

## OPTIMAL ENERGY BOUNDS IN SPHERICALLY SYMMETRIC $\alpha^2$ -DYNAMOS

BY

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**Abstract.** In kinematic dynamo theory energy bounds provide necessary conditions for dynamo action valid for every velocity field. When expressed by the magnetic Reynolds number  $R$  this number  $R_E$  may be compared with the critical Reynolds number  $R_c = R_c(\mathbf{v})$  indicating the onset of dynamo action for a given velocity field  $\mathbf{v}$ . Typically, there is an (often large) gap between both numbers, which suggests the question: are there better (energy) bounds or are the most critical velocity fields not yet known (or are both conjectures false)?

Here we answer this question in a simplified setting, viz. for spherically symmetric  $\alpha^2$ -mean-field dynamos, where the single scalar field  $\alpha$  takes the role of the velocity field and where spherical symmetry allows the reliable numerical solution of a non-linear variational problem. The non-linear problem arises from the simultaneous variation of magnetic field *and*  $\alpha$ -profile (measured in a suitable norm), which, in fact, yields an improved energy bound  $R_E^{opt} = 4.4717$  compared to the best hitherto known bound  $R_E = 3.0596$ . This bound is close to the best hitherto known critical Reynolds number  $R_c = 4.4934$ , which belongs to a constant  $\alpha$ -profile, and is, moreover, optimal since it is connected to an  $\alpha$ -profile whose critical Reynolds number exceeds  $R_E^{opt}$  by less than  $10^{-4}$ .

**1. Introduction.** In kinematic dynamo theory there was always interest in identifying the most efficient velocity fields (i.e., velocity fields with critical magnetic Reynolds numbers as low as possible) given the geometry of the fluid volume, boundary conditions, and possibly further kinematic restrictions. With the advent of dynamo experiments in the laboratory this interest even increased since the realization of efficient velocity fields

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can decide on success or failure of the experiment. The search for efficient velocity fields is, however, laborious and restricted so far in its reach. It started (to our knowledge) with the optimization of simple model flows in balls with respect to a few parameters modelling the shape of some profile functions describing the flow [LG96, Hol03]. Similar few-parameter-optimizations have been performed in the context of the Karlsruhe dynamo [Til97], the Riga dynamo [SGG99], and the Madison experiment [KBCWF12]. Only recently the search has been extended to high-dimensional parameter spaces representing in principle the space of all (divergence-free) velocity fields in a box with periodic [Wil12] or superconducting/pseudovacuum boundary conditions [CHJ15].

On the other hand, there are a few general lower bounds on critical magnetic Reynolds numbers which all are based on the energy balance of the induction equation: the energy transfer from the velocity field to the magnetic field must exceed the Ohmic loss of the electric current associated to the magnetic field to maintain the dynamo, i.e., formally the quotient

$$\frac{\int_V \mathbf{B}(\nabla\mathbf{v})\mathbf{B} dv}{\int_V \eta|\nabla\times\mathbf{B}|^2 dv} \quad (1.1)$$

must exceed one. Here,  $(\nabla\mathbf{v})$  denotes the (symmetrized) velocity-gradient-matrix of a flow field  $\mathbf{v}$  confined to the fluid volume  $V$ ,  $\mathbf{B}$  denotes the magnetic field satisfying some boundary conditions at  $\partial V$ , and  $\eta$  the magnetic diffusivity. When estimating  $(\nabla\mathbf{v})$  by the space-time-maximum of its maximum eigenvalue and  $\eta$  by its minimum  $\eta_0$  (defining this way the magnetic Reynolds number  $R$ ), (1.1) reduces to the variational expression

$$\frac{\int_V |\mathbf{B}|^2 dv}{\int_V |\nabla\times\mathbf{B}|^2 dv} \quad (1.2)$$

for  $\mathbf{B}$  only. Maximizing (1.2) with respect to those magnetic fields that satisfy the boundary conditions yields the well-known Backus-type bound on  $R$  for dynamo action [Bac58].<sup>1</sup>

When comparing the critical Reynolds numbers  $R_c$  of optimized dynamos with the corresponding Backus-type lower bound  $R_B$ , one finds typically that  $R_c$  lies still on an order of magnitude [Hol03] or more [CHJ15] above  $R_B$ . This triggers the question: can this large gap be diminished and if so, from which side? Optimizing in a restricted velocity space always runs the risk of having excluded the most efficient velocity fields, whereas searching the full velocity space is laborious with results whose reliability is not easy to assess. When looking at the other side of the gap, i.e., when looking for better lower bounds on  $R$ , one obvious possibility is not to estimate the velocity field beforehand but to maximize the quotient (1.1) simultaneously with respect to magnetic field *and* velocity field (together with some normalization for the latter field). The Euler-Lagrange equations associated with this enlarged variational problem are then non-linear and their correct solution is again hard to assess.

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<sup>1</sup>A similar bound on  $R$ , now based on the maximum of  $\mathbf{v}$  instead of  $(\nabla\mathbf{v})$ , and also using the maximum of (1.2) is usually attributed to Childress (cf. [Pro07]).

The present study refers, therefore, to a simplified setting, where the above question can be completely answered. In the  $\alpha^2$ -mean-field model the single scalar quantity  $\alpha$  takes the role of the velocity field and the relevant quotient analogous to (1.1) now reads

$$\frac{\int_{B_1} \alpha \nabla \times \mathbf{B} \cdot \mathbf{B} \, dv}{\|\alpha\| \int_{B_1} |\nabla \times \mathbf{B}|^2 \, dv}, \quad (1.3)$$

where the diffusivity has already been estimated by  $\eta_0$  (and omitted) and where a normalization  $\|\cdot\|$  for  $\alpha$  has been introduced. The fluid volume is here the unit ball  $B_1$  and we assume the usual vacuum boundary condition for  $\mathbf{B}$ , i.e., continuous matching at the unit sphere  $S_1$  to an exterior harmonic vector field vanishing at infinity. No boundary condition is imposed on  $\alpha$ .

The norm  $\|\cdot\|$  should be chosen such that (i) the expression (1.3) is bounded from above with respect to all admissible fields  $\mathbf{B}$  and  $\alpha$ , (ii) the resulting Euler-Lagrange equations are of “standard type” (e.g., of no more than second order), and (iii) comparison is possible to already existing lower bounds. The first two criteria are met by the standard 3-dimensional  $H^1$ -norm

$$\|\alpha\|_{\sim} := \left( \frac{3}{4\pi} \int_{B_1} (\alpha^2 + |\nabla \alpha|^2) \, dv \right)^{1/2}, \quad (1.4)$$

whereas the 1-dimensional norm

$$\|\alpha\|_s := \left( \int_0^1 (\alpha^2 + (\partial_r \alpha)^2) \, dr \right)^{1/2}, \quad r := |\mathbf{r}|, \quad (1.5)$$

appropriate in the case of spherical symmetry, meets all three criteria.

The most incisive assumption of our investigation is spherical symmetry of the  $\alpha$ -field with the profitable consequence that the Euler-Lagrange equations associated with the variation of (1.3) boil down to a non-linear eigenvalue problem represented by a finite system of ordinary differential equations. The numerical solution of this eigenvalue problem is quite standard and allows the certain determination of the optimum eigenvalue, which is in general a delicate problem. We do this for both norms and obtain this way optimized energy bounds  $R_E^{opt}$  and  $\tilde{R}_E^{opt}$  based on  $\|\alpha\|_s$  and  $\|\alpha\|_{\sim}$ , respectively. We determine, moreover, the critical Reynolds numbers  $R_c^{opt}$  and  $\tilde{R}_c^{opt}$  for the corresponding optimal  $\alpha$ -profiles and a comparison yields

$$\frac{R_c^{opt} - R_E^{opt}}{R_c^{opt}} < 10^{-4}, \quad \frac{\tilde{R}_c^{opt} - \tilde{R}_E^{opt}}{\tilde{R}_c^{opt}} \lesssim 10^{-3}. \quad (1.6)$$

So, for both norms the gap between optimized energy bound and (supposedly) most critical  $\alpha$ -profile is (almost) closed. These results may be compared to the gap between the energy bound  $R_E^{(m)}$  (based on the maximum-norm  $\|\alpha\|_m$  and obtained by maximizing (1.2)) and the lowest critical Reynolds number  $R_c^{(m)}$  (for constant  $\alpha$ -profiles) available so far,

$$R_c^{(m)} - R_E^{(m)} = 4.4934 - 3.5059, \quad (1.7)$$

which is moderate compared to the gap in the full problem mentioned above but large compared to (1.6). As  $\|\cdot\|_s$  dominates  $\|\cdot\|_m$ ,  $R_E^{(m)}$  implies an energy bound  $R_E$  in the  $\|\cdot\|_s$ -norm and the gap (1.7) becomes in this norm:

$$R_c - R_E = 4.4934 - 3.0596.$$

The position of  $R_E^{opt} = 4.4717$  in this gap is clearly at the upper end, which means that the well-known constant  $\alpha$ -profile is already close to the most efficient profile.

**2.  $\alpha^2$ -dynamos, variational problems, and energy bounds.** A mean-field dynamo with pure  $\alpha^2$ -mechanism confined to the unit ball  $B_1$  with vacuum boundary condition is described by the following initial-value problem in  $\mathbb{R}^3 \times (0, \infty)$  (cf. [KR80, p. 171ff]):

$$\left. \begin{aligned} \partial_t \mathbf{B} &= \nabla \times (\alpha \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}), \quad \nabla \cdot \mathbf{B} = 0 && \text{in } B_1 \times (0, \infty), \\ \nabla \times \mathbf{B} &= 0, \quad \nabla \cdot \mathbf{B} = 0 && \text{in } \widehat{B}_1 \times (0, \infty), \\ [\mathbf{B}] &= 0 && \text{on } S_1 \times (0, \infty), \\ |\mathbf{B}(\mathbf{r}, \cdot)| &\rightarrow 0 && \text{for } |\mathbf{r}| \rightarrow \infty, \\ \mathbf{B}(\cdot, 0) &= \mathbf{B}_0, \quad \nabla \cdot \mathbf{B}_0 = 0 && \text{on } B_1 \times \{t = 0\}. \end{aligned} \right\} \quad (2.1)$$

The fluid volume is here the unit ball  $B_1$  with boundary  $S_1$  and exterior (vacuum) region  $\widehat{B}_1$ . The scalar  $\alpha$ -field and the diffusivity  $\eta > 0$  are prescribed functions on  $B_1 \times (0, \infty)$ , which in this section are assumed to be continuous and bounded but are otherwise unrestricted. We define, in particular,

$$\left. \begin{aligned} \max_{B_1 \times (0, \infty)} |\alpha| &=: \alpha_{\max} < \infty, \\ \min_{B_1 \times (0, \infty)} \eta &=: \eta_0 > 0. \end{aligned} \right\} \quad (2.2)$$

$\mathbf{B}$  denotes the magnetic field with initial-value  $\mathbf{B}_0$  and  $[\mathbf{B}]$  denoting the jump of  $\mathbf{B}$  over  $S_1$ . The kinematic dynamo problem asks then for those initial-values  $\mathbf{B}_0$ , such that corresponding solutions  $\mathbf{B}$  of (2.1) do not decay in time (so-called dynamo solutions).

