QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXVI, NUMBER 4
DECEMBER 2018, PAGES 601–609
https://doi.org/10.1090/qam/1512
Article electronically published on June 26, 2018

WEIGHTED BECKMANN PROBLEM WITH BOUNDARY COSTS

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Abstract. We show that a solution to a variant of the Beckmann problem can be obtained by studying the limit of some weighted $p-$Laplacian problems. More precisely, we find a solution to the following minimization problem:

$$\min\left\{\int_\Omega k\,\mathrm{d}|w|+\int_{\partial\Omega} g^-\,\mathrm{d}\nu^- - \int_{\partial\Omega} g^+\,\mathrm{d}\nu^+ \;:\; w\in\mathcal{M}^d(\Omega),\, \nu\in\mathcal{M}(\partial\Omega),\, -\nabla\cdot w = f+\nu\right\},$$

where f, k, and $g^\pm$ are given. In addition, we connect this problem to a formulation with Kantorovich potentials with Dirichlet boundary conditions.

1. Introduction. In this paper we consider a variant of the flow-minimization problem introduced by Beckmann in 1952 [2] as a particular case of a wider class of a convex optimization problem, of the form $\min\{\int H(w)\,\mathrm{d}x \;:\; -\nabla\cdot w = f^+ - f^-\}$, for convex H. The case $H(z)=|z|$ is very interesting because of its equivalence with the Monge problem which deals with the optimal way of moving points from one mass distribution to another so that the total work done is minimized. In his work, the cost of moving one unit of mass from x to y is measured with the Euclidean distance $|x-y|$, even though many other cost functions were later studied.

Given two finite positive Borel measures f^+ and f^- on a compact convex domain $\Omega\subset\mathbb{R}^d$, satisfying the mass balance condition $f^+(\Omega)=f^-(\Omega)$, then, the classical Monge optimal transportation problem [12] is the following:

$$\inf\left\{\int_\Omega |x-T(x)|\,\mathrm{d}f^+ \;:\; T_\# f^+ = f^-\right\}, \tag{MP}$$

where $T_\# f^+ = f^- \Leftrightarrow f^-(A) = f^+(T^{-1}(A))$ for every Borel set $A\subset\Omega$. The existence of optimal maps was addressed by many authors [1], [5], [8], [14], and [17]. Although this problem may have no solutions, its relaxed setting (which is the Kantorovitch problem

Received September 15, 2017.
2010 *Mathematics Subject Classification.* Primary 35B65, 46N10, 49N60.
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[13]) always has one. The relaxed problem is the following:

$$\min\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma:\ \gamma\in\Pi(f^+,f^-)\right\}, \tag{KP}$$

where

$$\Pi(f^+,f^-):=\left\{\gamma\in\mathcal{M}^+(\Omega\times\Omega):\ (\Pi_x)_\#\gamma=f^+\ ,\ (\Pi_y)_\#\gamma=f^-\ \right\},$$

where Π_x and Π_y are the two projections of $\Omega\times\Omega$ onto Ω. The authors of [15,16] prove that the dual of (KP) is the following:

$$\max\left\{\int_\Omega u\,\mathrm{d}(f^+-f^-)\ :\ u\in\mathrm{Lip}_1(\Omega)\right\}. \tag{DP}$$

The equality of the two optimal values implies that optimal γ and u satisfy $u(x)-u(y)=|x-y|$ on the support of γ, which means that the potential u decreases at the rate one as we move along the transport ray $[x,y]$ (note that the gradient of u gives the direction of these transport rays). It is well known that there exists a non-negative Borel measure σ over Ω (which is called *transport density*) such that (σ,u) solves a particular PDE system, called the Monge-Kantorovitch system [15]:

$$\begin{cases}-\nabla\cdot(\sigma\nabla u)=f:=f^+-f^- & \text{in }\ \Omega,\\ \sigma\nabla u\cdot n=0 & \text{on }\ \partial\Omega,\\ |\nabla u|\le 1 & \text{in }\ \Omega,\\ |\nabla u|=1 & \sigma-\text{a.e.}\end{cases} \tag{1.1}$$

This measure σ represents the amount of transport taking place in each region of Ω, i.e., for a given Borel set A, $\sigma(A)$ stands for "how much" the transport takes place in A if particles move from their origin x to their destination y on transport rays.

