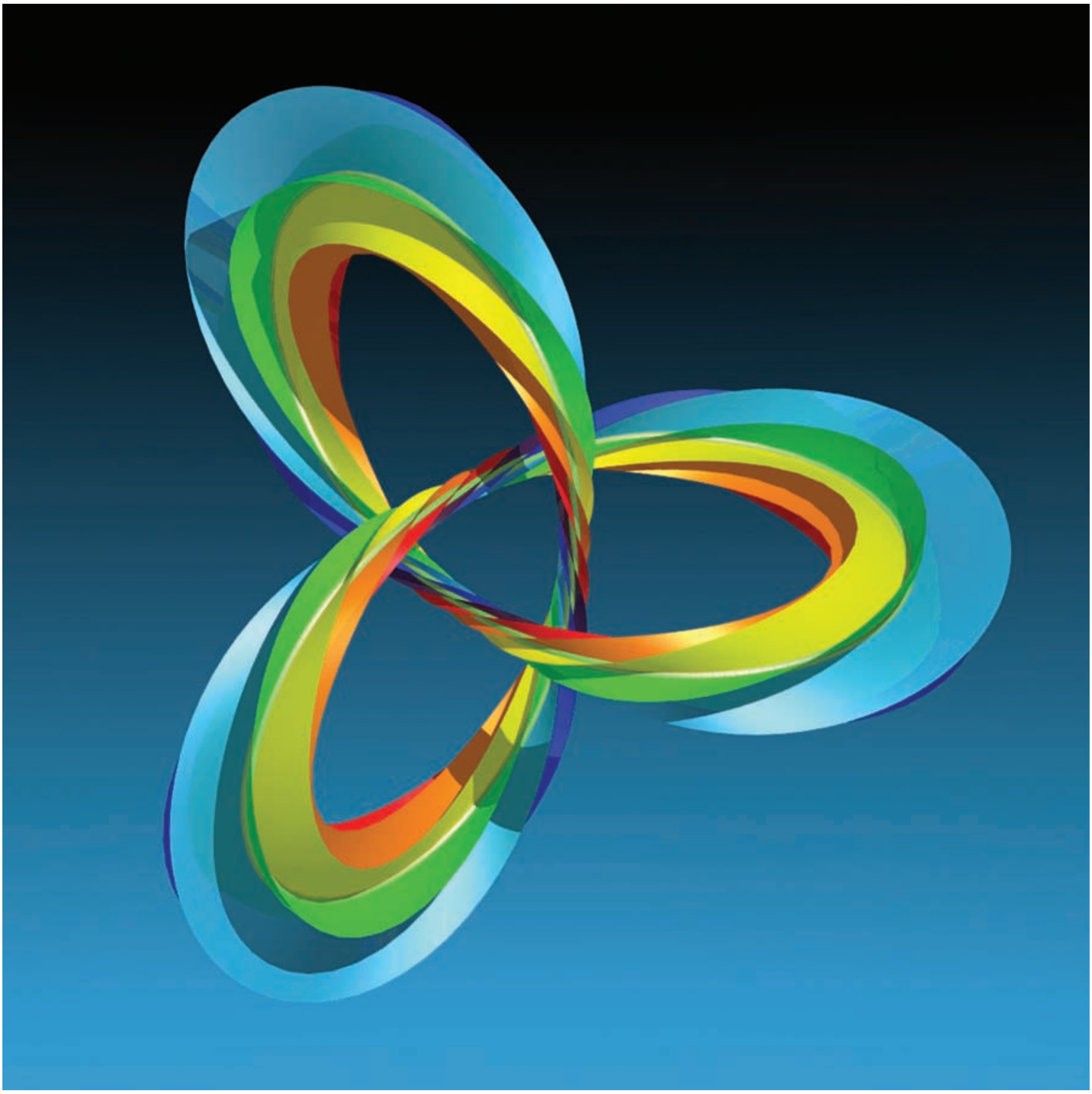


Fibered Knot. *If a knot is fibered, then a "fan" of surfaces can be defined, each one anchored to the knot and collectively filling out all of space. All Lorenz knots are fibered. (Figure courtesy of Jos Leys from the online article, "Lorenz and Modular Flows, a Visual Introduction.")*

A New Twist in Knot Theory

Étienne Ghys. *(Photo courtesy of Étienne Ghys.)*

WHETHER YOUR TASTE RUNS to spy novels or Shakespearean plays, you have probably run into the motif of the double identity. Two characters who seem quite different, like Dr. Jekyll and Mr. Hyde, will turn out to be one and the same.

This same kind of "plot twist" seems to work pretty well in mathematics, too. In 2006, Étienne Ghys of the École Normale Supérieure de Lyon revealed a spectacular case of double identity in the subject of knot theory. Ghys showed that two different kinds of knots, which arise in completely separate branches of mathematics and seem to have nothing to do with one another, are actually identical. Every *modular knot* (a curve that is important in number theory) is topologically equivalent to a *Lorenz knot* (a curve that arises in dynamical systems), and vice versa.

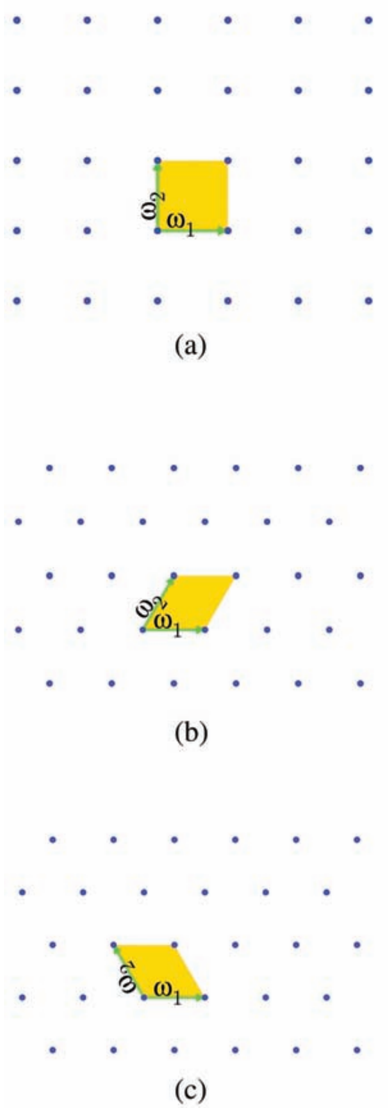
The discovery brings together two fields of mathematics that have previously had almost nothing in common, and promises to benefit both of them.