A necessary condition for dynamo action (i.e., for the existence of dynamo solutions) can be derived in close analogy to the full induction equation by consideration of the energy balance of (2.1). Multiplying (2.1)<sub>1</sub> by  $\mathbf{B}$  and intergration over  $B_1$  yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{B_1} |\mathbf{B}|^2 dv &= \int_{B_1} \nabla \times (\alpha \mathbf{B}) \cdot \mathbf{B} dv - \int_{B_1} \nabla \times (\eta \nabla \times \mathbf{B}) \cdot \mathbf{B} dv \\ &= \int_{B_1} \alpha \mathbf{B} \cdot (\nabla \times \mathbf{B}) dv - \int_{B_1} \eta |\nabla \times \mathbf{B}|^2 dv \\ &\quad - \int_{S_1} \eta (\nabla \times \mathbf{B}) \times \mathbf{B} \cdot \mathbf{r} ds, \end{aligned} \quad (2.3)$$

where we used integration by parts in the second line. As in [Bac58] the surface integral in (2.3) can be related to the time-derivative of the exterior magnetic energy:<sup>2</sup>

$$\frac{1}{2} \frac{d}{dt} \int_{\widehat{B}_1} |\mathbf{B}|^2 dv = \int_{S_1} \eta (\nabla \times \mathbf{B}) \times \mathbf{B} \cdot \mathbf{r} ds. \quad (2.4)$$

Summing up (2.3) and (2.4) and estimating  $\eta$  by (2.2)<sub>2</sub> yields then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\mathbf{B}|^2 dv &= \int_{B_1} \alpha \mathbf{B} \cdot (\nabla \times \mathbf{B}) dv - \int_{B_1} \eta |\nabla \times \mathbf{B}|^2 dv \\ &\leq \int_{B_1} \alpha \mathbf{B} \cdot (\nabla \times \mathbf{B}) dv - \eta_0 \int_{B_1} |\nabla \times \mathbf{B}|^2 dv \\ &= \left( \frac{\int_{B_1} \alpha \mathbf{B} \cdot (\nabla \times \mathbf{B}) dv}{\|\alpha\| \int_{B_1} |\nabla \times \mathbf{B}|^2 dv} - \frac{\eta_0}{\|\alpha\|} \right) \|\alpha\| \int_{B_1} |\nabla \times \mathbf{B}|^2 dv. \end{aligned} \quad (2.5)$$

In the last line we introduced the norm  $\|\cdot\|$  of  $\alpha$  to obtain the *homogeneous* variational expression (1.3). Fixing now some value  $\eta_0/\|\alpha\|$ , inequality (2.5) provides an obvious necessary condition for energy growth of the magnetic field, namely, that the supremum of the variational expression with respect to all admissible fields  $\mathbf{B}$  and  $\alpha$  must exceed this value  $\eta_0/\|\alpha\|$ .

Weaker conditions are obtained by partly estimating and partly maximizing expression (1.3). For example, with the estimate

$$\int_{B_1} \alpha \nabla \times \mathbf{B} \cdot \mathbf{B} dv \leq \alpha_{\max} \left( \int_{B_1} |\nabla \times \mathbf{B}|^2 dv \right)^{1/2} \left( \int_{B_1} |\mathbf{B}|^2 dv \right)^{1/2}, \quad (2.6)$$

inequality (2.5) with  $\|\alpha\| = \alpha_{\max}$  can be further estimated by

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\mathbf{B}|^2 dv \leq \left[ \left( \frac{\int_{B_1} |\mathbf{B}|^2 dv}{\int_{B_1} |\nabla \times \mathbf{B}|^2 dv} \right)^{1/2} - \frac{1}{R^{(m)}} \right] \alpha_{\max} \int_{B_1} |\nabla \times \mathbf{B}|^2 dv, \quad (2.7)$$

where we introduced the magnetic Reynolds number  $R^{(m)} := \alpha_{\max}/\eta_0$  based on the maximum norm. From (2.7) one reads off a necessary criterion, analogous to that of Backus or Childress for the induction equation, viz.

$$R^{(m)} > \left[ \sup_{0 \neq \mathbf{B} \in \mathcal{B}} \frac{\int_{B_1} |\mathbf{B}|^2 dv}{\int_{B_1} |\nabla \times \mathbf{B}|^2 dv} \right]^{-1/2}, \quad (2.8)$$

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<sup>2</sup>Note that Backus considers the case  $\eta = \text{const.}$  His manipulations, however, work for space-time-dependent  $\eta$  as well.

where  $\mathcal{B}$  denotes the space of admissible vector fields (see below). When replacing the variational expression in (2.8) by the larger expression

$$\frac{\int_{\mathbb{R}^3} |\mathbf{B}|^2 dv}{\int_{B_1} |\nabla \times \mathbf{B}|^2 dv},$$

one obtains the Backus-type bound  $R^{(m)} > \pi$  (see [Bac58]), whereas an exact calculation [Pro77] yields the sharper bound

$$R^{(m)} > \gamma = 3.5059, \tag{2.9}$$

where  $\gamma$  is the smallest positive zero of  $3j_1(\gamma) + 2\gamma j_0(\gamma) = 0$  with  $j_{0,1}$  being spherical Bessel functions.

Considering once more (2.5), the optimum bound that can be derived from the energy balance is apparently provided by the solution of the following enlarged variational problem:

$$\sup_{\substack{0 \neq \mathbf{B} \in \mathcal{B} \\ 0 \neq \alpha \in H^1}} \frac{\int_{B_1} \alpha \mathbf{B} \cdot (\nabla \times \mathbf{B}) dv}{\|\alpha\|_{\sim} \int_{B_1} |\nabla \times \mathbf{B}|^2 dv} =: \frac{1}{R_E^{opt}}, \tag{2.10}$$

where we have chosen the  $H^1$ -norm (1.4) to measure the strength of the  $\alpha$  field. Appropriate variational classes are then  $H^1$ , the space of [weakly]<sup>3</sup> differentiable functions  $\alpha : B_1 \rightarrow \mathbb{R}$  with

$$\int_{B_1} (\alpha^2 + |\nabla \alpha|^2) dv < \infty \tag{2.11}$$

and  $\mathcal{B}$ , the space of [weakly] differentiable vector fields  $\mathbf{B} : B_1 \rightarrow \mathbb{R}^3$  that are [weakly] divergence-free and have finite  $H^1$ -norm, which (due to the divergence constraint) is equivalent to

$$\int_{B_1} |\nabla \times \mathbf{B}|^2 dv < \infty. \tag{2.12}$$

To implement the non-local boundary condition (2.1)<sub>2-4</sub> we recall that such  $H^1$ -fields  $\mathbf{B}$  allow harmonic extensions  $\tilde{\mathbf{B}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , i.e., [weakly]

$$\nabla \cdot \tilde{\mathbf{B}} = 0 \quad \text{in } B_1 \cup \widehat{B_1}, \quad \nabla \times \tilde{\mathbf{B}} = 0 \quad \text{in } \widehat{B_1}, \tag{2.13}$$

which are, moreover, unique if [in the trace-sense]  $[\mathbf{r} \cdot \tilde{\mathbf{B}}] = 0$  on  $S_1$  (for more details see [Kai12]). In the following we identify these  $H^1$ -fields with their harmonic extensions and require for elements  $\tilde{\mathbf{B}} \in \mathcal{B}$  additionally [in the trace-sense]

$$[\tilde{\mathbf{B}}] = 0 \quad \text{on } S_1. \tag{2.14}$$

The tilde is henceforth omitted.

In these spaces the variational expression (1.3) turns out to be bounded from above,<sup>4</sup> i.e., problem (2.10) makes sense. Standard arguments imply, moreover, the existence of

<sup>3</sup>Readers not familiar with “weak differentiability” may ignore the specifications given in brackets in this paragraph without missing the essence of the presentation.

<sup>4</sup>A proof is based on Sobolev-type embedding results (see, e.g., [Ada75]). An explicit bound is given elsewhere.

a maximizing couple  $(\mathbf{B}, \alpha)$ , which satisfies [weakly] the corresponding Euler-Lagrange equations.<sup>5</sup>

To derive these equations we reformulate (2.10) as a constrained problem, i.e., we introduce Lagrange parameters  $\mu$  and  $\nu$  associated with (2.11) and (2.12), respectively, and Lagrange fields  $\rho$  and  $\boldsymbol{\sigma}$ , associated with (2.13) and perform variations of the extended functional

$$\begin{aligned} \mathcal{F}[\mathbf{B}, \alpha, \boldsymbol{\sigma}, \rho, \mu, \nu] := & \int_{B_1} \alpha \mathbf{B} \cdot \nabla \times \mathbf{B} \, dv \\ & - \int_{B_1 \cup \widehat{B}_1} \rho \nabla \cdot \mathbf{B} \, dv - \int_{\widehat{B}_1} \boldsymbol{\sigma} \cdot \nabla \times \mathbf{B} \, dv \\ & - \mu \left\{ \int_{B_1} |\nabla \times \mathbf{B}|^2 \, dv - 1 \right\} - \nu \left\{ \int_{B_1} (\alpha^2 + |\nabla \alpha|^2) \, dv - \frac{4\pi}{3} \right\} \end{aligned} \quad (2.15)$$

with respect to all its variables  $\mathbf{B}$  through  $\nu$  (cf. [CH53]):

$$\begin{aligned} \delta \mathcal{F} = & \int_{B_1} \left\{ \delta \alpha \mathbf{B} \cdot \nabla \times \mathbf{B} + \alpha \delta \mathbf{B} \cdot \nabla \times \mathbf{B} + \alpha \mathbf{B} \cdot \nabla \times \delta \mathbf{B} \right. \\ & - \delta \rho \nabla \cdot \mathbf{B} - \rho \nabla \cdot \delta \mathbf{B} - \delta \mu \left[ |\nabla \times \mathbf{B}|^2 - \frac{3}{4\pi} \right] - 2\mu \nabla \times \mathbf{B} \cdot \nabla \times \delta \mathbf{B} \\ & \left. - \delta \nu [(\alpha^2 + |\nabla \alpha|^2) - 1] - 2\nu (\alpha \delta \alpha + \nabla \alpha \cdot \nabla \delta \alpha) \right\} \, dv \\ & - \int_{\widehat{B}_1} \left\{ \delta \rho \nabla \cdot \mathbf{B} + \rho \nabla \cdot \delta \mathbf{B} + \delta \boldsymbol{\sigma} \cdot \nabla \times \mathbf{B} + \boldsymbol{\sigma} \cdot \nabla \times \delta \mathbf{B} \right\} \, dv. \end{aligned}$$