In addition, the flow $w:=\sigma\nabla u$ solves the Beckmann problem (see [15]), which is the following:

$$\min\left\{\int_\Omega \mathrm{d}|w|\ :\ w\in\mathcal{M}^d(\Omega),\ -\nabla\cdot w=f^+-f^-\right\} \tag{BP}$$

and we have the following equalities:

$$\min\,(\mathrm{BP})=\sup\,(\mathrm{DP})=\min\,(\mathrm{KP}).$$

An interesting variant of (KP), which is already present in [6,7,11], is to transport the mass f^+ to another one f^- (which do not have a priori the same total mass) with the possibility of transporting some mass to/from the boundary, paying the transport cost that is assumed to be given by the Euclidean distance $|x-y|$ plus an extra cost $g^-(y)$ for each unit of mass that comes out from a point $y\in\partial\Omega$ or $-g^+(x)$ for each unit of mass that enters at the point $x\in\partial\Omega$. Yet, it is reasonable to consider a distance d_k associated with a Riemannian metric k (where k is supposed to be positive and continuous), instead of the Euclidean distance, when we want to model a non-uniform cost for the movement (due to geographical obstacles or configurations). Recall that this distance d_k is defined as follows:

$$d_k(x,y):=\inf\left\{\int_0^1 k(\omega(t))|\omega'(t)|\,\mathrm{d}t\ :\ \omega\in\mathrm{Lip}([0,1],\Omega),\ \omega(0)=x,\ \omega(1)=y\right\}\ \forall\,x,\,y\in\Omega.$$

Then, set

$$\Pi b(f^+,f^-) := \left\{\gamma \in \mathcal{M}^+(\Omega\times\Omega) \, : \, ((\Pi_x)_\#\gamma)_{|\overset{\circ}{\Omega}} = f^+, \, ((\Pi_y)_\#\gamma)_{|\overset{\circ}{\Omega}} = f^-\right\}.$$

We minimize the quantity

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\gamma + \int_{\partial\Omega} g^-\,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g^+\,\mathrm{d}(\Pi_x)_\#\gamma \; : \; \gamma\in\Pi b(f^+,f^-)\right\}. \tag{KPb}$$

First of all, we assume that $g^\pm \in C(\partial\Omega)$ with

$$g^+(x) - g^-(y) < d_k(x,y), \text{ for all } x,\, y\in\partial\Omega. \tag{1.2}$$

Then, from [6], we can prove that (KPb) reaches a minimum and that its dual is the following:

$$\max\left\{\int_\Omega u\,\mathrm{d}(f^+-f^-) \, : \, |\nabla u|\le k,\, g^+\le u\le g^- \text{ on } \partial\Omega\right\}. \tag{DPb}$$

Note that, for this optimal transportation problem with boundary costs, the system (1.1) becomes

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f & \text{in } \Omega,\\ g^+\le u\le g^- & \text{on } \partial\Omega,\\ |\nabla u|\le k & \text{in } \Omega,\\ |\nabla u| = k & \sigma-\text{a.e.}, \end{cases} \tag{1.3}$$

and, the problem (BP) becomes

$$\min\left\{\int_\Omega k\,\mathrm{d}|w| + \int_{\partial\Omega} g^-\,\mathrm{d}\nu^- - \int_{\partial\Omega} g^+\,\mathrm{d}\nu^+ \; : \; (w,\nu) \in \mathcal{M}^d(\Omega)\times\mathcal{M}(\partial\Omega),\, -\nabla\cdot w = f+\nu\right\}. \tag{BPb}$$

