3)}(\mathbf{p}_L), \vec{v}^{\perp}_{(1,2)}(\mathbf{p}_L)+\vec{v}^{\perp}_{(1,3)}(\mathbf{p}_L)\right)}{\left\|\vec{v}^{\perp}_{(1,2)}(\mathbf{p}_L)+\vec{v}^{\perp}_{(1,3)}(\mathbf{p}_L)\right\|^2} = -\,\frac{8}{3}\,\frac{\mu^2+3}{\mu^2+5}$$

and then

$$\begin{aligned}\mathbf{p}_L &:= \mathbf{p}_L - \frac{8}{3}\,\frac{\mu^2+3}{\mu^2+5}\left(\vec{v}^{\perp}_{(1,2)}(\mathbf{p}_L)+\vec{v}^{\perp}_{(1,3)}(\mathbf{p}_L)\right)\\ &:= \frac{\partial}{\partial t} + \left(\frac{\mu^2+9}{3(\mu^2+5)}\right)\mu v_0\frac{\partial}{\partial x_1} - \left(\frac{\mu^2+1}{\mu^2+5}\right)v_0\frac{\partial}{\partial y_1}\\ &\quad +\frac{1}{3}\left(\frac{\mu^2+3}{\mu^2+5}\right)(\mu-\sqrt{3})\,v_0\frac{\partial}{\partial x_2} - \frac{\sqrt{3}}{3}\left(\frac{\mu^2+3}{\mu^2+5}\right)(\mu-\sqrt{3})\,v_0\frac{\partial}{\partial y_2}\\ &\quad +\frac{1}{3}\left(\frac{\mu^2+3}{\mu^2+5}\right)(\mu+\sqrt{3})\,v_0\frac{\partial}{\partial x_3} + \frac{\sqrt{3}}{3}\left(\frac{\mu^2+3}{\mu^2+5}\right)(\mu+\sqrt{3})\,v_0\frac{\partial}{\partial y_3}\,.\end{aligned} \tag{13}$$

Tedious but straightforward calculation shows that, when $-\sqrt{3} < \mu < \sqrt{3}$, the three values $\mathbf{p}_L(S_{(1,2)}), \mathbf{p}_L(S_{(1,3)}), \mathbf{p}_L(S_{(2,3)})$ are all positive. We can apply Step FINAL and then (13) is the outgoing right–velocity $\mathbf{p}_R$ of the system.

The right side of Fig. 3 illustrates the behavior of the system after the impact in the particular case $\mu = 1/\sqrt{3}$.

Once again some remarks about the results of this example are in order. The first is that once again, in case of initial spin of the cue disk, the algorithm does not affect the spin component of the cue ball. If the left-velocity is $\mathbf{p}_L = \frac{\partial}{\partial t} + \mu v_0 \frac{\partial}{\partial x_1} + v_0 \frac{\partial}{\partial y_1} + \omega_0 \frac{\partial}{\partial \vartheta_1}$, the right-velocity $\mathbf{p}_R$ is given by (13) with an additional component $\omega_0 \frac{\partial}{\partial \vartheta_1}$.

Moreover, Example 2 shows once again the difference between the multiple constraint viewed as formed by two constraints of codimension 1 and the multiple constraint viewed as a single constraint of codimension 2. In the second case, applying (7), we obtain:

$$
\begin{aligned}
\mathbf{p}_R &= \mathbf{p}_L - 2\,\vec{v}^{\perp}_{(1,2)/(1,3)}(\mathbf{p}_L) \\
&= \frac{\partial}{\partial t} + \frac{1}{3}\,\mu v_0 \frac{\partial}{\partial x_1} - \frac{1}{5}\,v_0 \frac{\partial}{\partial y_1} \\
&\quad + \left(\frac{\mu}{3} - \frac{\sqrt{3}}{5}\right) v_0 \frac{\partial}{\partial x_2} - \sqrt{3}\left(\frac{\mu}{3} - \frac{\sqrt{3}}{5}\right) v_0 \frac{\partial}{\partial y_2} \\
&\quad + \left(\frac{\mu}{3} + \frac{\sqrt{3}}{5}\right) v_0 \frac{\partial}{\partial x_3} + \sqrt{3}\left(\frac{\mu}{3} + \frac{\sqrt{3}}{5}\right) v_0 \frac{\partial}{\partial y_3}\,.
\end{aligned}
\tag{14}
$$

When $-\sqrt{3} < \mu < \sqrt{3}$, this is actually an outgoing velocity for $\mathcal{C}$ that is in general different from (13).