After integration by parts the necessary condition  $\delta \mathcal{F} = 0$  for critical points of  $\mathcal{F}$  yields the following Euler-Lagrange equations for  $\mathbf{B}$ ,  $\alpha$ ,  $\boldsymbol{\sigma}$  and  $\rho$  in  $B_1$  and  $\widehat{B}_1$ , together with jump relations over  $S_1$  and normalizations for  $\mathbf{B}$  and  $\alpha$ :

$$\left. \begin{aligned} \mu \nabla \times \nabla \times \mathbf{B} &= \alpha \nabla \times \mathbf{B} + \frac{1}{2} \nabla \alpha \times \mathbf{B} + \frac{1}{2} \nabla \rho, \quad \nabla \cdot \mathbf{B} = 0 && \text{in } B_1, \\ 2\nu(-\Delta \alpha + \alpha) &= \mathbf{B} \cdot \nabla \times \mathbf{B} && \text{in } B_1, \\ \nabla \times \mathbf{B} &= 0, \quad \nabla \cdot \mathbf{B} = 0 && \text{in } \widehat{B}_1, \\ \nabla \times \boldsymbol{\sigma} &= \nabla \rho && \text{in } \widehat{B}_1, \\ \mathbf{r} \cdot \nabla \alpha &= 0 && \text{on } S_1, \\ [\rho] &= 0 && \text{on } S_1, \\ \boldsymbol{\sigma}^+ \times \mathbf{r} &= (2\mu \nabla \times \mathbf{B} - \alpha \mathbf{B})^- \times \mathbf{r}, && \text{on } S_1, \\ \int_{B_1} |\nabla \times \mathbf{B}|^2 \, dv &= 1, \quad \int_{B_1} (\alpha^2 + |\nabla \alpha|^2) \, dv = \frac{4\pi}{3}. \end{aligned} \right\} \quad (2.16)$$

<sup>5</sup>In fact, using arguments similar to those in [KU09] weak solutions of the Euler-Lagrange system (2.20) can be shown to possess enough regularity, so that (2.20) is pointwise satisfied. Without going into more detail we focus henceforth on such “classical” solutions of (2.20).

The superscripts  $+$  and  $-$  in  $(2.16)_7$  denote the exterior and the interior side of  $S_1$ , respectively. Note that  $(2.16)_4$  and  $(2.16)_7$  with given right-hand sides constitute a well-posed exterior boundary-value problem for  $\boldsymbol{\sigma}$ , provided that  $\boldsymbol{\sigma} \rightarrow 0$  for  $r \rightarrow \infty$  and that the integrability condition

$$(\mathbf{r} \cdot (2.16)_4)^+ = \mathbf{r} \cdot \nabla \times \boldsymbol{\sigma}^+ = \nabla \cdot (\boldsymbol{\sigma}^+ \times \mathbf{r}) = \nabla \cdot (2.16)_7 \quad \text{on } S_1 \tag{2.17}$$

is satisfied. Using the radial component of  $(2.16)_1$ , equation (2.17) amounts to

$$\begin{aligned} (\mathbf{r} \cdot \nabla \rho)^+ &= \mathbf{r} \cdot (2\mu \nabla \times \nabla \times \mathbf{B} - \nabla \times (\alpha \mathbf{B}))^- \\ &= \mathbf{r} \cdot (\alpha \nabla \times \mathbf{B} + \nabla \rho)^- = (\mathbf{r} \cdot \nabla \rho)^- \quad \text{on } S_1. \end{aligned} \tag{2.18}$$

In the last step we made use of  $\mathbf{r} \cdot \nabla \times \mathbf{B} = 0$  on  $S_1$ , which follows from (2.14) and  $(2.16)_3$ . On the other hand taking the divergence of  $(2.16)_1$  and  $(2.16)_4$  yields

$$\left. \begin{aligned} \Delta \rho &= -\nabla \alpha \cdot \nabla \times \mathbf{B} && \text{in } B_1, \\ \Delta \rho &= 0 && \text{in } \widehat{B}_1. \end{aligned} \right\} \tag{2.19}$$

Taking into account (2.18) and (2.19),  $\boldsymbol{\sigma}$  may thus be eliminated from (2.16). Concerning asymptotic behaviour it is well known that  $|\mathbf{B}| = O(r^{-3})$ ,  $r \rightarrow \infty$  for monopole-free exterior harmonic vector fields; for exterior harmonic functions  $\rho$  we have generally  $\rho = \rho_0 r^{-1} + O(r^{-2})$  for  $r \rightarrow \infty$ . However, (2.18) and (2.19) imply

$$\begin{aligned} 0 &= \int_{S_1} \alpha \nabla \times \mathbf{B} \cdot \mathbf{r} \, ds = \int_{B_1} \nabla \alpha \cdot \nabla \times \mathbf{B} \, dv = - \int_{B_1} \Delta \rho \, dv \\ &= - \int_{S_1} \mathbf{r} \cdot \nabla \rho \, ds = - \int_{S_R} \mathbf{r}/r \cdot \nabla \rho \, ds = 4\pi \rho_0 - \int_{S_R} O(r^{-3}) \, ds, \end{aligned}$$

with  $R$  denoting an arbitrary radius  $> 1$ . The limit  $R \rightarrow \infty$  implies then  $\rho_0 = 0$  and hence  $\rho = O(r^{-2})$  for  $r \rightarrow \infty$ .

Finally, multiplying  $(2.16)_1$  by  $\mathbf{B}$  and  $(2.16)_2$  by  $\alpha$ , integrating over  $B_1$  and integrating by parts reveals by  $(2.16)_8$  that

$$\int_{B_1} \alpha \nabla \times \mathbf{B} \cdot \mathbf{B} \, dv = \mu = 2\nu \frac{4\pi}{3}.$$

Comparing with (2.10) thus yields the result

$$\widetilde{R}_E^{opt} = \inf \{ \mu^{-1} : \mu \text{ eigenvalue of system (2.20)} \}$$



with

$$\left. \begin{aligned}
 \mu \nabla \times \nabla \times \mathbf{B} &= \alpha \nabla \times \mathbf{B} + \frac{1}{2} \nabla \alpha \times \mathbf{B} + \frac{1}{2} \nabla \rho, \quad \nabla \cdot \mathbf{B} = 0 && \text{in } B_1, \\
 \mu(-\Delta \alpha + \alpha) &= \frac{4\pi}{3} \mathbf{B} \cdot \nabla \times \mathbf{B} && \text{in } B_1, \\
 -\Delta \rho &= \nabla \alpha \cdot \nabla \times \mathbf{B} && \text{in } B_1, \\
 \nabla \times \mathbf{B} = 0, \quad \nabla \cdot \mathbf{B} &= 0 && \text{in } \widehat{B}_1, \\
 \Delta \rho &= 0 && \text{in } \widehat{B}_1, \\
 [\mathbf{B}] &= 0 && \text{on } S_1, \\
 \mathbf{r} \cdot \nabla \alpha &= 0 && \text{on } S_1, \\
 [\rho] = 0, \quad [\mathbf{r} \cdot \nabla \rho] &= 0 && \text{on } S_1, \\
 |\mathbf{B}| = O(r^{-3}), \quad |\rho| &= O(r^{-2}) && \text{for } r \rightarrow \infty, \\
 \int_{B_1} (\alpha^2 + |\nabla \alpha|^2) dv &= \frac{4\pi}{3}.
 \end{aligned} \right\} \quad (2.20)$$

Note that contrary to the standard eigenvalue problem associated to (2.1) with stationary coefficients, eigenvalues of (2.20) are always real in accordance with their variational interpretation. Special cases of system (2.20) are related to the case of prescribed  $\alpha$ -field: (2.20)<sub>2</sub> and (2.20)<sub>7</sub> are then removed and the remaining equations constitute a linear eigenvalue problem, or to the case of constant  $\alpha$ : the auxiliary field  $\rho$  becomes trivial in this case and (2.20) boils down to the stationary case of the dynamo problem (2.1) with constant diffusivity  $\mu$ .

**3. Spherical symmetry.** Spherical symmetry of the  $\alpha$ -field greatly simplifies the solution of problem (2.10) in that it allows one to replace the non-linear system (2.20) of *partial* differential equations by a non-linear system of *ordinary* differential equations (in the radial variable  $r = |\mathbf{r}|$ ). The starting point is a (by spherical symmetry) reduced version of the variational expression (1.3). For this purpose we make use of the poloidal/toroidal decomposition for the magnetic field in the form

$$\mathbf{B} = -\Lambda T - \nabla \times \Lambda S, \tag{3.1}$$

where the poloidal scalar  $S$  and the toroidal one  $T$  are uniquely determined by

$$\mathbf{r} \cdot \mathbf{B} = -\mathcal{L} S, \quad \Lambda \cdot \mathbf{B} = -\mathcal{L} T, \tag{3.2}$$

provided that  $S$  and  $T$  have vanishing mean-value over every sphere  $S_r$ , i.e.,

$$\langle S \rangle_r = 0, \quad \langle T \rangle_r = 0, \quad \langle \dots \rangle_r := \frac{1}{4\pi r^2} \int_{S_r} \dots ds \tag{3.3}$$

and if  $\mathbf{B}$  is sufficiently regular.  $\Lambda$  denotes the non-radial-derivative operator  $\mathbf{r} \times \nabla$  and  $\mathcal{L} := \Lambda \cdot \Lambda$  is the Laplace-Beltrami-operator on  $S_1$ ;  $-\mathcal{L}$  is a positive symmetric operator with the spherical harmonics  $Y_{nm}$  as eigenfunctions:

$$-\mathcal{L} Y_{nm} = n(n+1)Y_{nm}; \tag{3.4}$$

moreover,  $\mathcal{L}$  is related to the Laplacian  $\Delta$  by

$$\Delta = \frac{1}{r} \partial_r^2 r + \frac{1}{r^2} \mathcal{L}, \tag{3.5}$$

where  $\partial_r$  denotes the radial derivative  $\mathbf{r}/r \cdot \nabla$ .