In [8], the authors prove that a solution to (1.1) (or equivalently, to (BP)) can be constructed by studying the $p-$Laplacian equation

$$-\nabla\cdot(|\nabla u_p|^{p-2}\nabla u_p) = f$$

in the limit as $p\to\infty$. In this paper, we prove that a solution to (1.3) (or equivalently, to (BPb)) can be constructed by studying the limit as $p\to\infty$ of the following weighted $p-$Laplacian problem:

$$\begin{cases} -\nabla\cdot(k^{-p}|\nabla u_p|^{p-2}\nabla u_p) = f & \text{in } \Omega,\\ \frac{\partial u_p}{\partial \mathbf{n}} = 0 & \text{on } \{g^+ < u_p < g^-\},\\ \frac{\partial u_p}{\partial \mathbf{n}} \ge 0 & \text{on } \{u_p = g^+\},\\ \frac{\partial u_p}{\partial \mathbf{n}} \le 0 & \text{on } \{u_p = g^-\},\\ g^+\le u_p\le g^- & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Using this approach, We also prove the following:

$$\min(\text{BPb}) = \sup(\text{DPb}) = \min(\text{KPb}).$$

This paper is organized as follows. In Section 2, we introduce the weighted $p-$Laplacian problems that we use to approximate a maximizer of (DPb). In Section 3, we prove the existence of a solution to (1.3). This means that we want to find a non-negative Borel measure σ such that (σ, u_∞) solves (1.3). Finally, in Section 4, we prove that min(BPb)= sup(DPb) and we find a minimizer to (BPb).

2. Uniform estimates on the solutions of the weighted $p-$Laplacian problems. In this section, the aim is to obtain estimates, independent of p, on the solution of (1.4). First of all, we note that the unique (may be up to a constant) weak solution u_p of (1.4) is found as the minimizer of the functional

$$\mathcal{J}_p(u) := \frac{1}{p}\int_\Omega k^{-p}|\nabla u|^p \,\mathrm{d}x - \int_\Omega uf \,\mathrm{d}x$$

over all $u \in W^{1,p}(\Omega)$, $g^+ \le u \le g^-$ on $\partial\Omega$. Under the assumption (1.2), we have the following.

Proposition 2.1. Let u_p be the solution of (1.4). Then, up to a subsequence, $u_p \to u_\infty$ uniformly as $p \to \infty$, where u_∞ solves (DPb).

Proof. Set

$$v(y) := \max_{x\in\partial\Omega}\{g^+(x) - d_k(x,y)\} \text{ for all } y \in \Omega.$$

Then, it is easy to see that v is Lip_1 according to the distance d_k and then, $|\nabla v| \le k$. In addition, (1.2) gives that

$$g^+ \le v < g^- \text{ on } \partial\Omega.$$

From the optimality of u_p, we have

$$\mathcal{J}_p(u_p) \le \mathcal{J}_p(v) \le \frac{|\Omega|}{p} + C,$$

where C is a constant independent of p. As

$$g^+ \le u_p \le g^- \text{ on } \partial\Omega,$$

then, it is easy to check that

$$||u_p||_{L^\infty(\Omega)} \le C(d, \text{diam}(\Omega))\, ||\nabla u_p||_{L^p(\Omega,\mathbb{R}^d)} + ||g||_{L^\infty(\partial\Omega)}.$$

Hence,

$$\begin{aligned}\int_\Omega k^{-p}|\nabla u_p|^p \,\mathrm{d}x &\le p\int_\Omega u_p f \,\mathrm{d}x + Cp \\ &\le Cp\Big(\int_\Omega k^{-p}|\nabla u_p|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} + Cp.\end{aligned}$$

Yet, this implies that

$$\Big(\int_\Omega k^{-p}|\nabla u_p|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \le (Cp)^{\frac{1}{p}}$$

and then, for $m < p$,

$$\left(\int_\Omega k^{-m}|\nabla u_p|^m \,\mathrm{d}x\right)^{\frac{1}{m}} \le (Cp)^{\frac{1}{p}} \, |\Omega|^{\frac{1}{m}-\frac{1}{p}}.$$

Hence, up to a subsequence, $u_p \rightharpoonup u_\infty$ in $W^{1,m}(\Omega)$ for all $m \in \mathbb{N}^\star$, and then, $u_p \to u_\infty$ uniformly in Ω. In addition, we have

$$\left(\int_\Omega k^{-m}|\nabla u_\infty|^m \,\mathrm{d}x\right)^{\frac{1}{m}} \le |\Omega|^{\frac{1}{m}} \text{ for all } m \in \mathbb{N}^\star$$

and then,

$$|\nabla u_\infty| \le k.$$

On the other hand, for any admissible function φ in (DPb), we have, from the optimality of u_p, that

$$-\int_\Omega u_p f \,\mathrm{d}x \le \mathcal{J}_p(u_p) \le \mathcal{J}_p(\varphi) \le \frac{|\Omega|}{p} - \int_\Omega \varphi f \,\mathrm{d}x.$$

