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ON ELLIPTIC EQUATIONS WITH SINGULAR POTENTIALS AND NONLINEAR BOUNDARY CONDITIONS

By

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Abstract. We consider the Laplace equation in the half-space satisfying a nonlinear Neumann condition with boundary potential. This class of problems appears in a number of mathematical and physics contexts and is linked to fractional dissipation problems. Here the boundary potential and nonlinearity are singular and of power-type, respectively. Depending on the degree of singularity of potentials, first we show a nonexistence result of positive solutions in $\mathcal{D}^{1,2}(\mathbb{R}^n_+)$ with a L^p-type integrability condition on $\partial\mathbb{R}^n_+$. After, considering critical nonlinearities and conditions on the size and sign of potentials, we obtain the existence of positive solutions by means of minimization techniques and perturbation methods.

1. Introduction. In this paper we consider the problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial u}{\partial \nu} = a(x)\frac{u}{|x|^\sigma} + g(u) & \text{on } \partial\mathbb{R}^n_+ \setminus \{0\}, \end{cases} \tag{1.1}$$

where $n \geq 3$, $g(u)$ is a power-type nonlinearity, and $\sigma > 0$. We analyze two classes of the weight $a(x)$, namely a is a bounded function in $\mathbb{R}^n$ and the angular case $a(x) = a\left(\frac{x}{|x|}\right) \in C^1(S^{n-2}, \mathbb{R})$. In the latter case the function $a(\xi)$ ($\xi \in S^{n-2}$) is the angular part of the potential.

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The problem (1.1) consists in an elliptic equation in the half-space with nonlinear Neumann condition with a singular boundary potential. Elliptic problems with Neumann boundary conditions (both linear and nonlinear) have been studied by several authors and appear in various mathematical and physics contexts. For instance, harmonic functions with Neumann boundary condition were considered in [5]. A study on harmonic functions with nonlinear boundary conditions linked to boundary vortices in thin magnetic films can be found in [21, 22]. Lewy [24] considered certain nonlinear boundary conditions connected to the study of hydrodynamical fluids. For subjects related to the study of Peierls-Nabarro and Benjamin-Ono equations and the Steklov problem, the reader is referred to [29,30]. In turn, singular potentials appear in a number of physical situations, such as quantum cosmology, nonrelativistic quantum mechanics, and molecular physics (see, e.g., [20], [23], [17]).

When $\sigma = 1$ the problem is more challenging and we have some kind of criticality. For instance, for general $a\left(\frac{x}{|x|}\right) \not\equiv 0$, the potential does not belong to the associated Kato class for (1.1) and has the same homogeneity degree of the equation. This kind of potential plays a central role on the qualitative properties of this class of equations and the solutions may inherit its singular behavior.

The equation (1.1) can also be critical with respect to the growth of the nonlinear part $g(u)$, indeed there is a balance between σ and the rate of growth of g in the case of a homogeneous nonlinearity as can be seen from Theorem 2.1. Moreover, when $\sigma = 1$ and $g(u) = u^{\frac{n}{n-2}}$ the equation is invariant with respect to the Kelvin transform, more precisely, in this case if u is a solution of (1.1), then $v(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right)$ also yields a solution.

Another important feature of (1.1) is that it is related to the fractional Laplacian in the sense that it is formally equivalent to the equation

$$(-\Delta)^{\frac{1}{2}} u = a(x)\frac{u}{|x|} + u^{\frac{n}{n-2}}, \quad \mathbb{R}^n \backslash \{0\},$$

via the so-called Caffarelli-Silvestre extension (see [10]). Notice that now one can see clearly the equality between the degree of homogeneity of the potential and the order of the differential operator. Nonlinear elliptic equations with the fractional Laplacian have been the subject of intense research in the last years; see for example [9, 10, 26].

Existence, regularity, symmetry and continuation property of solutions for elliptic equations with singular potential and fractional Laplacian can be found in [11, 14, 15]. The authors of [7] considered fractional elliptic equations with $a(x) \equiv 0$ and concave-convex nonlinearities. For the Laplacian case and Hardy and multipolar anisotropic potentials in the whole space, we quote the works [16, 17] whose approaches rely on variational methods combined with Hardy-type inequalities. These issues also have been considered in [18, 19] via contraction arguments and suitable singular weighted spaces.

Here we deal with (1.1) in essentially two ways. First we consider the case $a(x) = a\left(\frac{x}{|x|}\right) \in C^1(S^{n-2}, \mathbb{R})$ and prove an existence result using minimization techniques. In the other case, when $a(x)$ is a bounded function, we consider the problem with a small parameter multiplying the function a and we apply perturbation methods. In this case it is important to know some properties of the unperturbed problem, mainly the solutions of the linearized equation (see Lemma 4.1). This characterization is important on its own, see for instance [13], and can be applied to other problems that are close to the unperturbed problem in a suitable sense.