Standard manipulations then yield

$$\begin{aligned} \int_{S_1} \nabla \times \mathbf{B} \cdot \mathbf{B} \, ds &= \int_{S_1} \nabla \times (\Lambda T + \nabla \times \Lambda S) \cdot (\Lambda T + \nabla \times \Lambda S) \, ds \\ &= \int_{S_1} [\nabla \times \Lambda T \cdot \nabla \times \Lambda S + \nabla \times (\nabla \times \Lambda S) \cdot \Lambda T] \, ds \\ &= \int_{S_1} [\mathcal{L} T \mathcal{L} S + \Lambda \partial_r(rT) \cdot \Lambda \partial_r(rS) + \mathcal{L} T \Delta S r^2] \frac{1}{r^2} \, ds \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \int_{S_1} |\nabla \times \mathbf{B}|^2 \, ds &= \int_{S_1} |\nabla \times (\Lambda T + \nabla \times \Lambda S)|^2 \, ds \\ &= \int_{S_1} [|\nabla \times \Lambda T|^2 + |\nabla \times (\nabla \times \Lambda S)|^2] \, ds \\ &= \int_{S_1} [(\mathcal{L} T)^2 + |\Lambda \partial_r(rT)|^2 + |\Lambda \Delta S|^2 r^2] \frac{1}{r^2} \, ds, \end{aligned} \tag{3.7}$$

where  $ds$  denotes the area element on  $S_1$ . Next, we expand the variables  $rT$  and  $rS$  into spherical harmonics according to<sup>6</sup>

$$rT(\mathbf{r}) = \sum_{n,m} \tilde{T}_{nm}(r) Y_{nm}(\mathbf{r}/r), \quad rS(\mathbf{r}) = \sum_{n,m} \tilde{S}_{nm}(r) Y_{nm}(\mathbf{r}/r) \tag{3.8}$$

with  $|m| \leq n$ ,  $n \in \mathbb{N}$ . Inserting (3.8), the expressions (3.6) and (3.7) take by (3.4) and (3.5) the form

$$\int_{S_1} \nabla \times \mathbf{B} \cdot \mathbf{B} \, ds = \sum_{n,m} n(n+1) \left[ 2 \frac{n(n+1)}{r^2} \tilde{T}_{nm} \tilde{S}_{nm} + \tilde{T}'_{nm} \tilde{S}'_{nm} - \tilde{T}_{nm} \tilde{S}''_{nm} \right] \frac{1}{r^2} \tag{3.9}$$

and

$$\int_{S_1} |\nabla \times \mathbf{B}|^2 \, ds = \sum_{n,m} n(n+1) \left[ \frac{n(n+1)}{r^2} \tilde{T}_{nm}^2 + (\tilde{T}'_{nm})^2 + \left( \tilde{S}''_{nm} - \frac{n(n+1)}{r^2} \tilde{S}_{nm} \right)^2 \right] \frac{1}{r^2}, \tag{3.10}$$

where prime means the derivative  $\frac{d}{dr}$ . Abbreviating the brackets in (3.9) and (3.10) by  $\mathcal{N}_n[\tilde{T}_{nm}, \tilde{S}_{nm}]$  and  $\mathcal{D}_n[\tilde{T}_{nm}, \tilde{S}_{nm}]$ , respectively, the variational expression (1.3) now takes the form

$$\frac{\sum_{n,m} n(n+1) \int_0^1 \alpha \mathcal{N}_n[\tilde{T}_{nm}, \tilde{S}_{nm}] \, dr}{\sum_{n,m} n(n+1) \|\alpha\| \int_0^1 \mathcal{D}_n[\tilde{T}_{nm}, \tilde{S}_{nm}] \, dr}. \tag{3.11}$$

As to the supremum of (3.11) note that by proper choice of the relative sign of  $\tilde{T}_{nm}$  and  $\tilde{S}_{nm}$ , every term in the numerator of (3.11) can be made non-negative. Moreover,

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<sup>6</sup>By taking suitable linear combinations the  $Y_{nm}$  can be assumed to be real.

in view of Lemma 1 in Appendix A, we can select a single mode with maximal ratio, which at most increases (3.11). To determine the supremum of (3.11) it is thus sufficient to consider the single-mode problem

$$\sup_{n \in \mathbb{N}} \sup_{\substack{0 \neq \tilde{T}_n \in H_0^1 \\ 0 \neq \tilde{S}_n \in H_n^2 \\ 0 \neq \alpha_n \in H^1}} \frac{\int_0^1 \alpha_n \mathcal{N}_n[\tilde{T}_n, \tilde{S}_n] dr}{\|\alpha_n\|_s \int_0^1 \mathcal{D}_n[\tilde{T}_n, \tilde{S}_n] dr} =: \frac{1}{R_E^{opt}}, \quad (3.12)$$

where we have omitted the (now dummy) index  $m$  at  $\tilde{T}_n$  and  $\tilde{S}_n$ , and have added the index  $n$  at  $\alpha$  (to indicate this dependence). Appropriate variational classes corresponding to  $\mathcal{B}$  in problem (2.10) are  $H_0^1$ , the space of [weakly] differentiable functions  $\tilde{T}_n$  with finite  $H^1$ -norm and with zero boundary conditions at  $r = 0$  and  $r = 1$  and  $H_n^2$ , the space of twice [weakly] differentiable functions  $\tilde{S}_n$  with finite  $H^2$ -norm and with zero condition of  $\tilde{S}_n$  and  $\tilde{S}'_n$  at  $r = 0$  and the vacuum matching condition  $\tilde{S}'_n + n\tilde{S}_n = 0$  at  $r = 1$ . As to  $\alpha_n$  we consider the  $\|\cdot\|_s$ -norm (1.5) first. An appropriate variational class is then  $H^1$ , the space of [weakly] differentiable functions  $\alpha$  on  $(0, 1)$  with  $\|\alpha_n\|_s < \infty$ . The rough estimate

$$\begin{aligned} \alpha(x) &= \int_0^1 (\alpha(x) - \alpha(y)) dy + \int_0^1 \alpha(y) dy = \int_0^1 \int_y^x \alpha'(z) dz dy + \int_0^1 \alpha(y) dy \\ &\leq \int_0^1 |\alpha'(z)| dz + \int_0^1 \alpha(y) dy \leq \left( \int_0^1 (\alpha'(z))^2 dz \right)^{1/2} + \left( \int_0^1 (\alpha(y))^2 dy \right)^{1/2} \\ &\leq \sqrt{2} \left( \int_0^1 [(\alpha(y))^2 + (\alpha'(y))^2] dy \right)^{1/2} \quad \text{for all } x \in (0, 1) \end{aligned}$$

demonstrates the dominance of the  $\|\cdot\|_s$ -norm over the maximum norm:

$$\alpha_{\max} \leq C \|\alpha\|_s \quad (3.13)$$

with  $C = \sqrt{2}$ . The optimal (i.e., minimal) constant in (3.13) is itself a solution of a variational problem (see Lemma 2 in Appendix B) and reads

$$C_{opt} = \left( \frac{e + 1/e}{e - 1/e} \right)^{1/2} = 1.1459. \quad (3.14)$$

The bounds (2.8) and (3.13) imply by (3.9), (3.10), and (2.6) the boundedness from above of the variational expression (3.11) and hence the well-posedness of problem (3.12).

To derive the Euler-Lagrange equations associated to (3.12) we introduce analogously to section 2 an extended functional, viz.

$$\begin{aligned} \mathcal{F}_s[\tilde{T}_n, \tilde{S}_n, \alpha_n, \mu_n, \nu_n] &:= \int_0^1 \alpha_n \mathcal{N}_n[\tilde{T}_n, \tilde{S}_n] dr \\ &- \mu \left\{ \int_0^1 \mathcal{D}_n[\tilde{T}_n, \tilde{S}_n] dr - 1 \right\} - \nu \left\{ \int_0^1 (\alpha_n^2 + \alpha_n'^2) dr - 1 \right\} \end{aligned}$$

and perform variations with respect to  $\tilde{T}_n$  through  $\nu_n$ .  $\delta\mathcal{F}_s = 0$  yields then after integration by parts (using the boundary conditions implemented into the variational classes  $H_0^1$  and  $H_n^2$ ):

$$\left. \begin{aligned}
 \mu_n D_n \tilde{T}_n &= \alpha_n D_n \tilde{S}_n + \frac{1}{2} \alpha'_n \tilde{S}'_n && \text{in } (0, 1), \\
 \mu_n D_n^2 \tilde{S}_n &= -\alpha_n D_n \tilde{T}_n - \frac{3}{2} \alpha'_n \tilde{T}'_n - \frac{1}{2} \alpha''_n \tilde{T}_n && \text{in } (0, 1), \\
 2\nu_n(\alpha''_n - \alpha_n) &= -2 \frac{n(n+1)}{r^2} \tilde{T}_n \tilde{S}_n - \tilde{T}'_n \tilde{S}'_n + \tilde{T}_n \tilde{S}''_n && \text{in } (0, 1), \\
 \mu_n ((D_n \tilde{S}_n)' + n D_n \tilde{S}_n) + \alpha_n \tilde{T}'_n &= 0 && \text{at } r = 1, \\
 \alpha'_n &= 0 && \text{at } r = 0 \text{ and } 1, \\
 \int_0^1 \mathcal{D}_n[\tilde{T}_n, \tilde{S}_n] dr &= 1, \quad \|\alpha_n\|_s = 1.
 \end{aligned} \right\} \tag{3.15}$$

Here,  $D_n$  denotes the operator  $\frac{d^2}{dr^2} - \frac{n(n+1)}{r^2}$ . A calculation analogous to that in section 2 reveals that

$$\int_0^1 \alpha_n \mathcal{N}_n[\tilde{T}_n, \tilde{S}_n] dr = \mu_n = 2\nu_n$$

and comparison with (3.12) yields the result

$$\frac{1}{R_E^{opt}} = \sup_{n \in \mathbb{N}} \sup \{ \mu_n : \mu_n \text{ eigenvalue of system (3.17)} \} \tag{3.16}$$

with

$$\left. \begin{aligned}
 \mu_n D_n \tilde{T}_n &= \alpha_n D_n \tilde{S}_n + \frac{1}{2} \alpha'_n \tilde{S}'_n && \text{in } (0, 1), \\
 \mu_n D_n^2 \tilde{S}_n &= -D_n(\alpha_n \tilde{T}_n) + \frac{1}{2} (\alpha'_n \tilde{T}_n)' && \text{in } (0, 1), \\
 \mu_n(\alpha''_n - \alpha_n) &= \tilde{T}_n D_n \tilde{S}_n - \frac{n(n+1)}{r^2} \tilde{T}_n \tilde{S}_n - \tilde{T}'_n \tilde{S}'_n && \text{in } (0, 1), \\
 \tilde{T}_n = 0, \quad \tilde{S}_n = \tilde{S}'_n = 0, \quad \alpha'_n &= 0 && \text{at } r = 0, \\
 \tilde{T}'_n = 0, \quad \tilde{S}'_n + n \tilde{S}_n = 0, \quad \alpha'_n &= 0 && \text{at } r = 1, \\
 \mu_n ((D_n \tilde{S}_n)' + n D_n \tilde{S}_n) + \alpha_n \tilde{T}'_n &= 0 && \text{at } r = 1, \\
 \int_0^1 (\alpha_n^2 + (\alpha'_n)^2) dr &= 1.
 \end{aligned} \right\} \tag{3.17}$$