When $p \to \infty$, we infer that u_∞ solves (DPb). □

3. The limit of the weighted $p-$Laplacian problems. For all $p > d$, set

$$w_p := k^{-p} \, |\nabla u_p|^{p-2} \, \nabla u_p,$$

where u_p is the solution of (1.4). So, the aim of this section is to study the limit as $p \to \infty$ of $(w_p)_p$. In particular, we show that $w_p \rightharpoonup w$ in the sense of measures and that (σ, u_∞) solves (1.3) with $\sigma := k^{-1} \, |w|$.

LEMMA 3.1. For all $p > d$, there exists a measure ν_p, which is concentrated on the boundary of Ω, such that

$$\int_\Omega w_p \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega \varphi f \,\mathrm{d}x + \int_{\partial\Omega} \varphi \,\mathrm{d}\nu_p \text{ for every } \varphi \in W^{1,p}(\Omega).$$

In addition, we have

$$\operatorname{spt} \nu_p^\pm \subset \{u_p = g^\pm\}.$$

Proof. Take $\varphi \in C^\infty(\Omega)$ with

$$\operatorname{spt}(\varphi) \cap \{u_p = g^\pm\} = \emptyset.$$

As $u_p \in C(\Omega)$, then there exists $\varepsilon_0 > 0$ such that $g^+ \le u_p + \varepsilon\varphi \le g^-$ on $\partial\Omega$ for all $|\varepsilon| < \varepsilon_0$. Yet, from the optimality of u_p, we have

$$\mathcal{J}_p(u_p) \le \mathcal{J}_p(u_p + \varepsilon\varphi)$$

and when $\varepsilon \to 0$, we get

$$\int_\Omega w_p \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega \varphi f \,\mathrm{d}x.$$

Let $\varphi^{\pm} \geq 0$ be in $C^\infty(\Omega)$ with

$$\operatorname{spt}(\varphi^+) \cap \{u_p = g^-\} = \emptyset$$

and

$$\operatorname{spt}(\varphi^-) \cap \{u_p = g^+\} = \emptyset.$$

Working as above, we get

$$\int_\Omega w_p \cdot \nabla\varphi^+ \,\mathrm{d}x \,\geq\, \int_\Omega \varphi^+ f \,\mathrm{d}x$$

and

$$\int_\Omega w_p \cdot \nabla\varphi^- \,\mathrm{d}x \,\leq\, \int_\Omega \varphi^- f \,\mathrm{d}x.$$

Now, it is easy to conclude the proof. □

Under the assumption (1.2), we have the following.

Proposition 3.2. $w_p \rightharpoonup w$ and $\nu_p \rightharpoonup \nu$ in the sense of measures.

Proof. Set

$$v^+(x) := \min_{y\in\partial\Omega}\{g^-(y) + d_k(x,y)\} \text{ for all } x \in \Omega.$$

Then, it is clear that $g^+ < v^+ \leq g^-$ on $\partial\Omega$ and $|\nabla v^+| \leq k$. In addition, we have the following equality:

$$\int_\Omega w_p \cdot \nabla(u_p - v^+)\,\mathrm{d}x = \int_\Omega (u_p - v^+) f \,\mathrm{d}x \,+ \int_{\partial\Omega} (u_p - v^+)\,\mathrm{d}\nu_p.$$