The paper is structured as follows. In Section 2 we deduce some necessary conditions for the existence of nontrivial solutions. In Section 3 we prove an existence result using a minimization procedure and finally in Section 4 we prove an existence result by means of perturbation methods.

Notation.

$\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1},\ x_n > 0\}$;

$\partial\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1},\ x_n = 0\} \cong \mathbb{R}^{n-1}$;

$\overline{\mathbb{R}^n_+} = \mathbb{R}^n_+ \cup \partial\mathbb{R}^n_+$;

$B_R = \{x \in \mathbb{R}^n \mid |x| < R\}$;

$S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$;

$\mathcal{D}^{1,2}\left(\mathbb{R}^n_+\right)$ = the closure of $C^\infty_c(\overline{\mathbb{R}^n_+})$ in the norm $\|u\| = \left(\int_{\mathbb{R}^n_+} |\nabla u(x)|^2 dx\right)^{1/2}$;

$2_* = \frac{2(n-1)}{n-2}$.

2. Nonexistence results. In this section we prove the following result.

Theorem 2.1. Consider the equation (1.1) with

$$a(x) = a\left(\frac{x}{|x|}\right) \in C^1(S^{n-2}, \mathbb{R}).$$

(i) Suppose that $\sigma = 1$, $g(t) = t^p$, and that the problem (1.1) admits a nontrivial positive solution in $\mathcal{D}^{1,2}(\mathbb{R}^n_+) \cap L^{p+1}(\partial\mathbb{R}^n_+)$; then $p = \frac{n}{n-2}$.

(ii) Suppose that $\sigma \neq 1$, $g(t) = t^{\frac{n}{n-2}}$ and that the function $a(x)$ has a constant sign; then the problem (1.1) admits only the trivial solution in the space $\mathcal{D}^{1,2}(\mathbb{R}^n_+) \cap L^2(\partial\mathbb{R}^n_+, \frac{1}{|x|^\sigma})$.

Proof. The proof follows from a Pohozaev-type argument. Let us consider $0 < r < R$ and the domain $\Omega_{r,R} = B^+_R \setminus B^+_r$, where $B^+_R = B_R \cap \mathbb{R}^n_+$. Thus

$$\partial\Omega_{r,R} = S^+_R \cup S^+_r \cup A_{r,R},$$

where $S^+_R = \partial B_R \cap \mathbb{R}^n_+$, $S^+_r = \partial B_r \cap \mathbb{R}^n_+$, and $A_{r,R} = (B_R \setminus B_r) \cap \partial\mathbb{R}^n_+$.

Multiplying (1.1) by u and integrating over $\Omega_{r,R}$, it follows that

$$\int_{\Omega_{r,R}} |\nabla u|^2 = \int_{A_{r,R}} \left\{\frac{a(x)}{|x|^\sigma} u^2 + u^{p+1}\right\} + \int_{S^+_R \cup S^+_r} u\frac{\partial u}{\partial \nu}. \tag{2.1}$$

And if we multiply (1.1) by $\nabla u \cdot x$ and integrate over $\Omega_{r,R}$ we get

$$\int_{\Omega_{r,R}} \nabla u \cdot \nabla\left(\nabla u \cdot x\right) = \int_{\partial\Omega_{r,R}} \frac{\partial u}{\partial \nu} \nabla u \cdot x \tag{2.2}$$

$$-\frac{n-2}{2}\int_{\Omega_{r,R}} |\nabla u|^2 + \int_{\partial\Omega_{r,R}} \frac{1}{2}|\nabla u|^2 x\cdot\nu$$
$$= \int_{A_{r,R}} \left\{ \frac{a(x)}{|x|^\sigma} u\left(\nabla u\cdot x\right) + u^p\left(\nabla u \cdot x\right)\right\} + \int_{S_R^+\cup S_r^+} \frac{\partial u}{\partial \nu}\nabla u\cdot x \tag{2.3}$$

$$-\frac{n-2}{2}\int_{\Omega_{r,R}} |\nabla u|^2 + \int_{\partial\Omega_{r,R}} \frac{1}{2}|\nabla u|^2 x\cdot\nu = -\frac{n-1-\sigma}{2}\int_{A_{r,R}} \frac{a(x)}{|x|^\sigma}u^2 - \frac{n-1}{p+1}\int_{A_{r,R}} u^{p+1}$$
$$+\int_{\partial' A_{r,R}} \frac{1}{2}\frac{a(x)}{|x|^\sigma}u^2 x\cdot\nu' + \frac{u^{p+1}}{p+1}x\cdot\nu' + \int_{S_R^+\cup S_r^+}\frac{\partial u}{\partial\nu}\nabla u\cdot x, \tag{2.4}$$