As in the non-symmetric case solutions of system (3.17) are more regular than indicated by the variational classes, which, however, needs some effort to be demonstrated. As

to the behaviour at the origin there is a short-cut: with the assumption of a smooth magnetic field, i.e., smooth fields  $S(\mathbf{r})$  and  $T(\mathbf{r})$ , the representation<sup>7</sup>

$$S(\mathbf{r}) = \frac{1}{r} \tilde{S}_n(r) Y_{nm}(\mathbf{r}/r) = \left[ \frac{1}{r^{n+1}} \tilde{S}_n(r) \right] [r^n Y_{nm}(\mathbf{r}/r)] \quad (3.18)$$

implies for the first bracket on the right-hand side of (3.18) the Taylor expansion

$$\frac{1}{r^{n+1}} \tilde{S}_n(r) = \sum_{i=0}^I c_i r^{2i} + o(r^{2I}), \quad (3.19)$$

where  $I$  depends on the order of differentiability of  $S(\mathbf{r})$ . This can best be realized in Cartesian coordinates  $\mathbf{r} = (x, y, z)$ ; the second bracket in (3.18) is then a homogeneous (infinitely differentiable) polynomial of degree  $n$  and  $r$  is given by  $(x^2 + y^2 + z^2)^{1/2}$ , which is not differentiable at the origin. We note yet the following consequence of (3.19):

$$D_n \tilde{S}_n|_{r=0} = 0. \quad (3.20)$$

In order to reduce the order of system (3.17) and to allow a comparison with the non-symmetric problem (2.20) we introduce the auxiliary field  $\tilde{\rho}_n$  by the equation

$$D_n \tilde{\rho}_n = \frac{n(n+1)}{r} \alpha'_n \tilde{T}_n \quad \text{in } (0, 1) \quad (3.21)$$

and the boundary conditions

$$\tilde{\rho}_n = 0 \quad \text{at } r = 0, \quad \tilde{\rho}'_n + n \tilde{\rho}_n = 0 \quad \text{at } r = 1. \quad (3.22)$$

Note that (3.21) with the given right-hand side together with (3.22) determine  $\tilde{\rho}_n$  uniquely. Differentiating equation (3.21) one obtains

$$\frac{1}{2n(n+1)} D_n (r \tilde{\rho}'_n - \tilde{\rho}_n) = \frac{1}{2} (\alpha'_n \tilde{T}_n)',$$

which allows to reformulate equation (3.17)<sub>2</sub> as

$$D_n \left[ \mu_n D_n \tilde{S}_n + \alpha_n \tilde{T}_n - \frac{1}{2n(n+1)} (r \tilde{\rho}'_n - \tilde{\rho}_n) \right] = 0. \quad (3.23)$$

By (3.17)<sub>4-6</sub>, (3.19), (3.22), and  $D_n \tilde{\rho}_n|_{r=1} = 0$ , which follows from (3.21), one can verify the boundary conditions (3.22) for the bracket in (3.23); therefore, the operator  $D_n$  in

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<sup>7</sup>The same applies to  $T(\mathbf{r})$  and  $\rho(\mathbf{r})$  (see below).

front of the bracket may be removed with the result

$$\mu_n D_n \tilde{S}_n = -\alpha_n \tilde{T}_n + \frac{1}{2n(n+1)} (r\tilde{\rho}'_n - \tilde{\rho}_n). \tag{3.24}$$

Using (3.24) in (3.17)<sub>1,3</sub> we obtain, finally, the equivalent formulation of problem (3.17):

$$\left. \begin{aligned} \mu_n^2 D_n \tilde{T}_n &= -\alpha_n^2 \tilde{T}_n + \frac{1}{2} \mu_n \alpha'_n \tilde{S}'_n + \frac{1}{2n(n+1)} \alpha_n (r\tilde{\rho}'_n - \tilde{\rho}_n) && \text{in } (0, 1), \\ \mu_n D_n \tilde{S}_n &= -\alpha_n \tilde{T}_n + \frac{1}{2n(n+1)} (r\tilde{\rho}'_n - \tilde{\rho}_n) && \text{in } (0, 1), \\ D_n \tilde{\rho}_n &= \frac{n(n+1)}{r} \alpha'_n \tilde{T}_n && \text{in } (0, 1), \\ \mu_n^2 (\alpha''_n - \alpha_n) &= -\mu_n \left( \frac{n(n+1)}{r^2} \tilde{T}_n \tilde{S}_n + \tilde{T}'_n \tilde{S}'_n \right) && \text{in } (0, 1), \\ &\quad -\alpha_n \tilde{T}_n^2 + \frac{1}{2n(n+1)} \tilde{T}_n (r\tilde{\rho}'_n - \tilde{\rho}_n) \\ \tilde{T}_n = 0, \quad \tilde{S}_n = 0, \quad \tilde{\rho}_n = 0, \quad \alpha'_n = 0 &&& \text{at } r = 0, \\ \tilde{T}_n = 0, \quad \tilde{S}'_n + n \tilde{S}_n = 0, \quad \tilde{\rho}'_n + n \tilde{\rho}_n = 0, \quad \alpha'_n = 0 &&& \text{at } r = 1, \\ \int_0^1 (\alpha_n^2 + (\alpha'_n)^2) dr &= 1. \end{aligned} \right\} \tag{3.25}$$

Note that, formally, system (3.25) can be obtained from system (2.20) (after poloidal/toroidal decomposition) by the single-mode ansatz

$$\begin{aligned} r T(\mathbf{r}) &= \tilde{T}_n(r) Y_{nm}(\mathbf{r}/r), \quad r S(\mathbf{r}) = \tilde{S}_n(r) Y_{nm}(\mathbf{r}/r), \\ r \rho(\mathbf{r}) &= -\tilde{\rho}_n(r) Y_{nm}(\mathbf{r}/r) \end{aligned}$$

and projecting on  $Y_{nm}(\mathbf{r}/r)$  (with the exception of (2.20)<sub>2</sub>, where the spherical mean (3.3) has to be taken). The vacuum part of (2.20) can then be solved and eliminated, replacing thereby the matching conditions at  $S_1$  by the boundary conditions (3.25)<sub>6</sub>. (3.25)<sub>5</sub> arise from the regularity conditions (3.19), and (3.25)<sub>7</sub> refers to the  $\|\cdot\|_s$ -norm (instead of the  $\|\cdot\|_{\sim}$ -norm).

For the numerical implementation in a domain that contains the origin yet another reformulation is useful. According to (3.19) the variables

$$\begin{aligned} \bar{T}_n &:= \tilde{T}_n / (\mu_n r^{n+1}), \quad \bar{S}_n := \tilde{S}_n / (\mu_n r^{n+1}), \\ \bar{\rho}_n &:= \tilde{\rho}_n / (\mu_n^2 r^{n+1}), \quad \bar{\alpha}_n := \alpha_n / \mu_n \end{aligned} \tag{3.26}$$

are stripped off their zeros at the origin. They are governed by the equations

$$\left. \begin{aligned} \bar{T}_n'' &= -2 \frac{n+1}{r} \bar{T}_n' - \bar{\alpha}_n^2 \bar{T}_n + \frac{1}{2} \bar{\alpha}_n' (\bar{S}_n' + \frac{n+1}{r} \bar{S}_n) \\ &\quad + \frac{1}{2n(n+1)} \bar{\alpha}_n (r\bar{\rho}_n' + n\bar{\rho}_n), \\ \bar{S}_n'' &= -2 \frac{n+1}{r} \bar{S}_n' - \bar{\alpha}_n \bar{T}_n + \frac{1}{2n(n+1)} (r\bar{\rho}_n' + n\bar{\rho}_n), \\ \bar{\rho}_n'' &= -2 \frac{n+1}{r} \bar{\rho}_n' + \frac{n(n+1)}{r} \bar{\alpha}_n' \bar{T}_n, \\ \bar{\alpha}_n'' &= \bar{\alpha}_n - r^{2n+2} \left[ \frac{n(n+1)}{r^2} \bar{T}_n \bar{S}_n + \bar{T}_n' \bar{S}_n' + \frac{n+1}{r} (\bar{T}_n' \bar{S}_n + \bar{T}_n \bar{S}_n') \right. \\ &\quad \left. + \bar{\alpha}_n \bar{T}_n^2 - \frac{1}{2n(n+1)} \bar{T}_n (r\bar{\rho}_n' + n\bar{\rho}_n) \right] \end{aligned} \right\} \quad (3.27)$$

on the interval  $(0, 1)$  subject to the boundary conditions

$$\bar{T}_n' = 0, \quad \bar{S}_n' = 0, \quad \bar{\rho}_n' = 0, \quad \bar{\alpha}_n' = 0 \quad \text{at } r = 0, \quad (3.28)$$

which arise from (3.19) (and which are numerically convenient) and

$$\bar{T}_n = 0, \quad \bar{S}_n' + (2n+1)\bar{S}_n = 0, \quad \bar{\rho}_n' + (2n+1)\bar{\rho}_n = 0, \quad \bar{\alpha}_n' = 0 \quad \text{at } r = 1. \quad (3.29)$$

The eigenvalues  $\mu_n$  are then determined by

$$\mu_n = \|\bar{\alpha}_n\|_s^{-1} = \left( \int_0^1 (\bar{\alpha}_n^2 + (\bar{\alpha}_n')^2) dr \right)^{-1/2}. \quad (3.30)$$

For its numerical treatment, the system of four second order differential equations (3.27) is written as a system of eight first order differential equations. The general strategy is then to implement a shooting method which imposes at  $r = 0$  the four conditions (3.28) and searches for four additional conditions on  $\bar{T}_n$ ,  $\bar{S}_n$ ,  $\bar{\rho}_n$  and  $\bar{\alpha}_n$  at  $r = 0$  so that an integration yields a solution which fulfills the requirements (3.29) at  $r = 1$ . The differential equations are integrated with the standard Runge-Kutta-Fehlberg method of fifth order. However, the integration cannot start from exactly  $r = 0$  because of the singularity there and has instead to start a small distance  $\Delta r$  away from the origin. The search for the four unknown starting values which will lead to a solution satisfying the boundary conditions at  $r = 1$  is done with a Newton method in which the Jacobian is determined numerically from finite differences. This search finds a distinct root for every eigenvalue of (3.27) and the numerical problem is to find the smallest of these. Due to the non-linearity of the equations, there is no general method which guarantees finding the smallest eigenvalue. Depending on the initial guess for the four unknowns from which

Newton's method is started, it converges to one or the other root. However, a systematic scan of starting points for Newton's method reveals that every root has a broad attractor and that it is very unlikely that the smallest root was missed. The tolerances on the function values and search points in Newton's method together with  $\Delta r$  determine the accuracy of the numerical solution. All these numbers were set to  $10^{-8}$ .