However, when $\mu = 0$, that is, when the cue disk moves vertically with initial velocity $\mathbf{p}_L = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial y_1}$ and symmetrically strikes both the disks 2 and 3, the outgoing velocities (13) and (14) are the same. This is due to the fact that, when $\mu = 0$, there exists a coefficient $\beta$ such that $\vec{v}^{\perp}_{(1,2)/(1,3)}(\mathbf{p}_L) = \beta\left(\vec{v}^{\perp}_{(1,2)}(\mathbf{p}_L) + \vec{v}^{\perp}_{(1,3)}(\mathbf{p}_L)\right)$. The proportionality of the two vectors entails that the final right–velocity $\mathbf{p}_R$ can be indifferently obtained by the condition $\mathbf{p}_R = \mathbf{p}_L - 2\,\vec{v}^{\perp}_{(1,2)/(1,3)}(\mathbf{p}_L)$ or the condition $\mathbf{p}_R = \mathbf{p}_L - \lambda\left(\vec{v}^{\perp}_{(1,2)}(\mathbf{p}_L) + \vec{v}^{\perp}_{(1,3)}(\mathbf{p}_L)\right)$. The same situation happens in Example 1.2 if we consider the constraint $\mathcal{S}$ as intersection of the two constraints $\mathcal{S}_1 : y = 0$ and $\mathcal{S}_2 : z = 0$.

Finally, note that a procedure considering two consecutive impacts with the constraints $S_{(1,2)}$ and $S_{(1,3)}$ gives an evident incoherence even when $\mu = 0$. In fact this procedure gives different results if we consider first the impact with $S_{(1,2)}$ and then with $S_{(1,3)}$ or vice versa.

*Example 3. Split n.2.* Four equal disks of mass $M$ and radius $R$ can move in a plane. Labelling the disks with the numbers $0, 1, 2, 3$, the space–time configuration is described by 13 coordinates $(t, x_i, y_i, \vartheta_i), i = 0, \dots, 3$ where $(x_i, y_i)$ are the coordinates of the center of the $i$–th disk and $\vartheta_i$ is its orientation. The set of constraints is given by the functions

$$
S_{(i,j)} :\ (x_j - x_i)^2 + (y_j - y_i)^2 - 4R^2 \ = \ 0 \quad i, j = 0, \dots, 3,\ \ i < j.
$$

We consider the particular case represented in Fig. 4: the object disks $1, 2$ and 3 are at rest and in contact and the cue disk 0 moves and impacts 1. Let $\varphi$ be the angle formed by the direction of the velocity of the cue disk and the line determined by the centers of disks 0 and 1 (see Fig. 4). Moreover, let the direction of the velocity of the cue disk be orthogonal to the line determined by the centers of disks 2 and 3.

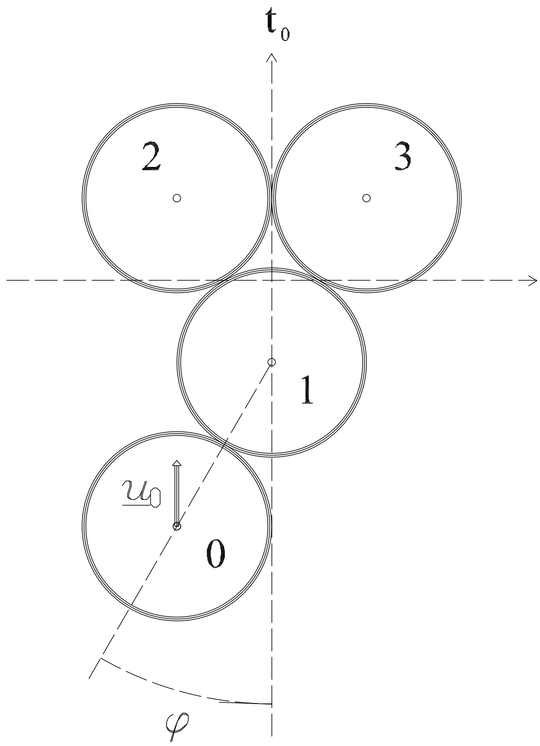


FIG. 4. Disk colliding with a disk at rest in contact with two other disks

In this case we have $cont(\pi(\mathbf{p}_L)) = \{(0,1),(1,2),(1,3),(2,3)\}$. Given the initial left–velocity $\mathbf{p}^0_L = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial y_0}$ with $u_0 > 0$, the application of the algorithm points out that:

- $\mathbf{p}^0_L$ has an incoming nature only for the constraint $S_{(0,1)}$. The first iteration of the algorithm gives a "second" left–velocity $\mathbf{p}^1_L$ such that disk 1 moves in the direction determined by the positions of the centers of the disks 0 and 1, and the cue disk moves orthogonally to the direction determined by the positions of the centers.
- The "second" left–velocity $\mathbf{p}^1_L$ has outgoing nature for both $S_{(0,1)}, S_{(2,3)}$. Moreover it has incoming nature for both $S_{(1,2)}$ and $S_{(1,3)}$ if and only if $-\sqrt{3} < \tan\varphi < \sqrt{3}$. In this case, $\mathbf{p}^1_L$ plays the same role of the initial velocity of Example 2. The second iteration of the algorithm gives a "third" left–velocity $\mathbf{p}^2_L$ analogous to (13) with additional components concerning the cue disk (unchanged with respect to those of $\mathbf{p}^1_L$).

- The "third" left–velocity $\mathbf{p}_L^2$ has outgoing nature for $S_{(1,2)}, S_{(1,3)}, S_{(2,3)}$. Moreover it has incoming nature for $S_{(0,1)}$ if and only if $-\sqrt{2\sqrt{3}-3} < \tan\varphi < \sqrt{2\sqrt{3}-3}$. In this case it is necessary that at least one additional iteration of the algorithm gives a "fourth" left–velocity $\mathbf{p}_L^3$.

Example 3 simultaneously presents some characteristics of Examples 1 and 2: it requires more than a single iteration of the algorithm as in Example 1 and it presents an effective multiple impact (if $-\sqrt{3} < \tan\varphi < \sqrt{3}$) as in Example 2. Fig. 5 shows the behavior of the system in the case $\varphi = 0$. The motion of the system turns out to be symmetric, disks 2 and 3 move in the direction determined by the positions of the centers of the disks in the impact configuration, disk 1 remains motionless and the cue disk bounces back with one fifth of the initial velocity.

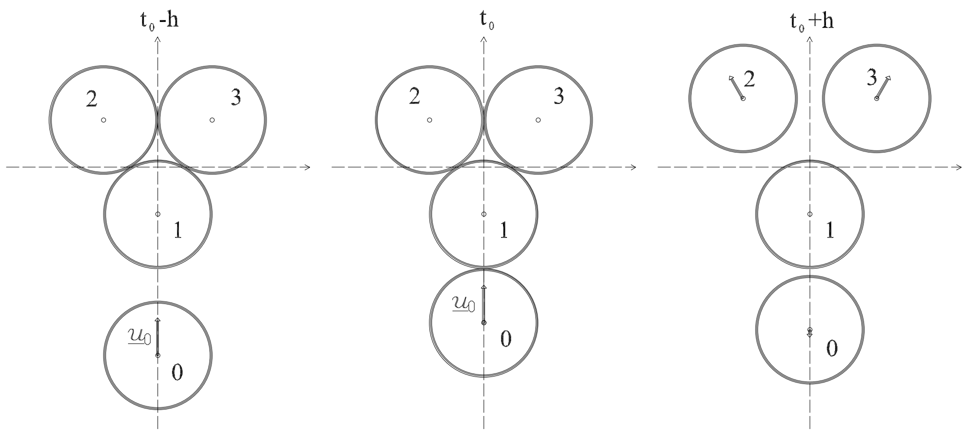


FIG. 5. Disk symmetrically colliding with a disk at rest in contact with two other disks

Of course Examples 2 and 3 can be viewed as simplified versions of billiard shots. In this sense, they give an idea of how complex can be the casuistry of behaviors even in a restricted context such as a billiard table. Nevertheless, the algorithm can be applied also, for instance, to the opening break shots illustrated in Fig. 6. The achievement of the result, in the form of right–velocity of the system, becomes simply a (hard) matter of calculation.

**5. Final remarks and future developments.** The approach, techniques and results presented in this paper can be considered a generalization to the case of multiple constraints of the analogous approach, techniques and results presented in [1] in the case of single constraint. However, due to the increased complexity of the multiple C/I phenomenon with respect to the single one, some aspects could be subject to an additional in-depth analysis.