where $\partial' A_{r,R}$ denotes the boundary of $A_{r,R}$ in $\mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$ and ν' denotes the unit outer normal vector field on $\partial' A_{r,R}$. By (2.1) and (2.4) we have

$$\left(\frac{n-2}{2} - \frac{n-1}{p+1}\right)\int_{A_{r,R}} u^{p+1} + \left(\frac{\sigma-1}{2}\right)\int_{A_{r,R}}\frac{a(x)}{|x|^\sigma}u^2 = -\frac{n-2}{2}\int_{S_R^+\cup S_r^+} u\frac{\partial u}{\partial \nu}$$
$$\int_{S_R^+\cup S_r^+}\left\{\frac{1}{2}|\nabla u|^2 x\cdot\nu - \frac{\partial u}{\partial\nu}\nabla u\cdot x\right\} - \int_{\partial' A_{r,R}}\left\{\frac{1}{2}\frac{a(x)}{|x|^\sigma}u^2 x\cdot\nu' + \frac{u^{p+1}}{p+1}x\cdot\nu'\right\}. \tag{2.5}$$

In both cases, (i) and (ii), we have

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 + |u|^{\frac{2n}{n-2}} < +\infty \quad \text{and} \quad \int_{\partial\mathbb{R}^n_+} \frac{a(x)}{|x|^\sigma}u^2 + u^{p+1} < +\infty$$

and hence the integrals on the right hand side of (2.5) go to zero (for suitable sequences $r_k \to 0$, $R_k \to +\infty$). Therefore

$$\left(\frac{n-2}{2} - \frac{n-1}{p+1}\right)\int_{\partial\mathbb{R}^n_+} u^{p+1} + \left(\frac{\sigma-1}{2}\right)\int_{\partial\mathbb{R}^n_+}\frac{a(x)}{|x|^\sigma}u^2 = 0$$

and the result follows. □

3. Existence results via minimization.

From now on we consider the problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial u}{\partial \nu} = a(\frac{x}{|x|})\frac{u}{|x|} + u^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+ \setminus \{0\}. \end{cases} \tag{3.1}$$

Consider the bilinear form

$$Q(u,v) = \int_{\mathbb{R}^n_+} \nabla u\cdot\nabla v - \int_{\mathbb{R}^{n-1}} \frac{a(x)}{|x|}uv, \qquad u,v\in\mathcal{D}^{1,2}(\mathbb{R}^n_+), \tag{3.2}$$

and the associated quadratic one

$$Q(u) := Q(u,u).$$

The existence of positive solutions to (3.1) is closely related to the sign of the following quantity:

$$\lambda(a) = \inf_{\substack{u\in\mathcal{D}^{1,2}(\mathbb{R}^n_+)\\ u(\cdot,0)\not\equiv 0}} \left\{ \frac{Q(u)}{\int_{\mathbb{R}^{n-1}} \frac{u^2}{|x|}} \right\}. \tag{3.3}$$

Our first existence result reads as follows.

THEOREM 3.1. Suppose that $\lambda(a) > 0$ and $\max_{S^{n-2}}\{a\} > 0$. Then

$$\inf_{\substack{u\in\mathcal{D}^{1,2}(\mathbb{R}^n_+)\\ u(\cdot,0)\not\equiv 0}} \left\{ \frac{Q(u)}{\|u(\cdot,0)\|^{\frac{2}{2_*}}_{L^{2_*}(\mathbb{R}^{n-1})}} \right\}$$

is achieved and therefore (3.1) has a positive solution.

To prove the above theorem we need some preliminary results. We shall argue as in [6] and [28].

By Kato's inequality one can see that $\lambda(a)$ is well defined. Next we prove that when $\lambda(a) > 0$ the quadratic form $Q(u)$ defines a norm in $\mathcal{D}^{1,2}(\mathbb{R}^n_+)$.

LEMMA 3.2. Suppose that $\lambda(a) > 0$; then $\sqrt{Q(u)}$ defines an equivalent norm in $\mathcal{D}^{1,2}(\mathbb{R}^n_+)$.