Finally, the eigenfunction is evaluated at  $10^4$  equidistant points. The function  $\bar{\alpha}_n$ , whose derivatives are zero both at  $r = 0$  and  $r = 1$ , is then represented as a natural cubic spline through these points. This spline interpolation is used to evaluate the integral (3.30) and to compute  $R_E^{opt}$  below.

Note that the transformation of system (3.25) into (3.27), although redundant from an analytical point of view, vastly improves the accuracy of the numerical solution. The reason for this is that the regularity condition (3.19) implies that higher derivatives of the variables in (3.25) are zero at  $r = 0$ . Since (3.25) is a second order system, only the variables and their first derivatives at  $r = 0$  can be imposed in a numerical procedure. Higher derivatives which ought to be zero acquire a finite value in the numerical approximation and mix unwanted contributions into the solution. In system (3.27) this problem obviously no longer occurs.

Fig. 1 shows  $1/\mu_n^{\max}$  plotted against  $n$  for  $n = 1, \dots, 12$ . Even though the monotonous dependence is obvious in Fig. 1, we were not able to prove it for all  $n \in \mathbb{N}$ . Instead we

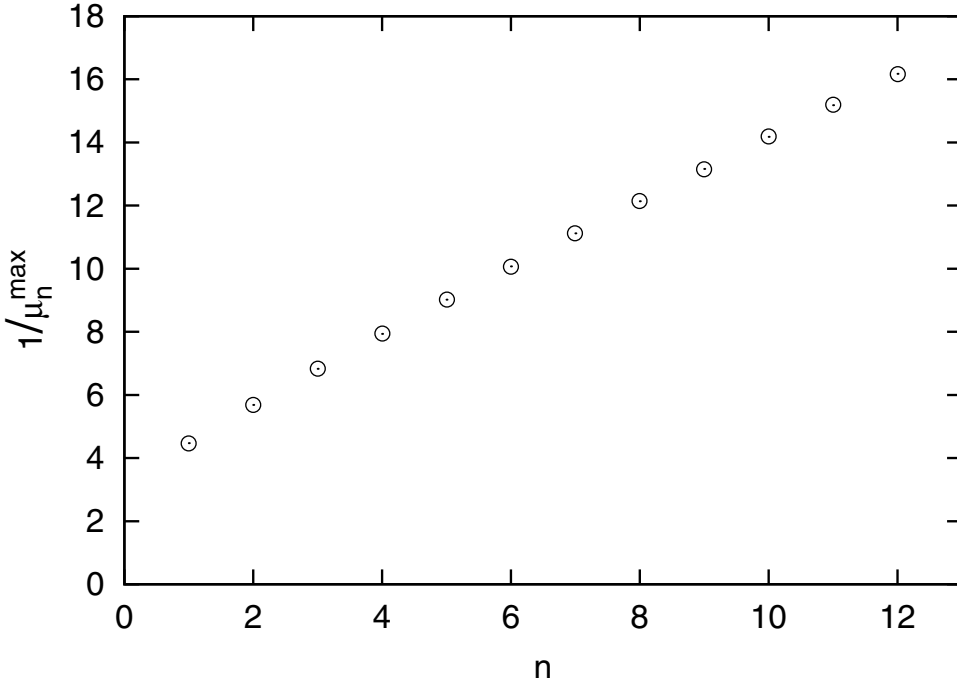


FIG. 1.  $1/\mu_n^{\max}$  as a function of  $n$ .



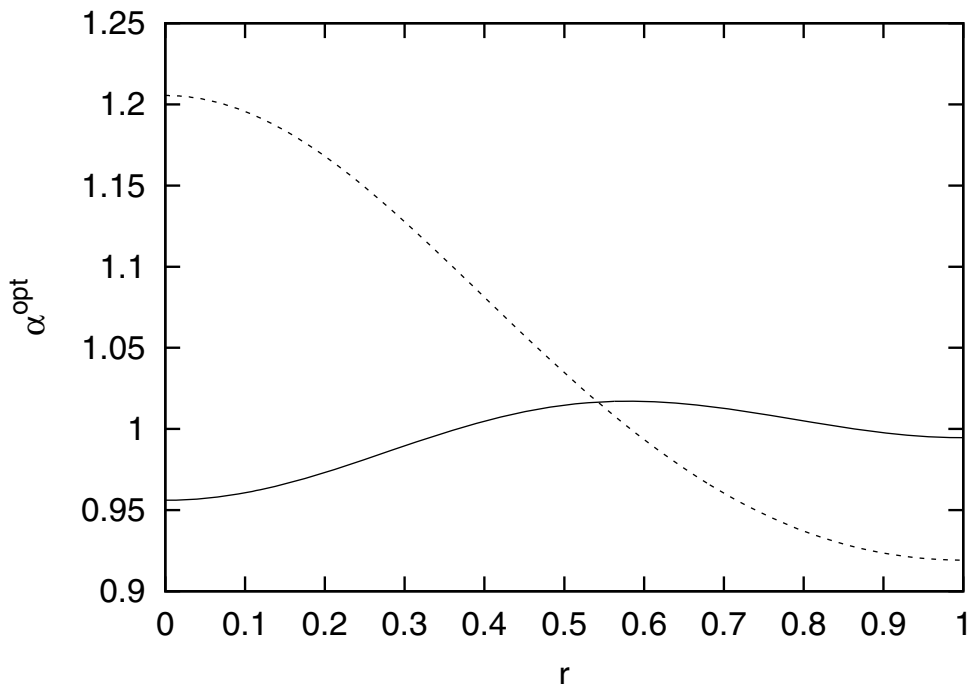


FIG. 2. Optimal profiles  $\alpha^{opt}$  (continuous line) and  $\tilde{\alpha}^{opt}$  (dashed line).

derive in Appendix A the estimate

$$\mu_n^{\max} \leq \max \left\{ \mu_1^{\max}, \frac{C_{opt}}{\sqrt{n(n+1) - 3/2}} \right\}, \quad n \in \mathbb{N} \quad (3.31)$$

with  $C_{opt}$  given in (3.14). For  $n \geq 5$  the right-hand side of (3.31) is dominated by  $\mu_1^{\max}$ ; we thus obtain for (3.16) (accurately to four decimals):

$$\frac{1}{R_E^{opt}} = \sup_{n \in \mathbb{N}} \mu_n^{\max} = \mu_1^{\max} = (4.4717)^{-1}. \quad (3.32)$$

The corresponding optimal profile  $\alpha^{opt}$  shown in Fig. 2 is close to the constant profile  $\alpha = 1$ .

The computation of the critical Reynolds number  $R_c^{opt}$  for  $\alpha^{opt}$  is a standard problem of (mean-field) dynamo theory that can be solved with great accuracy. Since  $R_c^{opt}$  is a real eigenvalue of a linear set of equations, it is enough to vary the Reynolds number along the real axis and to verify by a shooting method whether the boundary value problem has a solution for the chosen Reynolds number. The smallest Reynolds number with this property is the required  $R_c^{opt}$ . We obtain

$$R_c^{opt} = 4.4718, \quad (3.33)$$

which exceeds  $R_E^{opt}$  by less than  $10^{-4}$  when normalized to  $R_c^{opt}$ . For comparison we have also computed

$$R_c^{(\alpha=1)} = 4.4934, \quad (3.34)$$

which was (to our knowledge) the lowest hitherto known critical Reynolds number (cf. [KR80, p. 177]) and have, furthermore, translated the Proctor-bound (2.9) by means of (3.13), (3.14) from the max-norm to the  $\|\cdot\|_s$ -norm to obtain the necessary dynamo criterion:

$$R \geq C_{opt}^{-1} R^{(m)} > 3.0596. \quad (3.35)$$

These results may be contrasted with those of the alternative norm (1.4). The only difference in the governing system (3.27) - (3.30) concerns (3.27)<sub>4</sub>, which reads now

$$\begin{aligned} \bar{\alpha}_n'' = & -\frac{2}{3r^3} \bar{\alpha}_n' + \frac{1}{3r^2} \bar{\alpha}_n - \frac{1}{3} r^{2n} \left[ \frac{n(n+1)}{r^2} \bar{T}_n \bar{S}_n + \bar{T}_n' \bar{S}_n' \right. \\ & \left. + \frac{n+1}{r} (\bar{T}_n' \bar{S}_n + \bar{T}_n \bar{S}_n') + \bar{\alpha}_n \bar{T}_n^2 - \frac{1}{2n(n+1)} \bar{T}_n (r\bar{\rho}_n' + n\bar{\rho}_n) \right] \end{aligned}$$

and (3.30), which determines the eigenvalues  $\tilde{\mu}_n$ :

$$\tilde{\mu}_n = \left( 3 \int_0^1 (\bar{\alpha}_n^2 + (\bar{\alpha}_n')^2) r^2 dr \right)^{-1/2}.$$

The numbers corresponding to (3.32) and (3.33) (accurately to three decimals) read

$$\tilde{R}_E^{opt} = (\tilde{\mu}_1^{\max})^{-1} = 4.349 \quad (3.36)$$

and

$$\tilde{R}_c^{opt} = 4.353, \quad (3.37)$$

which makes a (normalized) difference of about  $10^{-3}$ . These numbers can again be compared to<sup>8</sup>

$$\tilde{R}_c^{(\alpha=1)} = 4.493, \quad (3.38)$$

but not to the Backus/Proctor-bound, since these bounds cannot be translated into the  $\|\cdot\|_{\sim}$ -norm (see Appendix B).

The optimal profile  $\tilde{\alpha}^{opt}$  is shown in Fig. 2.