A first remark concerns the algorithm described in Section 3: no termination analysis of the algorithm has been performed in detail. In Section 4 we showed meaningful examples where the algorithm, after a single or several iterations, terminates in a final

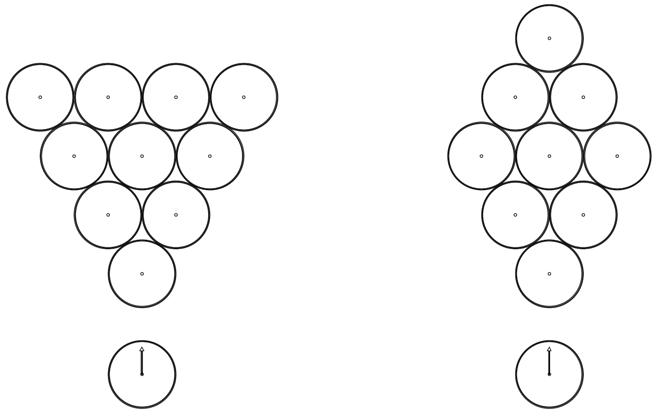

FIG. 6. Billiard opening break shots

ending state giving the right–velocity of the system after the C/I phenomenon. Nevertheless there are simple systems for which the termination is not so evident. Consider, for instance, a disk simultaneously impacting with two walls forming a suitable reentrant corner: could the disk "rebound" between the walls endlessly? Could it "rebound" so many times that the algorithm does not terminate after a reasonable number of iterations? The answers to these two questions are respectively no and yes, but this specific example is too complex to be analyzed here and it will be the subject of future works.

A second remark regards the examples presented in Section 4. In Example 1, the theoretical results given by the algorithm and the well-known ideal behavior of the system were briefly and satisfactorily compared. Nevertheless the Newton Cradle is a so classic and basic problem that it was tackled with several different techniques, also in non-ideal cases. We refer the interested reader to [12] and the references therein for a concise overview of the several results about the Newton Cradle.

Regarding the other examples, the qualitative analysis of the results obtained when the low complexity of the system allows explicit calculations makes us confident about the accordance between theoretical and experimental results. Nevertheless a detailed qualitative and quantitative comparison between theoretical and experimental results were performed only in simple cases. The consistency between them can be explored more thoroughly only once a sufficiently wide collection of experimental data is at our disposal.

Further enhancements can be sought by weakening the restrictions on $\mathcal{C}$, for example the conditions of being positional, ideal, isotropic. More specifically, following an approach analogous to [10, 13], generalizations of the results of this paper can be sought in case of:

- systems subject to ideal multiple unilateral constraints of both positional and kinetic nature, permanently or instantaneously acting on the system;
- systems subject to non–ideal frictionless multiple positional unilateral constraints, for which the kinetic energy is not preserved;

- systems subject to multiple positional unilateral constraints with friction;
- systems subject to anisotropic multiple constraints, for instance multiple constraints $\mathcal{C}$ whose elements $\mathcal{S}_\xi \in \mathcal{C}$ play different roles in the simultaneous impact.

**Appendix: Coordinate–free and frame–invariant approaches.** The usefulness of a coordinate–free approach is a well-known aspect of the study of Classical Mechanics. Then, the use of differential geometric techniques pertaining to a manifold $\mathcal{Q}$, the so–called configuration space associated to a mechanical system with a finite number of degrees of freedom, is currently a common expertise.

It is, or it should be, equally well known that the coordinate–free approach to Classical Mechanics is not related to the frame–independent approach, coordinates and frames being two different concepts: the first is related to the (local) description of the points of the configuration space $\mathcal{Q}$, the second, in a very wide sense, is related to a rule identifying the configuration space $\mathcal{Q}$ at different times.

In order to show the difference between these two concepts and the relevance of a frame–independent approach to Classical Impulsive Mechanics, let us recall the simplest coordinate–free description of the time evolution of the system: this is a curve $P : \mathbb{R} \to \mathcal{Q}$ that, using local coordinates $(q^1, \ldots, q^n)$ in $\mathcal{Q}$, is given by the rule $P(t) = (q^1(t), \ldots, q^n(t))$. The section $\gamma : \mathbb{R} \to \mathbb{R} \times \mathcal{Q}$ of the trivial product bundle $t : \mathbb{R} \times \mathcal{Q} \to \mathbb{R}$ such that $\gamma(t) = (t, q^1(t), \ldots, q^n(t)) = (t, P(t))$ has the same information of $P(t)$ and it involves the product bundle $\mathbb{R} \times \mathcal{Q}$ that fits more than the simple manifold $\mathcal{Q}$ for the time-dependent description of the geometry of the system.