Hence,

$$\begin{aligned}\int_{\partial\Omega}(v^+ - u_p)\,\mathrm{d}\nu_p + \int_\Omega k^{-p}|\nabla u_p|^p\,\mathrm{d}x \quad &= \quad \int_\Omega w_p\cdot\nabla v^+\,\mathrm{d}x + \int_\Omega (u_p - v^+) f\,\mathrm{d}x \\ &\leq \quad \int_\Omega w_p\cdot\nabla v^+\,\mathrm{d}x \,+\, C,\end{aligned}$$

where C is a constant independent of p. As $v^+ - g^+ \geq c > 0$ on $\partial\Omega$, then, by Lemma 3.1, we get

$$\begin{aligned}c\int_{\partial\Omega}\mathrm{d}\nu_p^+ + \int_\Omega k^{-p}|\nabla u_p|^p\,\mathrm{d}x \quad &\leq \quad \int_\Omega k^{-(p-1)}|\nabla u_p|^{p-2}\nabla u_p\cdot k^{-1}\nabla v^+\,\mathrm{d}x \,+\, C \\ &\leq \quad |\Omega|^{\frac{1}{p}}\left(\int_\Omega k^{-p}|\nabla u_p|^p\,\mathrm{d}x\right)^{1-\frac{1}{p}} \,+\, C \\ &\leq \quad \left(1 - \frac{1}{p}\right)\int_\Omega k^{-p}|\nabla u_p|^p\,\mathrm{d}x \,+\, C.\end{aligned}$$

Finally, we infer that

$$c\int_{\partial\Omega}\mathrm{d}\nu_p^+ \,+\, \frac{1}{p}\int_\Omega k^{-p}|\nabla u_p|^p\,\mathrm{d}x \leq C.$$

Therefore,

$$\int_{\partial\Omega}\mathrm{d}\nu_p^+ \leq C.$$

Replacing v^+ by v^-, where

$$v^-(y) := \max_{x\in\partial\Omega}\{g^+(x) - d_k(x,y)\} \text{ for all } y \in \Omega,$$

we also get that

$$\int_{\partial\Omega} \mathrm{d}\nu_p^- \leq C.$$

Yet, we have

$$\int_\Omega k^{-p}|\nabla u_p|^p \,\mathrm{d}x = \int_\Omega u_p f \,\mathrm{d}x + \int_{\partial\Omega} u_p \,\mathrm{d}\nu_p.$$

Hence, the sequence $(w_p)_p$ (resp., $(\nu_p)_p$) is bounded in $\mathcal{M}^d(\Omega)$ (resp., $\mathcal{M}(\partial\Omega)$) and so there exists a vector measure w (resp., a measure ν supported on $\partial\Omega$) such that $w_p \rightharpoonup w$ (resp. $\nu_p \rightharpoonup \nu$) in the sense of measures. □

We conclude this section by proving the existence of a solution to (1.3).

Proposition 3.3. There exists a non-negative Borel measure σ over Ω such that (σ, u_∞), where u_∞ is a maximizer for (DPb), is a solution to (1.3).

Proof. By Lemma 3.1 and Proposition 3.2, for all $\varphi \in C^1(\Omega)$, we have

$$\int_\Omega \nabla\varphi \cdot \mathrm{d}w = \int_\Omega \varphi f \,\mathrm{d}x + \int_{\partial\Omega} \varphi \,\mathrm{d}\nu. \tag{3.1}$$

Set

$$\sigma := k^{-1}|w|.$$

Now, consider a sequence $(\varphi_n)_n \subset C^\infty(\Omega)$ such that $\varphi_n \to u_\infty$ uniformly and $\nabla\varphi_n \to \nabla_\sigma u_\infty$ in $L^2_\sigma(\Omega, \mathbb{R}^d)$, where ∇_σ is the tangential gradient operator with respect to σ defined in [4].

By (3.1), we get

$$\begin{aligned}\int_\Omega \nabla_\sigma u_\infty \cdot \mathrm{d}w &= \int_\Omega u_\infty f \,\mathrm{d}x + \int_{\partial\Omega} u_\infty \,\mathrm{d}\nu \\ &= \int_\Omega u_\infty f \,\mathrm{d}x + \int_{\partial\Omega} g^+ \,\mathrm{d}\nu^+ - \int_{\partial\Omega} g^- \,\mathrm{d}\nu^-.\end{aligned}$$

Yet,

$$\begin{aligned}\int_\Omega k \,\mathrm{d}|w| &\leq \liminf_p \int_\Omega k\,|w_p| \,\mathrm{d}x \\ &= \liminf_p \int_\Omega k^{-(p-1)}|\nabla u_p|^{p-1} \,\mathrm{d}x \\ &\leq \liminf_p |\Omega|^{\frac{1}{p}} \left(\int_\Omega k^{-p}|\nabla u_p|^p \,\mathrm{d}x\right)^{1-\frac{1}{p}}.\end{aligned}$$