Proof. By Kato's inequality, it suffices to prove that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 \leqslant C\left(\int_{\mathbb{R}^n_+} |\nabla u|^2 - \int_{\mathbb{R}^{n-1}} \frac{a(x)}{|x|}u^2\right). \tag{3.4}$$

Let us suppose, by contradiction, that for every $k \in \mathbb{N}$ there is a $u_k \in \mathcal{D}^{1,2}(\mathbb{R}^n_+)$, with $\|u_k\|_{\mathcal{D}^{1,2}(\mathbb{R}^n_+)} = 1$, such that

$$1 > k\left(1 - \int_{\mathbb{R}^{n-1}} \frac{a(x)}{|x|}u_k^2\right). \tag{3.5}$$

From the definition of $\lambda(a)$ we have

$$\lambda(a)\int_{\mathbb{R}^{n-1}} \frac{u_k^2}{|x|} \leqslant 1 - \int_{\mathbb{R}^{n-1}} \frac{a(x)}{|x|}u_k^2. \tag{3.6}$$

Since $\lambda(a) > 0$, by (3.5) and (3.6) we get

$$\int_{\mathbb{R}^{n-1}} \frac{u_k^2}{|x|} < \frac{1}{\lambda(a)k}. \tag{3.7}$$

On the other hand, by (3.5) again, we have that

$$\frac{k-1}{k} < \int_{\mathbb{R}^{n-1}} \frac{a(x)}{|x|}u_k^2 \leqslant (1 + \|a\|_\infty)\int_{\mathbb{R}^{n-1}} \frac{u_k^2}{|x|}. \tag{3.8}$$

Therefore, by (3.7) and (3.8) we obtain a contradiction. $\square$

We shall solve (3.1) via a minimization problem, more precisely

$$S_a = \inf_{\substack{u\in\mathcal{D}^{1,2}(\mathbb{R}^n_+)\\ u(\cdot,0)\neq 0}} \left\{ \frac{Q(u)}{\|u(\cdot,0)\|^{\frac{2}{2_*}}_{L^{2_*}(\mathbb{R}^{n-1})}} \right\}. \tag{3.9}$$

PROPOSITION 3.3. Suppose that $\lambda(a) > 0$ and $S_a < S_0$ where S_0 is the best constant in the trace inequality. Then the infimum in (3.9) is achieved.

Proof. Since $\lambda(a) > 0$, by Lemma 3.2 we have $S_a > 0$. Let us consider a minimizing sequence u_k for (3.9) such that

$$\int_{\mathbb{R}^{n-1}} |u_k(x,0)|^{2_*}\, dx = 1. \tag{3.10}$$

Since the problem is invariant under the scaling $\mu^{\frac{n-2}{2}} u(\mu x)$, we can also assume that

$$\int_{|x|<2} |u_k(x,0)|^{2_*}\, dx = \frac{1}{2}. \tag{3.11}$$

By Lemma 3.2 u_k is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^n_+)$, hence we can assume that u_k converges weakly to some u_0 in $\mathcal{D}^{1,2}(\mathbb{R}^n_+)$ and that, for all $v \in \mathcal{D}^{1,2}(\mathbb{R}^n_+)$, we have

$$\int_{\mathbb{R}^n_+} \nabla u_k \cdot \nabla v - \int_{\mathbb{R}^{n-1}} \frac{a(x)}{|x|} u_k v = S_a \int_{\mathbb{R}^{n-1}} |u_k|^{2_*-1} v + o_k(1)\|v\|, \tag{3.12}$$

by Ekeland's variational principle.

By standard arguments, it suffices to show that $u_k \rightharpoonup u_0 \neq 0$ (see [28, Prop. 5.1]). So, suppose, by contradiction, that $u_k \rightharpoonup 0$. In this case we claim that

$$\int_{|x|<1} |u_k(x,0)|^{2_*}\, dx = o_k(1). \tag{3.13}$$

Let us assume (3.13) for a moment and consider $\phi \in C_c^\infty(\mathbb{R}^n\backslash\{0\})$ such that $\phi \equiv 1$ in $B_2\backslash B_1$. Notice that

$$\int_{\mathbb{R}^n_+} \nabla u_k \cdot \nabla(\phi^2 u_k) - \int_{\mathbb{R}^n_+} |\nabla(\phi u_k)|^2 = -\int_{\mathbb{R}^n_+} |\nabla\phi|^2 |u_k|^2 = o_k(1)$$

by the Rellich compactness theorem. Therefore, using (3.12) and Hölder's inequality, we get

$$\int_{\mathbb{R}^n_+} |\nabla(\phi u_k)|^2 - \int_{\mathbb{R}^{n-1}} \frac{a(x)}{|x|} |\phi u_k|^2 \leqslant S_a \left(\int_{\mathbb{R}^{n-1}} |\phi u_k|^{2_*} \right)^{\frac{2}{2_*}} + o_k(1). \tag{3.14}$$