The terminology "modular" refers to a classical and ubiquitous structure in mathematics, the *modular group*. This group consists of all 2×2 matrices, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, whose entries are all integers and whose determinant $(ad - bc)$ equals 1. Thus, for instance, the matrix $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$ is an element of the modular group. (See "Error-Term Roulette and the Sato-Tate Conjecture," on page 19 for another mathematical problem where the concept of modularity is central.)

This algebraic definition of the modular group hides to some extent its true significance, which is that it is the symmetry group of 2-dimensional lattices. You can think of a lattice as an infinitely large wire mesh or screen. The basic screen material that you buy at a hardware store has holes, or unit cells, that are squares. (See Figure 1a, next page.) However, you can create lattices with other shapes by stretching or shearing the material uniformly, so that the unit cells are no longer square. They will become parallelograms, whose sides are two vectors pointing in different directions (traditionally denoted ω_1 and ω_2). The points where the wires intersect form a polka-dot pattern that extends out to infinity. The pattern is given by all linear combinations of the two "basis vectors," $a\omega_1 + b\omega_2$, such that both a and b are integers.

Unlike hardware-store customers, mathematicians consider two lattices to be the same if they form the same pattern of intersection points. (The wires, in other words, are irrelevant, except to the extent that they define where the crossing points are.) This will happen whenever a lattice has basis vectors ω_1' and ω_2' that are linear combinations of ω_1 and ω_2 (i.e., $\omega_1' =$

$a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$, for some integers a, b, c, and d) and vice versa. These conditions hold precisely when the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in the modular group. Figures 1b and 1c show two different bases for a hexagonal lattice. The matrix that transforms one basis into the other would be a member of the modular group.



Figures 1a–1c. *(a) A hardware-store lattice and its basis vectors. (b) A triangular lattice and its basis vectors. (c) The same lattice can be generated by two different vectors. (Figures courtesy of Jos Leys from the online article, "Lorenz and Modular Flows, a Visual Introduction.")*

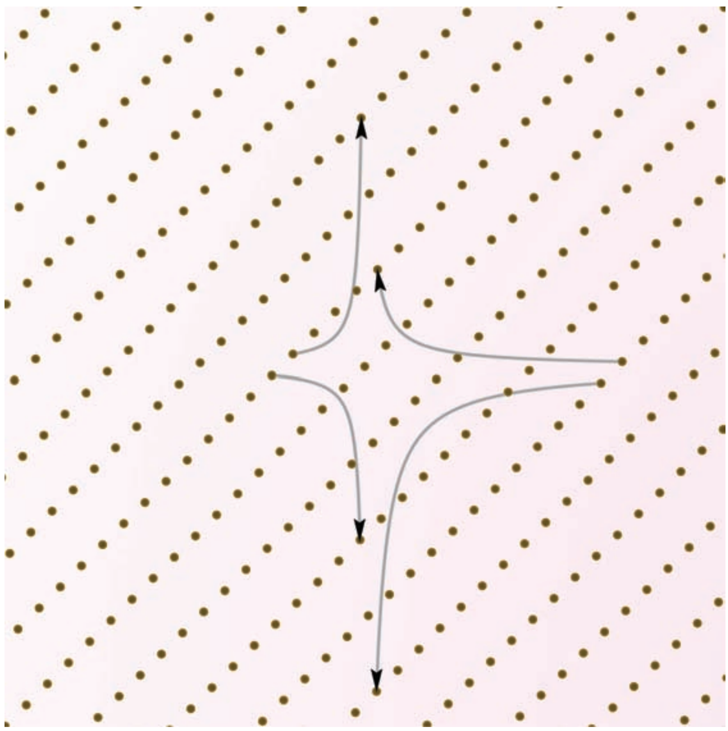


Figure 1d. *The modular flow gradually deforms the shape of a lattice, but brings it back after a finite time to the same lattice with different basis vectors. This figure illustrates the trajectories of four points in the lattice. Every point in the lattice moves simultaneously, and every point arrives at its "destination" in the lattice at the same time. Only the center point does not move at all. (Figure courtesy of Jos Leys from the online article, "Lorenz and Modular Flows, a Visual Introduction.")*

The matrix transformation distorts the underlying geometry of the plane, yet maps the lattice to itself. It accomplishes this transformation in one step. But there is also a way to produce the same effect gradually, by means of a smooth deformation. Imagine drawing a family of hyperbolas, with one hyperbola linking ω_1 to ω_1' and another linking ω_2 to ω_2'. (See Figure 1d.) Remarkably, it is possible to extend this *modular flow* to the entire plane, in such a way that all of the polka dots on the lattice flow along hyperbolas to different polka dots, no polka dots are left out, and the direction of motion of every polka dot at the beginning matches the direction of the new polka dot that comes to replace it at the end of the flow.

There is another way of visualizing the modular flow that emphasizes the special nature of matrices with integer entries. This method involves constructing an abstract "space of all lattices" (which turns out to be three-dimensional, and as described below, can be easily drawn by a computer). The modular flow defines a set of trajectories in this space, in the same way that water flowing in a stream generates a set of streamlines. Most of the streamlines do not form closed loops. However, those that do close up are called *modular knots*, and they correspond explicitly to elements of the modular group. This point of view subtly shifts the emphasis from algebra (the modular group is interesting because its elements have integer entries) to geometry (the modular group is interesting because it produces closed trajectories of the modular flow).