**4. Conclusions and outlook.** We determined in this paper improved energy bounds for the onset of dynamo action in the framework of spherically symmetric  $\alpha^2$ -dynamics by solution of a variational problem that considered both magnetic field and  $\alpha$ -profile. For the  $\alpha$ -profile we made use of the two different  $H^1$ -type norms (1.5) and (1.4) and computed numerically corresponding optimized energy bounds  $R_E^{opt}$  and  $\tilde{R}_E^{opt}$  as largest eigenvalues of certain non-linear systems of ordinary differential equations. These numbers have been compared with the critical Reynolds numbers  $R_c^{opt}$  and  $\tilde{R}_c^{opt}$  of the optimal

<sup>8</sup>Note that  $\|\cdot\|_{\sim}$  and  $\|\cdot\|_s$  have been normalized such that  $\|1\|_{\sim} = \|1\|_s = \|1\|_m = 1$ .

profiles, with  $R_c^{(\alpha=1)}$  of the constant profile, and with previous energy bounds, if available. In particular, we found  $R_E^{opt}$  and  $\tilde{R}_E^{opt}$  almost optimal (see (1.6)), although the difference in the  $\|\cdot\|_{\sim}$ -norm is beyond numerical inaccuracy (see (3.36) and (3.37)), which speaks against strict coincidence of  $\tilde{R}_E^{opt}$  and  $\tilde{R}_c^{opt}$ .

Numerical values and optimal profile shapes (see Fig. 2) depend on the chosen norm; they can be more or less close to already known efficient profiles (here the constant profile; see (3.33), (3.34), (3.37) and (3.38)). Comparison with previous bounds is only possible if the corresponding norm is weaker than the norm used for optimization. This is the case for the well-known max-norm bound and  $\|\cdot\|_s$ ;  $R_E^{opt}$  provides in fact a considerable improvement of the Proctor-bound (3.35). In this respect optimization directly in the max-norm would have been desirable, but to implement the max-norm in a variational approach seemed too difficult to us.

When the restriction to spherical symmetry is abandoned, an improved energy bound is according to (2.10) given by the largest eigenvalue of the system (2.20) of partial differential equations. A spherically symmetric  $\alpha$ -profile is no longer optimal as can be seen from equation (2.20)<sub>2</sub>, where the right-hand side is not spherically symmetric, and the reliable determination of the largest eigenvalue is a formidable task. An improvement compared to the Proctor-bound can be expected, since the latter is based on an estimate that is strict except for force-free magnetic fields. On the other hand, the improved bound will be strictly below the spherically symmetric optimal bound  $\tilde{R}_E^{opt}$  (see (3.36))<sup>9</sup>, since the spherically symmetric optimal solution can be used as a test function in (2.10) with value (3.36), which, however, is now no longer optimal.

Of course, it would be more interesting and more useful to derive improved energy bounds for the induction equation. Optimizing in the full velocity space requires supposedly a similar effort as the full-scale search for the most efficient velocity field. The variational ansatz, however, allows (other than the determination of critical Reynolds numbers) the combination of partial optimizations with estimations to obtain rigorous results. So, there is hope to obtain in this “cheap” way some improvement, whose extent, however, remains to be seen.

**Appendix A.** We deal in this appendix with two inequalities that are useful in section 3: we present an elementary one without proof (Lemma 1) and give the proof of inequality (3.31).

LEMMA 1. Let  $N \in \mathbb{N}$  and  $a_n \geq 0$ ,  $b_n > 0$  for  $1 \leq n \leq N$ . Then

$$\frac{\sum_{n=1}^N a_n}{\sum_{n=1}^N b_n} \leq \max_{1 \leq n \leq N} \frac{a_n}{b_n} \quad (\text{A1})$$

and equality holds if and only if  $a_n = b_n$  for all  $n$ .

---

<sup>9</sup>Note that here only the  $\|\cdot\|_{\sim}$ -norm makes sense.

Note that inequality (A1) remains valid in the limit  $N \rightarrow \infty$ .  
 In order to prove (3.31) we recall that

$$\mu_n^{\max} = \sup_{\substack{0 \neq \tilde{T}_n \in H_0^1 \\ 0 \neq \tilde{S}_n \in H_n^2 \\ 0 \neq \alpha_n \in H^1}} \frac{\int_0^1 \alpha_n \mathcal{N}_n[\tilde{T}_n, \tilde{S}_n] dr}{\|\alpha_n\|_s \int_0^1 \mathcal{D}_n[\tilde{T}_n, \tilde{S}_n] dr} \tag{A2}$$

with

$$\begin{aligned} \mathcal{N}_n[\tilde{T}_n, \tilde{S}_n] &= 2 \frac{n(n+1)}{r^2} \tilde{T}_n \tilde{S}_n + \tilde{T}'_n \tilde{S}'_n - \tilde{T}_n \tilde{S}''_n, \\ \mathcal{D}_n[\tilde{T}_n, \tilde{S}_n] &= \frac{n(n+1)}{r^2} \tilde{T}_n^2 + (\tilde{T}'_n)^2 + \left( \tilde{S}''_n - \frac{n(n+1)}{r^2} \tilde{S}_n \right)^2. \end{aligned}$$

We start with a reformulation of the denominator in (A2). Let  $\hat{S}_n$  with  $\hat{S}_n|_{(1, \infty)} := \tilde{S}_n(1) r^{-n}$  be the harmonic extension of  $\tilde{S}_n$  onto  $(0, \infty)$ . We have then by integration by parts

$$\begin{aligned} 0 &= \int_1^R \left( \hat{S}''_n - \frac{n(n+1)}{r^2} \hat{S}_n \right)^2 dr \\ &= \int_1^R \left[ (\hat{S}''_n)^2 - 2 \frac{n(n+1)}{r^2} \hat{S}_n \hat{S}''_n + \left( \frac{n(n+1)}{r^2} \right)^2 \hat{S}_n^2 \right] dr \\ &= \int_1^R \left[ (\hat{S}''_n)^2 + 2 \frac{n(n+1)}{r^2} (\hat{S}'_n)^2 + \frac{n(n+1)}{r^4} (n(n+1) - 6) \hat{S}_n^2 \right] dr \\ &\quad - 2 \frac{n(n+1)}{r^2} \hat{S}_n (\hat{S}'_n + \hat{S}_n/r) \Big|_1^R \end{aligned}$$

and analogously

$$\begin{aligned} &\int_0^1 \left( \hat{S}''_n - \frac{n(n+1)}{r^2} \hat{S}_n \right)^2 dr \\ &= \int_0^1 \left[ (\hat{S}''_n)^2 + 2 \frac{n(n+1)}{r^2} (\hat{S}'_n)^2 + \frac{n(n+1)}{r^4} (n(n+1) - 6) \hat{S}_n^2 \right] dr \\ &\quad - 2 \frac{n(n+1)}{r^2} \hat{S}_n (\hat{S}'_n + \hat{S}_n/r) \Big|_0^1. \end{aligned}$$

Thus, in the limit  $R \rightarrow \infty$ :

$$\begin{aligned} &\int_0^1 \left( \tilde{S}''_n - \frac{n(n+1)}{r^2} \tilde{S}_n \right)^2 dr = \int_0^1 \left( \hat{S}''_n - \frac{n(n+1)}{r^2} \hat{S}_n \right)^2 dr \\ &= \int_0^\infty \left[ (\hat{S}''_n)^2 + 2 \frac{n(n+1)}{r^2} (\hat{S}'_n)^2 + \frac{n(n+1)}{r^4} (n(n+1) - 6) \hat{S}_n^2 \right] dr \\ &=: \int_0^\infty \hat{\mathcal{D}}_n[\hat{S}_n] dr. \end{aligned}$$

With  $\tilde{\mathcal{D}}_n[\tilde{T}_n] := \frac{n(n+1)}{r^2} \tilde{T}_n^2 + (\tilde{T}'_n)^2$ , (A2) now can equivalently be formulated as

$$\mu_n^{\max} = \sup_{\substack{0 \neq \tilde{T}_n \in H_0^1 \\ 0 \neq \hat{S}_n \in H_\infty^2 \\ 0 \neq \alpha_n \in H^1}} \frac{\int_0^1 \alpha_n \mathcal{N}_n[\tilde{T}_n, \hat{S}_n] dr}{\|\alpha_n\|_s \left\{ \int_0^1 \tilde{\mathcal{D}}_n[\tilde{T}_n] dr + \int_0^\infty \hat{\mathcal{D}}_n[\hat{S}_n] dr \right\}}, \quad (\text{A3})$$

where  $H_\infty^2$  denotes the space of twice [weakly] differentiable functions  $\hat{S}_n$  on  $(0, \infty)$  with finite  $H^2$ -norm and with zero boundary conditions of  $\hat{S}_n$  and  $\hat{S}'_n$  at  $r = 0$ . Note that elements of  $H_\infty^2$  need not be harmonic in  $(1, \infty)$  and need no longer satisfy the ( $n$ -dependent) boundary condition at  $r = 1$ ; both properties are consequences of the Euler-Lagrange equations associated to (A3), which proves the equivalence of problems (A2) and (A3).

Next we decompose the numerator and denominator in (A3) according to

$$\begin{aligned} \int_0^1 \alpha_n \mathcal{N}_n[\tilde{T}_n, \hat{S}_n] dr &= \int_0^1 \alpha_n \left( 2 \frac{n(n+1)}{r^2} \tilde{T}_n \hat{S}_n + \tilde{T}'_n \hat{S}'_n - \tilde{T}_n \hat{S}''_n \right) dr \\ &= \int_0^1 \alpha_n \left( \frac{4}{r^2} \tilde{T}_n \hat{S}_n + \tilde{T}'_n \hat{S}'_n - \tilde{T}_n \hat{S}''_n \right) dr + \int_0^1 \alpha_n 2 \frac{n(n+1)-2}{r^2} \tilde{T}_n \hat{S}_n dr \\ &= \int_0^1 \alpha_n \mathcal{N}_1[\tilde{T}_n, \hat{S}_n] dr + (n(n+1)-2) \int_0^1 \alpha_n \frac{2}{r^2} \tilde{T}_n \hat{S}_n dr \end{aligned}$$

and

$$\begin{aligned} \int_0^1 \tilde{\mathcal{D}}_n[\tilde{T}_n] dr + \int_0^\infty \hat{\mathcal{D}}_n[\hat{S}_n] dr &= \int_0^1 \left( \frac{n(n+1)}{r^2} \tilde{T}_n^2 + (\tilde{T}'_n)^2 \right) dr \\ &\quad + \int_0^\infty \left[ (\hat{S}''_n)^2 + 2 \frac{n(n+1)}{r^2} (\hat{S}'_n)^2 + \frac{n(n+1)}{r^4} (n(n+1)-6) \hat{S}_n^2 \right] dr \\ &= \int_0^1 \left( \frac{2}{r^2} \tilde{T}_n^2 + (\tilde{T}'_n)^2 \right) dr + \int_0^\infty \left[ (\hat{S}''_n)^2 + \frac{4}{r^2} (\hat{S}'_n)^2 - \frac{8}{r^4} \hat{S}_n^2 \right] dr \\ &\quad + \int_0^1 \frac{n(n+1)-2}{r^2} \tilde{T}_n^2 dr \\ &\quad + \int_0^\infty \left[ 2 \frac{n(n+1)-2}{r^2} (\hat{S}'_n)^2 + \frac{n(n+1)-2}{r^4} (n(n+1)-4) \hat{S}_n^2 \right] dr \\ &= \int_0^1 \tilde{\mathcal{D}}_1[\tilde{T}_n] dr + \int_0^\infty \hat{\mathcal{D}}_1[\hat{S}_n] dr \\ &\quad + (n(n+1)-2) \left\{ \int_0^1 \frac{1}{r^2} \tilde{T}_n^2 dr + \int_0^\infty \left( \frac{2}{r^2} (\hat{S}'_n)^2 + \frac{n(n+1)-4}{r^4} \hat{S}_n^2 \right) dr \right\}. \end{aligned}$$