On the other hand, recalling that $\phi \in C_c^\infty(\mathbb{R}^n\backslash\{0\})$ and using the Rellich compactness theorem once more, we have

$$\int_{\mathbb{R}^n_+} |\nabla(\phi u_k)|^2 - \int_{\mathbb{R}^{n-1}} \frac{a(x)}{|x|} |\phi u_k|^2 \geqslant S_0 \left(\int_{\mathbb{R}^{n-1}} |\phi u_k|^{2_*} \right)^{\frac{2}{2_*}} + o_k(1), \tag{3.15}$$

and hence

$$\begin{aligned} S_0 \left(\int_{\mathbb{R}^{n-1}} |\phi u_k|^{2_*} \right)^{\frac{2}{2_*}} &\leqslant S_a \left(\int_{\mathbb{R}^{n-1}} |\phi u_k|^{2_*} \right)^{\frac{2}{2_*}} + o_k(1) \\ (S_0 - S_a) \left(\int_{\mathbb{R}^{n-1}} |\phi u_k|^{2_*} \right)^{\frac{2}{2_*}} &= o_k(1) \\ \left(\int_{1<|x|<2} |u_k|^{2_*} \right)^{\frac{2}{2_*}} &= o_k(1), \end{aligned} \tag{3.16}$$

which contradicts (3.11) and (3.13).

It remains to prove (3.13). For that, consider $\eta \in C_c^\infty(B_2)$ such that $\eta \equiv 1$ in B_1 and apply (3.12) to $\eta^2 u_k$ in order to obtain

$$\begin{aligned} &\int_{\mathbb{R}^n_+} |\nabla(\eta u_k)|^2 - \int_{\mathbb{R}^{n-1}} \frac{a(x)}{|x|} |\eta u_k|^2 = S_a \int_{\mathbb{R}^{n-1}} |u_k|^{2_*-2} |\eta u_k|^2 + o_k(1) \\ &\leqslant S_a \left(\int_{B_2} |u_k|^{2_*} \right)^{\frac{2_*-2}{2_*}} \left(\int_{B_2} |\eta u_k|^{2_*} \right)^{\frac{2}{2_*}} + o_k(1) \\ &\leqslant \frac{S_a}{2^{\frac{2_*-2}{2_*}}} \left(\int_{\mathbb{R}^{n-1}} |\eta u_k|^{2_*} \right)^{\frac{2}{2_*}} + o_k(1). \end{aligned} \tag{3.17}$$

On the other hand, we have

$$S_a \left(\int_{\mathbb{R}^{n-1}} |\eta u_k|^{2_*} \right)^{\frac{2}{2_*}} \leqslant \int_{\mathbb{R}^n_+} |\nabla(\eta u_k)|^2 - \int_{\mathbb{R}^{n-1}} \frac{a(x)}{|x|} |\eta u_k|^2. \tag{3.18}$$

Therefore, by (3.17) and (3.18), it follows that

$$0 \leqslant S_a \left(1 - \frac{1}{2^{\frac{2_*-2}{2_*}}} \right) \left(\int_{\mathbb{R}^{n-1}} |\eta u_k|^{2_*} \right)^{\frac{2}{2_*}} \leqslant o_k(1)$$

and, since $\eta \equiv 1$ in B_1, (3.13) follows. The proof is now complete. $\square$

Now we prove Theorem 3.1.

Proof of Theorem 3.1. Since $\max_{S^{n-2}}\{a\} > 0$ there exists some $e_0 \in S^{n-2}$ such that $a(e_0) > 0$ and hence, for some $0 < r < 1$, $a_0 = \inf_{B_r(e_0)}\{a\} > 0$, where $B_r(e_0) \subset \mathbb{R}^{n-1}$. Then

$$\frac{a(x)}{|x|} \geqslant \frac{a_0}{1-r} = \mu \quad \text{for all } x \in B_r(e_0).$$

We can choose $r > 0$ small enough such that

$$\mu = \frac{a_0}{1-r} < \mu_r, \tag{3.19}$$

where μ_r is the square root of the first eigenvalue of $-\Delta$ in $B_r(e_0)$ with Dirichlet boundary condition. Now, using [26, Prop. 4.2], we can infer that

$$S(a) < S_0.$$

Then, by Proposition 3.3, the infimum in (3.9) is achieved and therefore (3.1) has a positive solution. $\square$

4. Existence results via perturbation methods. In this section we consider the problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial u}{\partial \nu} = \varepsilon \frac{a(x)}{|x|} u + u^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+ \setminus \{0\}, \end{cases} \tag{4.1}$$

where $\varepsilon \in \mathbb{R}$ is a small parameter and $a \in L^\infty(\mathbb{R}^{n-1})$. Here our aim is to prove the existence of positive solutions for (4.1) under some additional conditions on the function $a(x)$ using perturbation methods as in [2, 3, 12]; see also [4] for a nice exposition of the subject.