Unfortunately, the product bundle $\mathbb{R} \times \mathcal{Q}$ is appropriate for a coordinate–free description of the time evolution of the mechanical system, but it is not appropriate for a frame–independent description. In fact the product bundle $\mathbb{R} \times \mathcal{Q}$ brings along a natural identification of the configuration space $\mathcal{Q}$ at different times: for each choice of a coordinate system $(q^1, \ldots, q^n)$ on $\mathcal{Q}$ and different times $t_0, t_1$ of $\mathbb{R}$, the map $\varphi$ such that $\varphi(t_0, q^1, \ldots, q^n) = (t_1, q^1, \ldots, q^n)$ is a diffeomorphism of the fibres $\{t_0\} \times \mathcal{Q}$ and $\{t_1\} \times \mathcal{Q}$ of the product bundle $\mathbb{R} \times \mathcal{Q}$. Using a more modern language, the invariant (with respect to fibred changes of coordinates in $\mathbb{R} \times \mathcal{Q}$) vector field $\frac{\partial}{\partial t}$ determines a fibre–preserving 1–parameter group of diffeomorphisms of the product bundle.

It is elementarily known that velocity and kinetic energy do not have a frame–independent mechanical meaning, and that they become meaningful mechanical quantities only when referred to a frame of reference. With this in mind, it is clear that the common idea of “vector velocity” $v^i \frac{\partial}{\partial q^i} \in \mathbb{R} \times T(\mathcal{Q})$ (where $T(\mathcal{Q})$ is the tangent space of $\mathcal{Q}$), is not a possible velocity of the system (that is clearly a meaningless concept), but only a possible velocity of the system in the frame of reference (associated to) $\frac{\partial}{\partial t}$.

These well-known remarks are full of more or less evident consequences. An evident one is that results obtained by using the geometric model given by the product bundle $\mathbb{R} \times \mathcal{Q}$ and regarding the velocity of a mechanical system, or deduced involving the concept of velocity, are obtained in an intrinsically “fixed frame” context. Therefore the frame invariance of these results, if significant, should be proved as an additional feature.

Classical C/I Mechanics of constrained systems is one of the most important branches of Classical Mechanics where this aspect should be taken into account. By the very nature of the C/I phenomenon, every characterization of the reaction exerted on the system by the constraints in a C/I phenomenon necessarily involves the concept of velocity. In fact the velocity is directly involved if we use the orthogonal and/or tangent components of the velocity with respect to the constraints. This happens, for instance, in Coulomb's model of frictional impacts, or in Newton's model of restitution. The velocity is indirectly involved if we use arguments of balance consideration, for instance, the conservation of kinetic energy.

It is a matter of fact that the majority of the results obtained, even in recent papers, about Impulsive Mechanics described in the geometric context of the product bundle $\mathbb{R} \times \mathcal{Q}$ (or worse in $\mathbb{R} \times \mathbb{R}^n$), when involving the concept of velocity, are not proved to be frame–independent. This should be even more clear by observing that the majority of the results are obtained for the mechanical system where the constraints are at rest; then in the specific (if it exists) rest frame of the constraints. The following example of a mechanical system having a multiple impact highlights the critical aspect of a naive geometric approach.

**Example**. A sphere of radius $R$ moves in contact with a horizontal floor and has a multiple impact with two vertical and orthogonal walls. Using a naive (but effective) language, we can geometrize the floor with the condition $z = 0$, one of the walls with $x = 0$, the other with $y = 0$ (see Fig. 7). However, the system is such that the points of the two walls are not steady: the $x = 0$ wall is a sort of shutter, whose points go up and down, for example with a harmonic motion $z(t) = A_z \cos(\omega_z t + \varphi_z)$; the $y = 0$ wall is a sort of sliding door, whose points go left and right, for example with a harmonic motion $x(t) = A_x \cos(\omega_x t + \varphi_x)$.

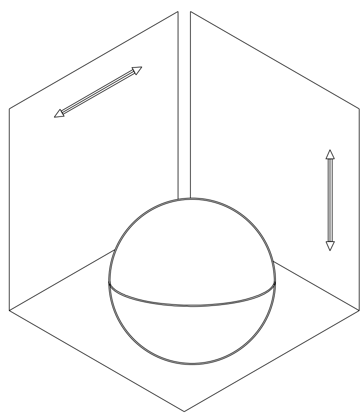

FIG. 7. Multiple impact with time-dependent constraints

The usual configuration space for this system is a five–dimensional manifold $\mathcal{Q}$ locally described by coordinates $(x, y, \psi, \varphi, \theta)$ where $(x, y)$ are the coordinates of the center of the sphere and $(\psi, \varphi, \theta)$ are its Euler angles. The unilateral constraints are described by the inequalities $x \geq R, y \geq R$. Unfortunately, this geometrization is too naive to deal with the problem of multiple impact. In fact a coordinate–free approach concerns only the coordinates of $\mathcal{Q}$. In particular, a change of coordinates in $\mathcal{Q}$ can modify only the geometric expressions of the configurations of the sphere and the geometric expressions of the points of the walls, but it cannot express in any way the "motion" of the points of the walls.