In addition, we have

$$\begin{aligned}\int_\Omega k^{-p}|\nabla u_p|^p\,\mathrm{d}x &= \int_\Omega u_p f\,\mathrm{d}x + \int_{\partial\Omega} u_p\,\mathrm{d}\nu_p\\ &= \int_\Omega u_p f\,\mathrm{d}x + \int_{\partial\Omega} g^+\,\mathrm{d}\nu_p^+ - \int_{\partial\Omega} g^-\,\mathrm{d}\nu_p^-\\ &\to \int_\Omega u_\infty f\,\mathrm{d}x \,+\, \int_{\partial\Omega} g^+\,\mathrm{d}\nu^+ \,-\, \int_{\partial\Omega} g^-\,\mathrm{d}\nu^- = \int_\Omega \nabla_\sigma u_\infty\cdot\mathrm{d}w.\end{aligned}$$

Finally, we get

$$\int_\Omega k\,\mathrm{d}|w| \le \int_\Omega \nabla_\sigma u_\infty\cdot\mathrm{d}w.$$

Since $|\nabla u_\infty|\le k$, hence

$$\int_\Omega \nabla_\sigma u_\infty\cdot\mathrm{d}w = \int_\Omega k\,\mathrm{d}|w|$$

and

$$w = \sigma\,\nabla_\sigma u_\infty\,,\ |\nabla_\sigma u_\infty| = k \quad \sigma-\text{a.e.}$$

□

4. Producing a solution to a variant of the Beckmann problem.

Now, we are ready to find a solution to (BPb). Let w (resp., ν) be the limit of $(w_p)_p$ (resp., $(\nu_p)_p$) as in Proposition 3.2. Then, we have the following.

PROPOSITION 4.1. (w,ν) solves the problem (BPb). Moreover, the minimal value of (BPb) equals the maximal value of (DPb).

Proof. We start from min(BPb) $\ge$ sup(DPb). In order to do so, take an arbitrary function $\varphi\in C^1(\Omega)$ with $|\nabla\varphi|\le k$ and $g^+\le\varphi\le g^-$ on $\partial\Omega$. Consider that for any $(v,\chi)\in\mathcal{M}^d(\Omega)\times\mathcal{M}(\partial\Omega)$ with $-\nabla\cdot v = f+\chi$, we have

$$\int_\Omega k\,\mathrm{d}|v| \ge \int_\Omega \nabla\varphi\cdot\mathrm{d}v = \int_\Omega \varphi\,\mathrm{d}(f+\chi) \ge \int_\Omega \varphi f\,\mathrm{d}x + \int_{\partial\Omega} g^+\,\mathrm{d}\chi^+ - \int_{\partial\Omega} g^-\,\mathrm{d}\chi^-.$$

By an approximation argument, we can infer that

$$\int_\Omega k\,\mathrm{d}|v| + \int_{\partial\Omega} g^-\,\mathrm{d}\chi^- - \int_{\partial\Omega} g^+\,\mathrm{d}\chi^+ \ge \sup(\mathrm{DPb}) = \min(\mathrm{KPb})$$

for any admissible (v,χ), i.e., min(BPb) $\ge$ sup(DPb). Yet, by Proposition 3.3, we have

$$\int_\Omega k\,\mathrm{d}|w| + \int_{\partial\Omega} g^-\,\mathrm{d}\nu^- - \int_{\partial\Omega} g^+\,\mathrm{d}\nu^+ = \int_\Omega u_\infty f\,\mathrm{d}x.$$

Hence, (w,ν) solves (BPb) and we have min(BPb) = sup(DPb) = min(KPb). □

REMARK 4.1. Note that, from [10], we have $\sigma\in L^1$ as soon as $f\in L^1$ and $k\in C^{1,1}$, and then, the following problem:

$$\min\left\{\int_\Omega k|w|\,\mathrm{d}x + \int_{\partial\Omega} g^-\,\mathrm{d}\nu^- - \int_{\partial\Omega} g^+\,\mathrm{d}\nu^+ \;:\; w\in L^1(\Omega,\mathbb{R}^d),\ \nu\in\mathcal{M}(\partial\Omega),\ -\nabla\cdot w = f+\nu\right\}$$

reaches a minimum.