The unperturbed problem, i.e., with $\varepsilon = 0$, is

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial u}{\partial \nu} = u^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{4.2}$$

and has the following family of solutions:

$$U_{\mu,\xi}(x) = \frac{1}{\mu^{\frac{n-2}{2}}} U\left(\frac{x-\xi}{\mu}\right), \qquad \mu > 0 \text{ and } \xi \in \mathbb{R}^{n-1}, \tag{4.3}$$

where

$$U(x) = \left(\frac{n-2}{(1+x_n)^2 + |x'|^2}\right)^{\frac{n-2}{2}}. \tag{4.4}$$

It is known that all regular nonnegative solutions of (4.2) are of the form (4.3); see for example [25].

In order to find positive solutions to (4.1) we consider the functional

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^n_+} |\nabla u|^2 dx - \frac{1}{2_*}\int_{\partial\mathbb{R}^n_+} u_+^{2_*} dx - \varepsilon \frac{1}{2}\int_{\partial\mathbb{R}^n_+} \frac{a(x)}{|x|} u^2 dx \\ &= I_0(u) - \varepsilon \frac{1}{2} G(u). \end{aligned} \tag{4.5}$$

We have the following n-dimensional manifold of solutions to (4.2):

$$\mathcal{Z} = \{U_{\mu,\xi} \ : \ \mu > 0, \ \xi \in \mathbb{R}^{n-1}\}.$$

In order to apply the perturbation methods we need to prove that this manifold is non-degenerate in the following sense.

Lemma 4.1. For every $U_{\mu,\xi}$, the space of solutions of the linearized equation

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial v}{\partial \nu} = \frac{n}{n-2} U_{\mu,\xi}^{\frac{2}{n-2}} v & \text{on } \partial\mathbb{R}^n_+, \\ v \in \mathcal{D}^{1,2}(\mathbb{R}^n_+), \end{cases} \tag{4.6}$$

is spanned by the functions

$$\frac{\partial}{\partial \mu} U_{\mu,\xi}, \ \frac{\partial}{\partial \xi_i} U_{\mu,\xi}, \ \ i = 1, \dots, n-1.$$

Moreover, the operator $I_0''(U_{\mu,\xi})$ is Fredholm of index zero.

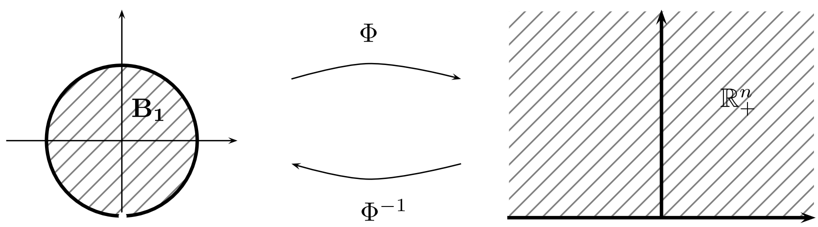
Proof. We proceed as in [1] (see also [13]). Without loss of generality, we prove the result for $\mu = \frac{1}{2}$ and $\xi = 0$. In this way the equation (4.6) becomes

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial v}{\partial \nu} = \dfrac{n}{2} \dfrac{1}{\frac{1}{4} + |x|^2} v & \text{on } \partial \mathbb{R}^n_+, \\ v \in \mathcal{D}^{1,2}(\mathbb{R}^n_+). \end{cases} \tag{4.7}$$

Now, as in [27], we use the conformal maps between $\mathbb{R}^n_+$ and the unit ball B_1, namely

$$\Phi : B_1 \to \mathbb{R}^n_+ \ ; \qquad x \mapsto \Phi(x) = \frac{x + \mathrm{e}_n}{|x + \mathrm{e}_n|^2} - \frac{1}{2}\mathrm{e}_n \tag{4.8}$$

$$\Phi^{-1} : \mathbb{R}^n_+ \to B_1 \ ; \qquad y \mapsto \Phi^{-1}(y) = \frac{y + \frac{1}{2}\mathrm{e}_n}{|y + \frac{1}{2}\mathrm{e}_n|^2} - \mathrm{e}_n. \tag{4.9}$$