How should we visualize the "space of all lattices"? This turns out to be a crucial question.

How should we visualize the "space of all lattices"? This turns out to be a crucial question. One traditional approach identifies a lattice with the *ratio* of its two basis vectors, $\tau = \omega_2/\omega_1$. In order for this definition to make sense, the basis vectors have to be considered as complex numbers (i.e., numbers with both a real and imaginary part). The ratio τ will be a complex number $x + iy$ whose imaginary part (y) can be assumed to be positive. Thus τ lies in the upper half of the xy-plane. In fact, it can be pinned down more precisely than that. As explained above, any given lattice can have many different pairs of basis vectors with different ratios τ, but it turns out that there is only one pair of basis vectors whose ratio τ lies in the shaded region of Figure 2 (next page). This "fundamental region" can therefore be thought of as representing the space of all lattices, with each lattice corresponding to one point in the region. For example, the screen you buy in the hardware store, with square holes, corresponds to the ratio $0 + 1i$ or the point $(0, 1)$ in the fundamental region.

However, there is some ambiguity at the edge of the fundamental region. The points τ on the left-hand boundary correspond to the same lattices as the points τ' on the right-hand side. This means that the two sides of the fundamental region should be "glued together" to form a surface that looks like an infinitely long tube with an oddly shaped cap on one end. This two-dimensional surface is called the *modular surface*.

As mentioned above, each lattice is represented by a single point on the modular surface. As the lattice deforms under the modular flow, its corresponding point travels along a circular arc in the upper half-plane. Because of the way the boundaries of the fundamental domain are glued together, each time the curve exits one side of the fundamental domain it re-emerges on the opposite side, and the result is a trajectory with multiple

Legend. F = Fundamental Domain

$r = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

Mapping	Description	Formula	Matrix
X	Rotation about *i* by π radians	$\frac{0z-1}{1z+0}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
Y	Rotation about *r* by 2π/3 radians	$\frac{0z-1}{1z+1}$	$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$
Z	Translation by 1 (to the right)	$\frac{z+1}{0z+1}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

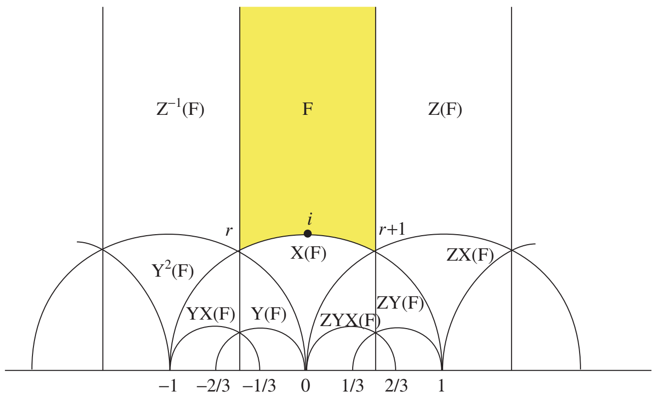


Figure 2. *Any lattice has a basis* ω_1, ω_2 *whose ratio lies in the fundamental domain F. Any change of basis corresponds to a linear fractional transformation (see column labeled "Formula") or to a matrix (see column labeled "Matrix"). The set of all such transformations is known as the* modular group*. The simple transformations X, Y and Z, listed here, generate the rest of the modular group. Note that the images X(F), Y(F), XY(F), ..., cover a half-plane. The* modular surface *is the quotient space of the half-plane by the modular group. It can be visualized as the region F with its sides glued together. (The bottom is also folded in half and glued together.)*

pieces, somewhat like the path of a billiard ball (see Figure 3). (In fact, the modular flow has sometimes been called *Artin's billiards*, after the German mathematician Emil Artin who studied it in the 1920s.) Most billiard trajectories do not close up, but a few of them do, and these are called *closed geodesics*. They are almost, but not quite, the same as modular knots; the difference is that they lie in a two-dimensional surface, but modular knots are defined in three-dimensional space.

The "missing" dimension arises because there are really four dimensions that describe any lattice: two dimensions of shape, one dimension of orientation, and one of mesh size. The modular surface ignores the last two dimensions. In other words, two lattices correspond to the *same point* in the modular surface if they have the same shape but different orientations. For everyday applications, it makes sense to consider such lattices to be equivalent. If you wanted a screen window made up of diamond shapes instead of square shapes, you wouldn't take your screen back to the store and exchange it; you would simply rotate the lattice 45 degrees.

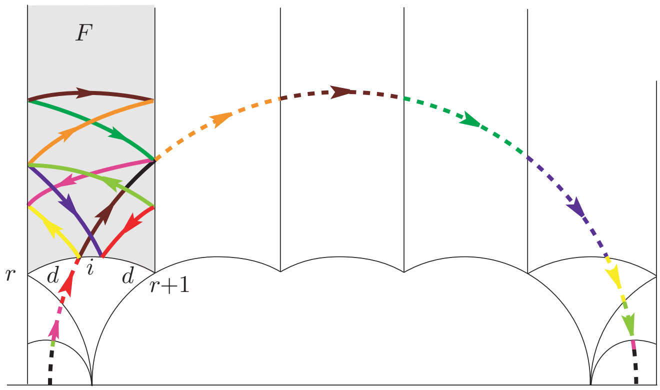


Figure 3. *When a lattice is deformed as by the modular flow, its corresponding basis vectors define a path through the half-plane illustrated in Figure 2. Each segment of this path (indicated here by different colors) may be mapped back to the fundamental region F by a transformation in the modular group.*

However, this common-sense reduction throws away some valuable information about the modular flow. If the orientation is not ignored, the "space of all lattices" becomes three-dimensional.[1] Amazingly, it is *simpler* to visualize this space than it is to visualize the modular surface. Using some elegant formulas from the theory of elliptic curves, Ghys showed that the space of all lattices is topologically equivalent to an ordinary three-dimensional block of wood, with a wormhole bored out of it in the shape of a trefoil knot. (See Figure 4, next page.) Modular knots, therefore, are simply curves in space that avoid passing through the forbidden zone, the trefoil-shaped wormhole in space.