By (A1), (3.13) and abbreviating the braces in the last line by  $\mathcal{B}_n[\tilde{T}_n, \hat{S}_n]$  we can then estimate

$$\begin{aligned} \mu_n^{\max} &= \sup_{\tilde{T}_n, \hat{S}_n, \alpha_n} \frac{\int_0^1 \alpha_n \mathcal{N}_n[\tilde{T}_n, \hat{S}_n] dr}{\|\alpha_n\|_s \left\{ \int_0^1 \tilde{\mathcal{D}}_n[\tilde{T}_n] dr + \int_0^\infty \hat{\mathcal{D}}_n[\hat{S}_n] dr \right\}} \\ &\leq \sup_{\tilde{T}_n, \hat{S}_n, \alpha_n} \max \left\{ \frac{\int_0^1 \alpha_n \mathcal{N}_1[\tilde{T}_n, \hat{S}_n] dr}{\|\alpha_n\|_s \left\{ \int_0^1 \tilde{\mathcal{D}}_1[\tilde{T}_n] dr + \int_0^\infty \hat{\mathcal{D}}_1[\hat{S}_n] dr \right\}}, \frac{\int_0^1 \alpha_n \frac{2}{r^2} \tilde{T}_n \hat{S}_n dr}{\|\alpha_n\|_s \mathcal{B}_n[\tilde{T}_n, \hat{S}_n]} \right\} \\ &\leq \max \left\{ \sup_{\tilde{T}_1, \hat{S}_1, \alpha_1} \frac{\int_0^1 \alpha_1 \mathcal{N}_1[\tilde{T}_1, \hat{S}_1] dr}{\|\alpha_1\|_s \left\{ \int_0^1 \tilde{\mathcal{D}}_1[\tilde{T}_1] dr + \int_0^\infty \hat{\mathcal{D}}_1[\hat{S}_1] dr \right\}}, \right. \\ &\qquad \qquad \qquad \left. \sup_{\tilde{T}_n, \hat{S}_n, \alpha_n} \frac{(\alpha_n)_{\max} \int_0^1 \frac{2}{r^2} |\tilde{T}_n \hat{S}_n| dr}{\|\alpha_n\|_s \mathcal{B}_n[\tilde{T}_n, \hat{S}_n]} \right\} \\ &\leq \max \left\{ \mu_1^{\max}, C_{opt} \sup_{\tilde{T}_n, \hat{S}_n} \frac{\int_0^1 \frac{2}{r^2} |\tilde{T}_n \hat{S}_n| dr}{\mathcal{B}_n[\tilde{T}_n, \hat{S}_n]} \right\}. \end{aligned}$$

Finally, by the inequality

$$\int_0^\infty \hat{S}^2 \frac{dr}{r^4} \leq \frac{4}{5} \int_0^\infty (\hat{S}')^2 \frac{dr}{r^2},$$

which follows by the substitution  $\hat{S} = rS$  from Hardy's inequality

$$\int_0^\infty S^2 \frac{dr}{r^2} \leq 4 \int_0^\infty (S')^2 dr$$

(see [HLP88, p. 175]), we can estimate  $\mathcal{B}_n[\tilde{T}_n, \hat{S}_n]$  by

$$\begin{aligned} \mathcal{B}_n[\tilde{T}_n, \hat{S}_n] &\geq \int_0^1 \frac{1}{r^2} \tilde{T}_n^2 dr + \int_0^\infty \left( \frac{5}{2r^4} \hat{S}_n^2 + \frac{n(n+1)-4}{r^4} \hat{S}_n^2 \right) dr \\ &\geq \int_0^1 \frac{1}{r^2} \tilde{T}_n^2 dr + \left( n(n+1) - \frac{3}{2} \right) \int_0^\infty \frac{1}{r^4} \hat{S}_n^2 dr, \end{aligned}$$

and  $\int_0^1 \frac{2}{r^2} |\tilde{T}_n \hat{S}_n| dr$  by Cauchy-Schwartz:

$$\begin{aligned} \int_0^1 \frac{2}{r^2} |\tilde{T}_n \hat{S}_n| dr &\leq \int_0^1 2 \frac{|\tilde{T}_n|}{r} \frac{|\hat{S}_n|}{r^2} dr \\ &\leq \left( n(n+1) - \frac{3}{2} \right)^{-1/2} \left\{ \int_0^1 \frac{1}{r^2} \tilde{T}_n^2 dr + \left( n(n+1) - \frac{3}{2} \right) \int_0^\infty \frac{1}{r^4} \hat{S}_n^2 dr \right\}. \end{aligned}$$

Therefore,

$$C_{opt} \frac{\int_0^1 \frac{2}{r^2} |\tilde{T}_n \widehat{S}_n| dr}{\mathcal{B}_n[\tilde{T}_n, \widehat{S}_n]} \leq \frac{C_{opt}}{\sqrt{n(n+1) - 3/2}},$$

which proves (3.31).

**Appendix B.** We determine in this appendix the optimal constant (3.14) in the estimate  $\alpha_{\max} \leq C \|\alpha\|_s$  (Lemma 2) and give a counterexample to the estimate  $\alpha_{\max} \leq C \|\alpha\|_{\sim}$ .

LEMMA 2. For every  $f \in H^1((0, 1))$  the inequality

$$\max_{[0,1]} |f| \leq C_{opt} \left( \int_0^1 (f^2 + (f')^2) dx \right)^{1/2} \quad (\text{B1})$$

holds with

$$C_{opt} = \left( \frac{e + 1/e}{e - 1/e} \right)^{1/2}.$$

$C_{opt}$  is the minimal constant in (B1); it is attained by the function  $f_{\min}(x) = \cosh x / \cosh 1$ , which is unique up to a sign and up to a reflection at  $x = 1/2$ .

*Proof.* Observe that the right-hand side in (B1) is the standard norm in  $H^1((0, 1))$ , which implies by Sobolev's embedding theorems that  $f \in H^1((0, 1))$  is (Hölder-)continuous in  $[0, 1]$ ; the left-hand side in (B1) is thus well defined and boundary-values exist in the classical sense.  $C_{opt}$  is then given by

$$\frac{1}{C_{opt}} = \inf_{0 \neq f \in H^1((0,1))} \frac{\left( \int_0^1 (f^2 + (f')^2) dx \right)^{1/2}}{\max_{[0,1]} |f|}. \quad (\text{B2})$$

As the variational expression in (B2) is invariant under the transformations  $f \rightarrow -f$  and  $f(x) \rightarrow f(1-x)$ , it is sufficient to consider functions  $f \in H^1((0, 1))$  with

$$\max_{[0,1]} f = 1 \quad \text{and} \quad f(0) \leq f(1). \quad (\text{B3})$$

To determine the infimum it is, furthermore, sufficient to assume  $f \geq 0$  in  $(0, 1)$ , since otherwise the function  $\max(f, 0)$  has this property, is admissible and at most decreases the variational expression in (B2). Finally, it can be assumed that the maximum is attained at the boundary, which in view of (B3) means that  $f(1) = 1$ ; this assumption is justified since any function with  $f(x_0) = 1$ ,  $x_0 \notin \{0, 1\}$  can be replaced by

$$\tilde{f} := \begin{cases} \max(f(x+x_0) + f(0) - f(1), 0) & \text{for } 0 \leq x \leq 1 - x_0, \\ f(x - 1 + x_0) & \text{for } 1 - x_0 < x \leq 1, \end{cases}$$

which is admissible, i.e.,  $\tilde{f}$  is non-negative, satisfies (B3) and  $\tilde{f} \in H^1((0, 1))$ , and has, additionally, the property  $\tilde{f}(1) = 1$  and does not increase the variational expression in (B2). Problem (B2) is thus equivalent to the variational problem

$$\frac{1}{C_{opt}} = \inf_{\{f \in H^1((0,1)): f(1)=1\}} \left( \int_0^1 (f^2 + (f')^2) dx \right)^{1/2}, \quad (\text{B4})$$

which is of standard type. In particular, there exists a unique minimizer  $f_{\min} := \cosh x / \cosh 1$  of problem (B4), which is the unique solution of the Euler-Lagrange equations

$$f'' - f = 0 \quad \text{in } (0, 1), \quad f'(0) = 0, \quad f(1) = 1,$$

associated to (B4). Inserting  $f_{\min}$  into (B4) we find

$$\frac{1}{C_{opt}} = \left( \frac{e - 1/e}{e + 1/e} \right)^{1/2},$$

which is the asserted optimal constant. □

In contrast to (B1) there is no constant  $C > 0$ , such that

$$\max_{[0,1]} |f| \leq C \left( \int_0^1 (f^2 + (f')^2) x^2 dx \right)^{1/2}$$

holds for all weakly differentiable functions  $f : (0, 1) \rightarrow \mathbb{R}$ . This is evident from the sequence of functions  $(f_n)_{n \in \mathbb{N}}$  with

$$f_n := \begin{cases} 1 & \text{for } 0 \leq x \leq \frac{1}{n}, \\ \frac{1}{nx} & \text{for } \frac{1}{n} < x \leq 1, \end{cases}$$

for which one finds  $\max_{[0,1]} |f_n| = 1$  for all  $n \in \mathbb{N}$  but

$$\int_0^1 (f_n^2 + (f_n')^2) x^2 dx = \frac{1}{n} \left( 1 - \frac{2}{3n^2} \right),$$

which vanishes for  $n \rightarrow \infty$ . This is in accordance with well-known results about the embedding of the space of (Hölder-)continuous functions into  $H^1$ : it works in one dimension but not in three (see [Ada75, p. 97]).

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