However, the motion of the walls is crucial in case of (multiple) impact with friction, since the "tangential" velocities of the contact points depend on the "underlying" velocities of the points of the constraints. The too basic geometrization given by $\mathbb{R} \times \mathcal{Q}$ and $\mathbb{R} \times T(\mathcal{Q})$ gives a clear meaning only to the velocity of the sphere with respect to "the" rest frame $\frac{\partial}{\partial t}$ of the constraint $z = R$, but it does not take into account the motion of the walls. Then it is not suitable to deal with frictional impact problems of this system. Problems of this kind require the introduction (if possible) of a notion of rest frame of the whole set of constraints.

Note that, unlike frictional impacts, a frictionless impact does not involve the tangential velocities of the points of the ball in contact with the vertical walls. In this case there is no need of introducing "the" rest frame of the constraints (if ever it exists). However, the mere geometric description $\{x \geq R, y \geq R\}$ of the constraint is enough to introduce the set $\mathcal{H}$ of "all possible rest frames" of the constraint. As has been also shown, this set is indispensable to have a correct notion of orthogonal component of the velocity ([1]).

Far from being purely artificial, the example above models in a very simplified way several possible C/I phenomena, for instance in billiard situations or more generally for granular materials.

## References


[1] Stefano Pasquero, *Ideality criterion for unilateral constraints in time-dependent impulsive mechanics*, J. Math. Phys. **46** (2005), no. 11, 112904, 20, DOI 10.1063/1.2121247. MR2186772

[2] Bernard Brogliato (ed.), *Impacts in mechanical systems*, Lecture Notes in Physics, vol. 551, Springer-Verlag, Berlin, 2000. Analysis and modelling; Papers from the Euromech Colloquium 397 held in Grenoble, June 30–July 2, 1999. MR1843040

[3] F. Pfeiffer and C. Glocker. *Multibody dynamics with unilateral contacts*, volume 421. Springer Science & Business Media, 2000.

[4] K. L. Johnson. *Contact mechanics*. Cambridge University Press, Cambridge, 1985.

[5] W. J. Stronge. *Impact Mechanics*. Cambridge University Press., Cambridge, 2000.

[6] D. J. Saunders, *The geometry of jet bundles*, London Mathematical Society Lecture Note Series, vol. 142, Cambridge University Press, Cambridge, 1989. MR989588

[7] J.-F. Pommaret, *Systems of partial differential equations and Lie pseudogroups*, Mathematics and its Applications, vol. 14, Gordon & Breach Science Publishers, New York, 1978. With a preface by André Lichnerowicz. MR517402

[8] Enrico Massa and Enrico Pagani, *Classical dynamics of nonholonomic systems: a geometric approach* (English, with French summary), Ann. Inst. H. Poincaré Phys. Théor. **55** (1991), no. 1, 511–544. MR1130215

[9] Enrico Massa and Enrico Pagani, *A new look at classical mechanics of constrained systems* (English, with English and French summaries), Ann. Inst. H. Poincaré Phys. Théor. **66** (1997), no. 1, 1–36. MR1434114

[10] Stefano Pasquero, *Nonideal unilateral constraints in impulsive mechanics: a geometric approach*, J. Math. Phys. **49** (2008), no. 4, 042902, 17, DOI 10.1063/1.2890382. MR2412290

[11] Stefano Pasquero, *Determining the internal reactions due to permanent constraints acting on ideal impulsive systems*, Meccanica **47** (2012), no. 1, 235–244, DOI 10.1007/s11012-011-9442-0. MR2875459

[12] V. Ceanga and Y. Hurmuzlu, *A new look at an old problem: Newton's cradle*, Journal of Applied Mechanics **68** (2001), no. (4), 575–583.

[13] Stefano Pasquero, *On the simultaneous presence of unilateral and kinetic constraints in time-dependent impulsive mechanics*, J. Math. Phys. **47** (2006), no. 8, 082903, 19, DOI 10.1063/1.2234728. MR2258597