These maps induce the following isometries:

$$\Phi^* : \mathcal{D}^{1,2}(\mathbb{R}^n_+) \to H^1(B_1), \qquad (\Phi^*)^{-1} : H^1(B_1) \to \mathcal{D}^{1,2}(\mathbb{R}^n_+), \tag{4.10}$$

$$v \mapsto \Phi^* v(x) = \frac{v(\Phi(x))}{|x + \mathrm{e}_n|^2}, \qquad u \mapsto (\Phi^*)^{-1} u(y) = \frac{u(\Phi^{-1}(y))}{|y + \frac{1}{2}\mathrm{e}_n|^2}, \tag{4.11}$$

where we consider $H^1(B_1)$ with the norm

$$\int_{B_1} |\nabla v|^2 + \frac{n-2}{2} \int_{\partial B_1} v^2 d\sigma.$$

Thus, if v is any solution of (4.7), then $u = \Phi^* v \in H^1(B_1)$ is a weak solution of

$$\begin{cases} \Delta u = 0 & \text{in } B_1, \\ \dfrac{\partial u}{\partial \nu} = u & \text{on } \partial B_1. \end{cases} \tag{4.12}$$

It is known that the space of solutions of (4.12) has dimension n and is spanned by the coordinate functions $u = y_i$ for $i = 1, \dots, n$. Therefore, we get that the space of solutions of (4.6) has dimension n and this finishes the first part of the proof.

Finally, to prove that the operator $I_0''(U_{\mu,\xi})$ is Fredholm of index zero, one can proceed as in [8, Lemma 2.3] and check that $I_0''(U_{\mu,\xi})$ is a compact perturbation of the identity. $\square$

Then we are in position to apply the following result.

Proposition 4.2 ([4]). Given a compact subset $K \subset \mathbb{R}_+ \times \mathbb{R}^{n-1}$, there exists $\varepsilon_0 > 0$ such that for all $(\mu, \xi) \in K$ and $0 < |\varepsilon| < \varepsilon_0$ there exists a unique $w_\varepsilon(\mu, \xi) \in T_{U_{\mu,\xi}}\mathcal{Z}$ such that

(i) $\|w_\varepsilon(\mu, \xi)\| = O(|\varepsilon|)$ as $\varepsilon \to 0$, uniformly with respect to $(\mu, \xi) \in K$;
(ii) $w_\varepsilon(\cdot, \cdot)$ is of class C^1.

The function $\Phi_\varepsilon : K \to \mathbb{R}$ defined by $\Phi_\varepsilon(\mu, \xi) = I_\varepsilon(U_{\mu,\xi} + w_\varepsilon(\mu, \xi))$ is of class C^1 and

$$\Phi'_\varepsilon(\mu^*, \xi^*) = 0 \quad \Longrightarrow \quad I'_\varepsilon\left(U_{\mu^*,\xi^*} + w_\varepsilon(\mu^*, \xi^*)\right) = 0.$$

Moreover, one has

$$\Phi_\varepsilon(\mu, \xi) = c_0 - \varepsilon \frac{1}{2} G(U_{\mu,\xi}) + o(\varepsilon), \tag{4.13}$$

where $c_0 = I_0(U_{\mu,\xi})$.

Therefore we need to study the finite dimensional functional

$$\Gamma(\mu, \xi) = G(U_{\mu,\xi}) = \int_{\mathbb{R}^{n-1}} \frac{a(x)}{|x|} U^2_{\mu,\xi}(x)dx. \tag{4.14}$$

Lemma 4.3. Suppose that $\frac{a(x)}{|x|} \in L^1(\mathbb{R}^{n-1}) \cap L^p(\mathbb{R}^{n-1})$ for some $p > n-1$; then $\lim_{\mu+|\xi|\to\infty} \Gamma(\mu, \xi) = 0$.

Proof. Let us first consider the case where $\mu \to 0$ and $|\xi| \to \infty$. Then we have

$$\begin{aligned} |\Gamma(\mu, \xi)| &\leqslant \frac{1}{\mu^{n-2}} \int_{\mathbb{R}^{n-1}} \left| \frac{a(x)}{|x|} \right| U^2 \left(\frac{x-\xi}{\mu} \right) dx \\ &\leqslant \frac{1}{\mu^{n-2}} \left(\int_{\mathbb{R}^{n-1}} \left| \frac{a(x)}{|x|} \right|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^{n-1}} U^{\frac{2p}{p-1}} \left(\frac{x-\xi}{\mu} \right) dx \right)^{\frac{p-1}{p}} \\ &\leqslant \mu^{\frac{(n-1)(p-1)}{p} - (n-2)} \left\| \frac{a(x)}{|x|} \right\|_{L^p(\mathbb{R}^{n-1})} \|U\|^2_{L^{\frac{2p}{p-1}}(\mathbb{R}^{n-1})} \longrightarrow 0 \end{aligned} \tag{4.15}$$