Acknowledgment. The author would like to thank the FSMP for the financial support of the publication of this article.

References

[1] L. Ambrosio, *Lecture notes on optimal transport problems*, Mathematical aspects of evolving interfaces (Funchal, 2000), Lecture Notes in Math., vol. 1812, Springer, Berlin, 2003, pp. 1–52, DOI 10.1007/978-3-540-39189-0_1. MR2011032

[2] M. Beckmann, *A continuous model of transportation*, Econometrica **20** (1952), 643–660, DOI 10.2307/1907646. MR0068196

[3] S. Bianchini and F. Cavalletti, *The Monge problem for distance cost in geodesic spaces*, Comm. Math. Phys. **318** (2013), no. 3, 615–673, DOI 10.1007/s00220-013-1663-8. MR3027581

[4] G. Bouchitté and G. Buttazzo, *Characterization of optimal shapes and masses through Monge-Kantorovich equation*, J. Eur. Math. Soc. (JEMS) **3** (2001), no. 2, 139–168, DOI 10.1007/s100970000027. MR1831873

[5] L. A. Caffarelli, M. Feldman, and R. J. McCann, *Constructing optimal maps for Monge's transport problem as a limit of strictly convex costs*, J. Amer. Math. Soc. **15** (2002), no. 1, 1–26, DOI 10.1090/S0894-0347-01-00376-9. MR1862796

[6] S. Dweik, *Optimal transportation with boundary costs and summability estimates on the transport density*, J. Convex Anal. **25** (2018), no. 1, 135–160. MR3756930

[7] S. Dweik and F. Santambrogio, *Summability estimates on transport densities with Dirichlet regions on the boundary via symmetrization techniques,* preprint arXiv:1606.00705, 2016.

[8] L. C. Evans and W. Gangbo, *Differential equations methods for the Monge-Kantorovich mass transfer problem*, Mem. Amer. Math. Soc. **137** (1999), no. 653, viii+66, DOI 10.1090/memo/0653. MR1464149

[9] M. Feldman and R. J. McCann, *Uniqueness and transport density in Monge's mass transportation problem*, Calc. Var. Partial Differential Equations **15** (2002), no. 1, 81–113, DOI 10.1007/s005260100119. MR1920716

[10] M. Feldman and R. J. McCann, *Monge's transport problem on a Riemannian manifold*, Trans. Amer. Math. Soc. **354** (2002), no. 4, 1667–1697, DOI 10.1090/S0002-9947-01-02930-0. MR1873023

[11] J. M. Mazón, J. D. Rossi, and J. Toledo, *An optimal transportation problem with a cost given by the Euclidean distance plus import/export taxes on the boundary*, Rev. Mat. Iberoam. **30** (2014), no. 1, 277–308, DOI 10.4171/RMI/778. MR3186940

[12] G. Monge, *Mémoire sur la théorie des déblais et des remblais,* Histoire de l'Académie Royale des Sciences de Paris (1781), 666-704.

[13] L. Kantorovitch, *On the translocation of masses*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **37** (1942), 199–201. MR0009619

[14] V. N. Sudakov, *Geometric problems in the theory of infinite-dimensional probability distributions*, Proc. Steklov Inst. Math. **2** (1979), i–v, 1–178. Cover to cover translation of Trudy Mat. Inst. Steklov **141** (1976). MR530375

[15] F. Santambrogio, *Optimal transport for applied mathematicians*, Progress in Nonlinear Differential Equations and their Applications, vol. 87, Birkhäuser/Springer, Cham, 2015. Calculus of variations, PDEs, and modeling. MR3409718

[16] C. Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, 2003. MR1964483

[17] N. S. Trudinger and X.-J. Wang, *On the Monge mass transfer problem*, Calc. Var. Partial Differential Equations **13** (2001), no. 1, 19–31, DOI 10.1007/PL00009922. MR1854255