Every matrix in the modular group corresponds to a modular knot, and simpler matrices tend to correspond to simpler knots. For instance, Ghys showed that the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ corresponds to the green loop shown in Figure 5a, page 9. This loop is unknotted, but it does form a nontrivial link with the forbidden trefoil knot. (That is, it cannot be pulled away from the forbidden zone without passing through it.) The matrix $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$ corresponds to a knot that winds around the forbidden zone several times, and is actually a trefoil knot itself. The matrix $\begin{bmatrix} 3997200840707 & 2471345320891 \\ 9088586606886 & 5619191248961 \end{bmatrix}$ corresponds to the white knot in Figure 5b, page 9, which is a bit of a mess. Thus two natural questions arise: What kinds of knots can arise as modular knots? And how many times do they wind around

[1]This space should properly be called the space of all *unimodular* lattices because the area of the unit cells is still ignored (or, equivalently, assumed to be equal to 1).

the forbidden zone? Remarkably, Ghys answered both of these questions. But the answer requires a detour into a completely different area of mathematics.

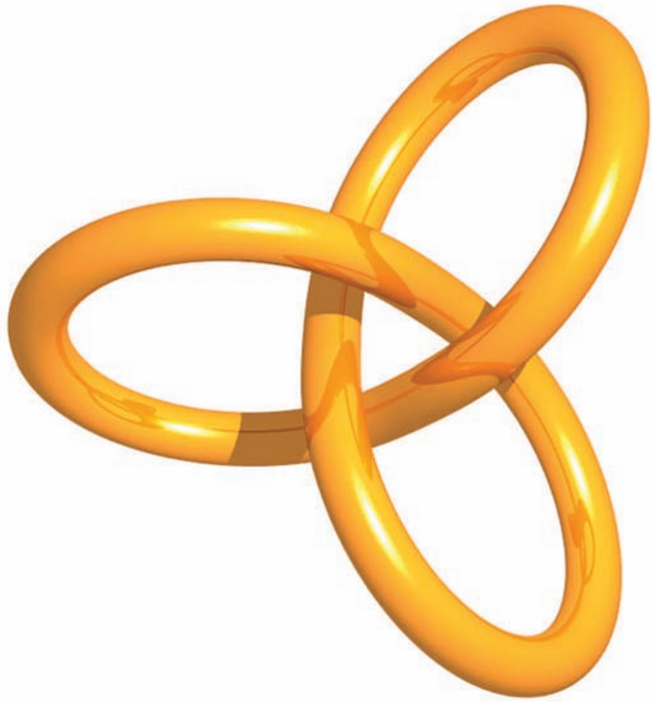


Figure 4. *Ghys realized that the conventional approach to defining the modular surface omits information about the orientation of a lattice. Therefore he defined a modular space, which is topologically equivalent to the exterior of a trefoil knot, as shown here. Closed geodesics, like the one in Figure 3, lift to* modular knots, *which wind around the trefoil but never intersect it. (Figure courtesy of Jos Leys from the online article, "Lorenz and Modular Flows, a Visual Introduction.")*

In 1963, a mathematician and meteorologist named Edward Lorenz was looking for a simple model of convection in the atmosphere, and came up with a set of differential equations that have become iconic in the field of dynamical systems. The equations are these: $\frac{dx}{dt} = 10(y - x)$, $\frac{dy}{dt} = 28x - y - xz$, $\frac{dz}{dt} = xy - \frac{8}{3}z$.

The specific meaning of the variables x, y, z is not too important. They are three linked variables, each a function of time (t), which in Lorenz's model represented the temperature and amount of convection at time t in a fictitious atmosphere. The equations describe how the atmosphere evolves over time. Because there are only three variables, unlike the millions of variables necessary to describe the real atmosphere, the solutions can be plotted easily as trajectories in three-dimensional space.

Lorenz noticed a phenomenon that is now known as deterministic chaos or the "butterfly effect." Even though the equations are completely deterministic—there is no randomness in this fictitious atmosphere—nevertheless it is impossible to forecast the weather forever. No matter what

Figure 5. *Two different modular knots and the corresponding elements of the modular group. (Figures courtesy of Jos Leys from the online article, "Lorenz and Modular Flows, a Visual Introduction.")*

starting point (x, y, z) you choose, even the slightest deviation from this initial condition corresponding to a slight experimental error in measuring the temperature or convection) will eventually lead to completely different weather conditions. The name "butterfly effect" refers to an often-cited analogy: the flapping of a butterfly's wings today in Borneo could lead to a typhoon next month in Japan.

A look at the trajectories of the Lorenz equation (Figure 6) explains why this is so. The trajectories concentrate around two broad, roughly circular tracks that, ironically, bear some resemblance to a pair of butterfly wings. You can think of one loop as predicting dry, cold weather and the other as predicting rainy, warm weather. Each time the trajectory circles one ring of the track (one "day"), it returns to the intricately interwoven region in the center, where it "decides" which way to go the next day. After a few dozen circuits, all information about the starting position is effectively lost, and the trajectory might as well be picking its direction at random.[2] Thus the full trajectory is, for all practical purposes, unknowable.