as $\mu \to 0$ because $p > n-1$. Now, if $\mu \to \mu_0 > 0$ and $|\xi| \to \infty$, the limit

$$\Gamma(\mu, \xi) = \frac{1}{\mu^{n-2}} \int_{\mathbb{R}^{n-1}} \frac{a(x)}{|x|} U^2 \left(\frac{x-\xi}{\mu} \right) dx \longrightarrow 0$$

is obtained from the Dominated Convergence Theorem, since $\frac{a(x)}{|x|} \in L^1(\mathbb{R}^{n-1})$.

Finally, if $\mu \to \infty$, then

$$|\Gamma(\mu, \xi)| \leqslant \frac{1}{\mu^{n-2}} \left\|U^2\right\|_{L^\infty(\mathbb{R}^{n-1})} \left\| \frac{a(x)}{|x|} \right\|_{L^1(\mathbb{R}^{n-1})} \longrightarrow 0. \tag{4.16}$$

□

Now we can prove an existence result for (4.1).

Theorem 4.4. Suppose that $a(x) \in L^\infty(\mathbb{R}^{n-1})$, $\frac{a(x)}{|x|} \in L^1(\mathbb{R}^{n-1}) \cap L^p(\mathbb{R}^{n-1})$ for some $p > n-1$ and that $\int_{\mathbb{R}^{n-1}} \frac{a(x)}{|x|} dx \neq 0$. Then, (4.1) possesses a positive solution for $|\varepsilon|$ small enough.

Proof. By (4.15) we can extend the function $\Gamma(\mu,\xi)$ continuously to $\{0\}\times\mathbb{R}^{n-1}$ with $\Gamma(0,\xi)\equiv 0$. Also, by Lemma 4.3 we already know that

$$\lim_{\mu+|\xi|\to\infty}\Gamma(\mu,\xi)=0.$$

Now, we fix $\xi=0$ and evaluate

$$\lim_{\mu\to\infty}\int_{\mathbb{R}^{n-1}}\frac{a(x)}{|x|}U^2\left(\frac{x}{\mu}\right)dx=U^2(0)\int_{\mathbb{R}^{n-1}}\frac{a(x)}{|x|}dx\neq 0$$

and hence $\lim_{\mu\to\infty}\mu^{n-2}\Gamma(\mu,0)\neq 0$, which implies that $\Gamma\not\equiv 0$.

Therefore Γ attains a maximum or a minimum at some (μ_0,ξ_0) with $\mu>0$ and there exists a neighborhood $\mathcal{N}\subset\mathbb{R}_+\times\mathbb{R}^{n-1}$ of (μ_0,ξ_0) such that

$$\max_{\partial\mathcal{N}}\Gamma<\Gamma(\mu_0,\xi_0)\qquad\text{or}\qquad\min_{\partial\mathcal{N}}\Gamma>\Gamma(\mu_0,\xi_0).\tag{4.17}$$

It follows from (4.13) and (4.17) that Φ_ε has a local maximum or a local minimum for $|\varepsilon|$ small and so, by Proposition 4.2, I_ε has a critical point of the form

$$u_\varepsilon=U_{\mu_\varepsilon,\xi_\varepsilon}+w_\varepsilon(\mu_\varepsilon,\xi_\varepsilon)\tag{4.18}$$

with $(\mu_\varepsilon,\xi_\varepsilon)$ closes to (μ_0,ξ_0).

It remains to prove that the solution u_ε is positive. Since $I'_\varepsilon(u_\varepsilon)=0$ we have

$$0=I'_\varepsilon(u_\varepsilon)u_\varepsilon^-=\|u_\varepsilon^-\|^2-\varepsilon\int_{\partial\mathbb{R}^n_+}\frac{a(x)}{|x|}|u_\varepsilon^-|^2dx.$$

Hence, using Kato's inequality, we get

$$\|u_\varepsilon^-\|^2\leqslant|\varepsilon|\|a\|_{L^\infty}C_n\|u_\varepsilon^-\|^2$$
$$(1-|\varepsilon|\|a\|_{L^\infty}C_n)\|u_\varepsilon^-\|^2\leqslant 0,$$

which implies that $u_\varepsilon^-\equiv 0$ for $|\varepsilon|$ small. Finally, by the maximum principle and Hopf's lemma, we arrive at

$$u_\varepsilon>0\quad\text{in}\quad\overline{\mathbb{R}^n_+}\setminus\{0\}.$$

□

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