Figure 6. *The Lorenz attractor (yellow). One particular trajectory is shown in blue. It is a closed orbit of the Lorenz differential equations, or a* Lorenz knot. *(Figure courtesy of Jos Leys from the online article, "Lorenz and Modular Flows, a Visual Introduction.")*

However, suppose we aren't being practical. Suppose we can prescribe the initial position with unrealistic, infinite precision.

[2]Incidentally, Lorenz chose the coefficient 28, in the equation for dy/dt, to postpone the onset of chaos as long as possible. A trajectory starting near the origin $(0,0,0)$ will loop around one track of the Lorenz attractor 24 times before it finally switches to the other side. If the coefficient 28 was made either larger or smaller, the number of consecutive "sunny days" at the beginning of the trajectory would decrease.

Can we find a trajectory of the Lorenz equations that is the epitome of predictability—a closed loop? If so, the weather on day 1 would be repeated exactly on day 7, on day 13, and so on. Order, not chaos, would reign in the world.

This was the question that Bob Williams (now retired from the University of Texas) began asking in the late 1970s, together with Joan Birman of Columbia University. Over a thirty-year period, it has gradually become clear that closed trajectories do exist and that they form a variety of nontrivial knots. It is natural to call them *Lorenz knots.* For example, the trajectory shown in Figure 7a is topologically equivalent to a trefoil knot.

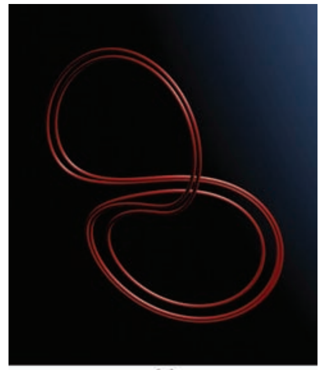
Even though perfectly periodic weather conditions can never be realized in practice, nevertheless they "give you a feeling of how tangled up this flow is," Birman says. It is amazing to discover that the simple pair of butterfly wings seen in Figure 6 actually contains infinitely many different Lorenz knots, all seamlessly interwoven without ever intersecting one another. (See Figure 7b.) However, there are also many knots that do *not* show up as trajectories of the Lorenz equations. For example, the second-simplest nontrivial knot, the figure-eight knot (Figure 7c) is *not* a Lorenz knot. (The simplest nontrivial knot, the trefoil knot, is a Lorenz knot because we have seen it in Figure 7a.)

In a paper written in 1982, Birman and Williams derived a host of criteria that a knot must satisfy in order to be a Lorenz knot. For example, they are *fibered* knots—an extremely unusual property that means that it is possible to fill out the rest of space with surfaces whose boundaries all lie on the knot. The figure on page 3, "The Fibered Knot," illustrates this difficult-to-visualize property. Using Birman and Williams' criteria, Ghys has showed that only eight of the 250 knots with ten or fewer "overpasses" or "underpasses" are Lorenz knots. In other words, even though infinitely many different Lorenz knots exist, they are rather uncommon in the universe of all knots.

Lorenz knots are extremely difficult to draw because the very nature of chaos conspires against any computer rendering program. Even the slightest roundoff error makes the knot fail to close up, and eventually it turns into a chaotic tangle. Thus, without theoretical results to back them up, we would not know that computer-generated pictures like Figure 7a represent actual closed trajectories.

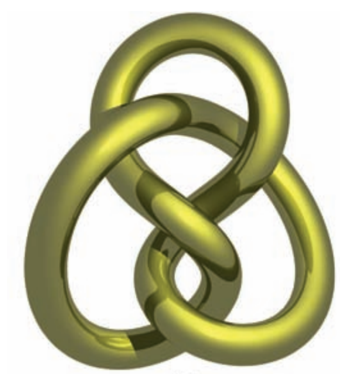
Birman and Williams originally proved their theorems about Lorenz knots under one assumption. Earlier, Williams had constructed a figure-eight-shaped surface, a *geometric template*, which (he believed) encoded all the dynamics of a Lorenz knot. In essence, he argued that the butterfly wings of Figure 6 are real (see also Figure 8, p. 12), and not just a trick of the eye. Although he and Clark Robinson of Northwestern University produced strong numerical evidence to support this belief, a proof remained elusive. In fact, this problem appeared on a list of leading "problems for the 21^{st} century" compiled by Stephen Smale in 1998.

As it turned out, Smale (and Birman and Williams) did not have long to wait. In 2002, Warwick Tucker of Uppsala University proved the conjecture by using interval arithmetic, a hybrid



(a)

(b)



(c)

Figure 7. *(a) This Lorenz knot is topologically equivalent to a trefoil knot. (b) Different Lorenz knots interlace with each other in a phenomenally complex way, never intersecting one another. (c) Not all topological knots are Lorenz knots. For example, no orbit of the Lorenz equations is topologically equivalent to a figure-eight knot, shown here. (Figures courtesy of Jos Leys from the online article, "Lorenz and Modular Flows, a Visual Introduction.")*

technique that combines computer calculations with rigorous proofs that the results are robust under roundoff error.

Even though the actual trajectories of the Lorenz flow do not lie on the template, Tucker's work guarantees that they can be mashed down onto the template without altering the topological type of the knot. In other words, none of the strands of the knot pass through each other or land on top of one another as a result of the mashing process. Thus, the topology of Lorenz knots can be studied simply by drawing curves on the template, which is a much easier job than solving Lorenz's equations.

Once the knots have been pressed onto the template, the shape of the knot is determined by the series of choices the trajectory makes as it passes through the central region. Each time it chooses to veer either left or right. The trefoil knot in Figure 7a would correspond to the string of decisions "right, left, right, left, right," or simply the string of letters RLRLR. (Different ways of reducing the three-dimensional dynamics to a one-dimensional "return map" had been noted by other mathematicians too, including John Guckenheimer of Cornell University.)

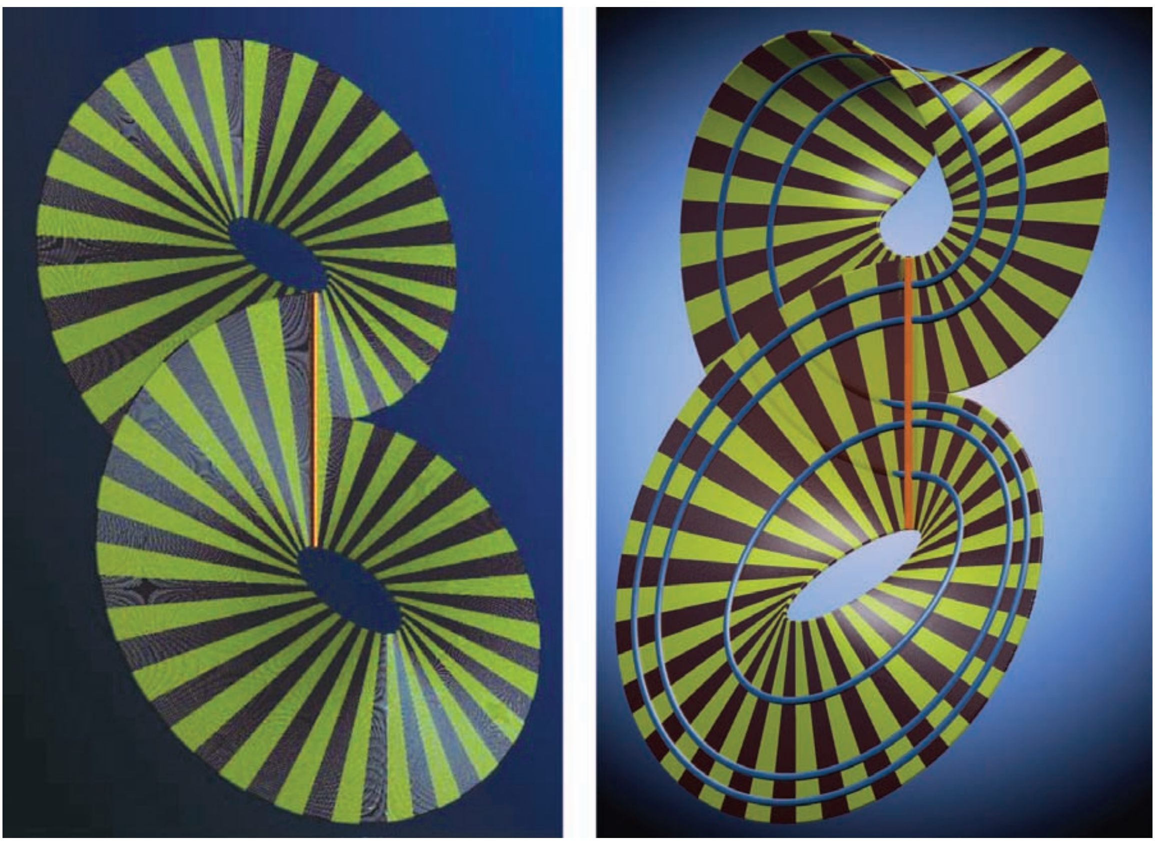


Figure 8. *(left) Orbits of the Lorenz equations can be classified by flattening them down onto a* template, *a sort of paper-and-scissors model of the Lorenz attractor. (right) A related set of equations, Ghrist's dynamical system, has a template that looks like the Lorenz template with an extra half-twist to one of the lobes. Amazingly, this slight modification is enough to guarantee that* every *knot appears as a closed orbit. (Figures courtesy of Jos Leys from the online article, "Lorenz and Modular Flows, a Visual Introduction.")*

The idea of templates has led to other surprising discoveries in dynamical systems. In a 1995 doctoral dissertation, Robert Ghrist, who is now at the University of Illinois, showed that some dynamical systems allow a much richer set of closed trajectories—in fact, they contain *every* closed knot. Not just every known knot, but every knot that will ever be discovered! Ghrist points out that his example can be made quite concrete. "Suppose I take a loop of wire," he says, "and bend it in the shape of a figure-eight knot [Figure 7c]. I run an electric current through it, and look at the induced magnetic field. Assuming it doesn't have any singularities, it will have closed field lines of all knot types." Once again, the proof involved the construction of a geometric template, which turned out to be very similar to the butterfly-shaped Lorenz template, except that one of the lobes is given an extra twist. (See Figure 8.)

Although Ghrist's result surprised Birman and Williams, who had conjectured that a "universal template" containing all knots was impossible, it confirmed the idea that some flows are more chaotic than others, and that studying closed trajectories is a good way of telling them apart. "With the wisdom of hindsight, the existence of a small number of knots in a flow is like the onset of chaos," Birman says. Lorenz's equations seem to define a relatively mild form of chaos, while Ghrist's equations seem to represent a full-blown case.

When Ghys started thinking about modular knots, he ironically made the opposite guess to Birman and Williams. He thought that the class of modular knots was probably universal—in other words, that every knot can be found somewhere in the modular flow. "The first time I thought of these questions, I wanted to understand modular knots, and I had no idea they were connected with the Lorenz equations," Ghys says. But then he made a remarkable discovery. There is a copy of the Lorenz template hidden within the modular flow! He originally made a schematic drawing that shows the template looking very much like a pair of spectacles straddling the trefoil-shaped "forbidden zone." Later, with the help of graphic artist Jos Leys of Belgium, he produced beautiful animations that show how any modular knot can be deformed onto the template. (See Figure 9, next page.) "A proof for me is not always fully formalized," he says. "I had it all clear in my head. I was sure it was true, I knew why it was true, and I was beginning to write it down. But for me, putting it in a picture was a confirmation that it was more true than I thought."

Whether by formal argument or a "proof by picture," the conclusion immediately follows that all modular knots are Lorenz knots. "A lot of people look at Étienne's result and see it as a result about how complicated the modular flow is, or the number theory,"says Ghrist. "I take a contrarian view. Étienne's work is showing that there is a certain parsimony in modular flow. Working under the constraints of complicated dynamics, it's got the simplest representation possible. Likewise, with the Lorenz equations being one of the first dynamical systems investigated, it's not surprising that they also have the simplest kind of chaotic dynamics."

Bob Williams. *(Photo courtesy of R. F. Williams.)*

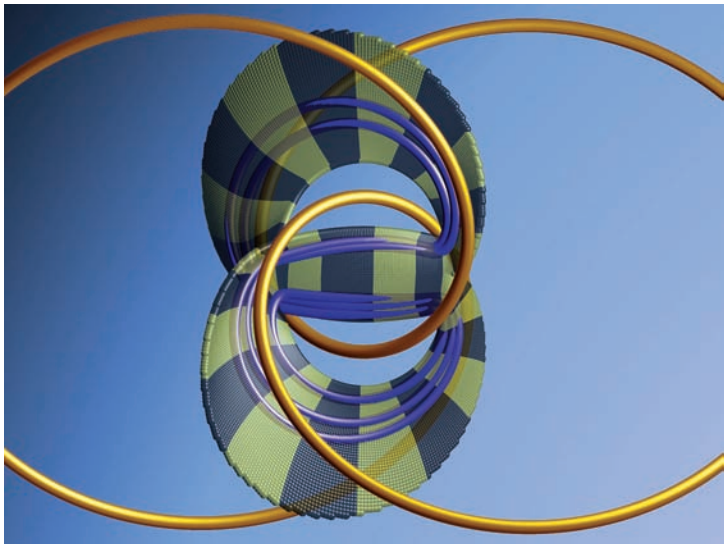


Figure 9. *The Lorenz template can be deformed in such a way that it straddles the "forbidden" trefoil from Figure 4. This insight is a key part of the proof that any Lorenz knot is a modular knot, and vice versa. (Figure courtesy of Jos Leys from the online article, "Lorenz and Modular Flows, a Visual Introduction.")*

The converse—that all Lorenz knots are modular—also holds true, once it is shown that the modular group allows all possible sequences of "left turns" and "right turns" within the template. In fact, Ghys found a direct connection between the sequence of turns and a previously known function in number theory. Ghys calls it the "Rademacher function," although he comments that so many mathematicians have discovered and rediscovered it that he is not sure whether to name it after "Arnold, Atiyah, Brooks, Dedekind, Dupont, Euler, Guichardet, Hirzebruch, Kashiwara, Leray, Lion, Maslov, Meyer, Rademacher, Souriau, Vergne, [or] Wigner"! The Rademacher function assigns to each matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in the modular group an integer. The classical, and not very intuitive way, of defining this integer goes as follows: First you sum two nested infinite series of complex numbers that are not integers. After computing the sums (called g_2 and g_3), next you compute $(g_2)^3 - 27(g_3)^2$ (the "Weierstrass discriminant"), and take its 24^{th} root (the "Dedekind eta function"). Finally, you take the complex logarithm of this function. It is well known that complex logarithms have an ambiguity that is an integer multiple of $2\pi i$. When you traverse the closed geodesic defined by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and come back to the starting point, the logarithm of the Dedekind eta function will not necessarily come back to its original value. It will change by $2\pi i$ times an integer—and that integer is the Rademacher function of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

A variety of other ways to compute this function were known, but none of them can be said to be really easy. Ghys' work gives it a new topological interpretation that does not require such an elaborate definition and that makes its meaning completely clear. As explained above, the idea is to use modular knots in three-space instead of closed geodesics in the modular surface. Every matrix in the modular group defines a modular knot. From Ghys's work, it follows that you can press this knot down onto the Lorenz template. Then the Rademacher function is simply the number of left turns minus the number of right turns! It's hard to imagine a more elegant or a more concrete description.

Ghys's result opens up new possibilities both for number theory and for dynamical systems. One reason mathematicians care so much about two-dimensional lattices is that they are the next step up from one-dimensional lattices. In one dimension, up to scaling, there is only one lattice: the set of integers, $\{\dots, -2, -1, 0, 1, 2, \dots\}$. The discipline of number theory (properties of the integers) is an exploration of its properties.

In fact, there is a precise analogy between number theory and the modular surface, which has yet to be fully understood. One of the most important functions in number theory is the Riemann zeta function $\zeta(s)$, which relates the distribution of prime numbers to the distribution of a mysterious set of non-integers, the points s_n where $\zeta(s_n) = 0$. These points s_n are known as zeroes of the zeta function.

The most famous open problem in number theory, the Riemann Hypothesis (see *What's Happening in the Mathematical Sciences*, Volumes 4 and 5), asks for a proof that the numbers s_n all lie on a single line in the complex plane. This kind of tight control over their distribution would imply a host of "best possible" results about the distribution of prime numbers.

One of the many pieces of evidence in favor of the Riemann Hypothesis is a very similar theorem for two-dimensional lattices, called the *Selberg trace formula*, that was proved in 1956 by Atle Selberg (a Norwegian mathematician who died recently, in 2007). It involves a *Selberg zeta function*, whose zeroes can be described as the energy levels of waves on the modular surface, and whose formula looks eerily similar to the formula for the Riemann zeta function. And what plays the role of prime numbers in that formula? The answer is: the lengths of closed geodesics in the modular surface. To make a long story short, the Selberg trace formula says that these lengths are dual to energy levels of waves on the modular surface, in exactly the same way that prime numbers are thought to be dual to zeroes of the Riemann zeta function. In fact, this analogy has led some mathematicians and physicists to suggest that the Riemann zeroes may also turn out to be energy levels of some yet undiscovered quantum-mechanical oscillator.

In any event, closed geodesics on the modular surface are clearly very relevant to number theory. And Ghys' result suggests that there is much more information to be obtained by going up a dimension and looking at modular knots. The Rademacher function is only the tip of the iceberg. It represents the simplest topological invariant of a modular knot, namely the "linking number," which describes how many times it wraps around the forbidden trefoil. Knot theory offers many

The Rademacher function is only the tip of the iceberg. It represents the simplest topological invariant of a modular knot, namely the "linking number," which describes how many times it wraps around the forbidden trefoil.

Ghys' theorem also implies that modular knots, because they are Lorenz knots, have the same properties that Lorenz knots do. For instance, modular knots are fibered—a fact that was not previously known.

more possible invariants for a modular knot. Could some of these also have analogues in number theory?

"I must say I have thought about many aspects of these closed geodesics, but it had never crossed my mind to ask what knots are produced," says Peter Sarnak, a number theorist at Princeton University. "By asking the question and by giving such nice answers, Ghys has opened a new direction of investigation which will be explored much further with good effect."

Ghys' theorem also implies that modular knots, because they are Lorenz knots, have the same properties that Lorenz knots do. For instance, modular knots are fibered—a fact that was not previously known. Ghys is currently looking for a more direct proof of this fact.

The double identity of modular and Lorenz knots also raises new questions for dynamical systems. For starters, modular knots are vastly easier to generate than Lorenz knots because the trajectories are parametrized by explicit functions. These trajectories are not literally solutions of the Lorenz flow, and yet somehow they capture an important part, perhaps all, of its dynamical properties. How faithfully does the modular flow really reflect the Lorenz flow? Tali Pinsky of Technion in Israel recently showed that there is a trefoil knot in space that forms a "forbidden zone" for the Lorenz flow, analogous to the forbidden zone for the modular flow. More generally, what kinds of dynamical systems have templates, and when is a template for a solution just as good as the solution itself? How can you tell whether a template is relatively restrictive, like Lorenz's, or allows for lots of different behaviors, like Ghrist's?

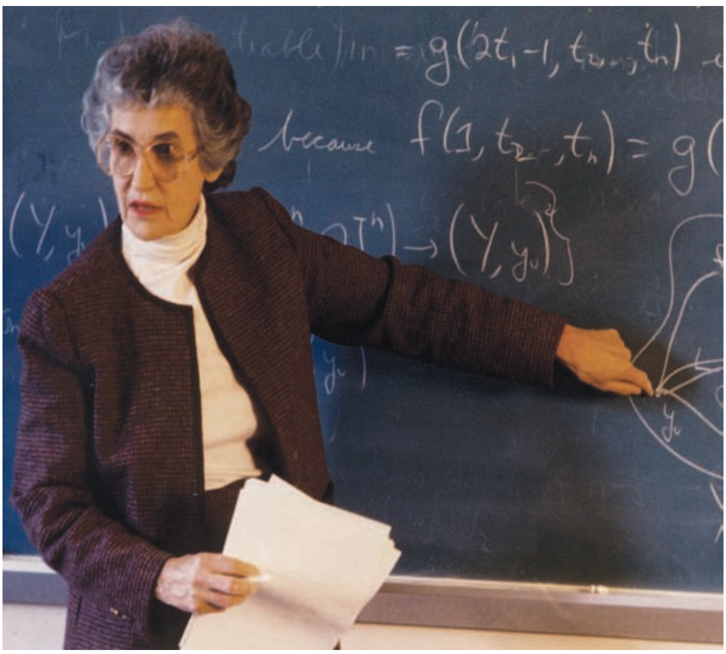
Ghys' theorem has already inspired Birman to take a fresh look at Lorenz knots. With Ilya Kofman of the College of Staten Island, she has recently come up with a complete topological description of them. It was already known that all torus knots—in other words, curves that can be drawn on the surface of a torus—are Lorenz knots. (For example, a trefoil knot can be drawn as a curve that goes around the torus three times in the "short" direction while going twice around in the "long" direction.) However, the converse is not true—many Lorenz knots are not torus knots.

Birman and Kofman have shown that Lorenz knots are nevertheless related to torus knots by a simple twisting procedure. The idea of twisting is to take several consecutive strands of the knot—any number you want—and pull them tight so that they lie parallel to each other. Cut all of the parallel pieces at the top and bottom, to get a skein. Now give the whole skein a twist by a rational multiple of 360°, so that the bottom ends once again lie directly below the top ends. Then sew them back up to the main knot exactly where they were cut off in the first step.

Any Lorenz knot, Birman and Kofman showed, can be obtained from a torus knot by repeated application of this twisting procedure.[3]

In fact, topologists were already aware that twisted torus knots have some special properties. In 1999, topologists Patrick Callahan, John Dean, and Jeff Weeks proved that their Jones polynomials (a knot invariant discovered in the 1980s by Vaughan Jones) are unusually simple. Another important topological invariant of knots, discovered by William Thurston in the 1970s, is the hyperbolic volume of their complement. Callahan, Dean and Weeks showed that twisted torus knots tend to have unusually small volumes. Their results, together with the work of Ghys and Birman, suggest that twisted torus knots arise naturally in problems outside topology because they are the simplest, most fundamental non-torus knots.

"What I'm most pleased about is that Ghys' work is reminding a new generation of mathematicians of what Joan Birman and Bob Williams did back in the 1980s," Ghrist says. "That was incredibly beautiful and visionary work that they did. I'm delighted to see someone of Ghys' stature and talent coming in, revisiting those ideas and finding new things."



Joan Birman. *(Photo courtesy of Joan S. Birman.)*

[3]There is one mild technical condition, which is that the twists all have to be in the same direction.