Robert Osserman received a Ph.D. from Harvard University in 1955, working in Riemann surface theory under the guidance of Lars Ahlfors. He then joined the mathematics department at Stanford University, where his research interests shifted towards differential geometry. In recent years he has made numerous research contributions to the theory of minimal surfaces and other topics in geometry. His elegant book A Survey of Minimal Surfaces *has surely contributed to the geometry renaissance he describes in his article.*

The Geometry Renaissance in America: 1938–1988

ROBERT OSSERMAN

Freeman Dyson, in his 1972 Gibbs lecture to the AMS: "Missed Opportunities," and in a 1981 lecture to the Humboldt Foundation: "Unfashionable Pursuits," urges us to look beyond the narrow confines of those subjects and pursuits that happen to be in fashion at a given time. Thinking back to my days as a graduate student I have no trouble in distinguishing what was fashionable from what was not, although at the time I would probably have been shocked to hear the word "fashion" used to describe what seemed to be simply the important and exciting areas of research. Certainly Bourbaki was the height of fashion. Also, anything "algebraic," whether topology, geometry, or analysis, had added panache. At the other end of the spectrum there were subjects such as partial differential equations and functions of one complex variable, that had been declared dead for so long that it would be hard even to find mourners.

And somewhere past them, beyond the pale, was differential geometry. It was simply not an option. At least in the case of partial differential equations and complex variables there were faculty members active in those areas, with whom one could write a dissertation. I doubt if there were any Ph.D. theses in differential geometry at Harvard for a period of some 20 years, from the late thirties to the late fifties. During my five years in residence I recall a course in the subject being offered just once. It was given by Ahlfors, and was an excellent course, although not designed to lead to research in the subject. Ahlfors' own work is infused with a deep geometric sense. However, he was not a geometer, in the sense of making contributions to the field; what he did do was to make brilliant use of differential geometry in various parts of function theory. A simple, but penetrating example, was his far-reaching generalization of Schwarz' Lemma. Pick had extracted the geometric content of Schwarz' Lemma by interpreting it as a statement about arc length in the Poincaré metric. Ahlfors showed that it applied much more widely to metrics with certain curvature constraints. At the same time, he de-geometricized the lemma to a certain extent by revealing its roots in partial differential equations. The generality of the method allowed the result to be extended to higher dimensions and to many different classes of mappings. Similarly, Ahlfors' various geometric approaches to Nevanlinna theory made possible the many subsequent higher-dimensional generalizations. But once again, Ahlfors was not primarily a geometer, and none of his Ph.D. students wrote a thesis on differential geometry.

In a broader context, the picture at other major research centers, such as Princeton, Chicago, and MIT was not very different from that at Harvard. At Princeton, Eisenhart had become Dean of the Graduate School in 1933. Between 1938 and his retirement in 1945, his only publications were two introductory textbooks. Veblen's interests were very broad, and included a period of concentration on differential geometry in the twenties during which J. H. C. Whitehead and T. Y. Thomas were his students. But Veblen left Princeton University for the Institute for Advanced Study in 1932, and retired in 1950 at age 70. There was no apparent move to replace either Eisenhart or Veblen with a young geometer. The one locus of geometric activity was Bochner. Like Ahlfors, he was primarily an analyst, but unlike Ahlfors, he had begun in the late forties to work actively in certain areas of geometry, to which he made significant contributions. His Ph.D. students included Rauch in 1947 and Calabi in 1950. However, it should be noted that Rauch's thesis was in analysis. His seminal work in differential geometry — the Rauch comparison theorem and its application to his theorem that a compact manifold whose curvature is close to that of a sphere must be homeomorphic to a sphere—stemmed from a postdoctoral year spent at Zürich, where Heinz Hopf apparently posed the question.

Like Eisenhart at Princeton, there was Struik at MIT and Lane in Chicago — geometers of an earlier generation who during the forties were not doing work destined to have a major impact on the future course of geometry.

Writing on "Fifty years of American Mathematics" for the AMS Semicentennial Publication, G.D. Birkhoff describes the situation in 1938 in these words:

> It must be admitted ... that there are few of our younger men who occupy themselves with algebraic or classical differential geometry, or any other of the geometric questions which seemed most vital fifty years ago.

There is no doubt that the figures who were active in the twenties and thirties and who were to affect most profoundly the future course of geometry were to be found in Europe. First and foremost (in retrospect) there was Élie Cartan in France. In Switzerland, Heinz Hopf, though primarily a topologist, was a powerful force in geometry. Germany maintained a strong geometric tradition, led at the time by Blaschke. There was simply nothing comparable in the United States.

When I say "nothing comparable," I am referring to mathematicians of the stature of Blaschke and Cartan who would classify themselves (and be classified by others) as differential geometers. If one broadens the scope just a bit, one would clearly want to include such major figures as Marston Morse, with his calculus of variations in the large, Hassler Whitney, the inventor of differentiable manifolds and sphere bundles, and Jesse Douglas, winner of one of the first two Fields Medals in 1936, whose solution of Plateau's problem had a geometric component, although it would have to be viewed primarily as analysis. In Morse's case, his application of "Morse theory" to the study of geodesics on a Riemannian manifold would certainly count as a major contribution to differential geometry.

At the Institute for Advanced Study there was also Hermann Weyl. In a curious footnote to the history of geometry, he provided a crucial link between the classical and the general Gauss–Bonnet theorems, after he attended a lecture in 1938 at the Princeton Mathematics Club by the statistician Harold Hotelling. In order to analyze a certain statistical problem, Hotelling wanted formulas for the volumes of tube domains around submanifolds of Euclidean space or of a sphere. Weyl's treatment of the problem led directly to the papers by Allendoerfer and Chern, as well as to later work on submanifolds and integral geometry.

When we look for those core geometers in America whose theorems we still quote, we find them scattered about and isolated. Carl Allendoerfer was at Haverford College and Sumner Myers at the University of Michigan. There was also J. L. Synge, originally at the University of Toronto and in the mid-forties at Ohio State and Carnegie Tech. His well-known geometric results date from the twenties. He later turned to more applied areas, including a period during World War II, when, like Allendoerfer and a number of other mathematicians, he deferred pursuing his own research in order to participate directly in the war effort. Another rather idiosyncratic figure was Herbert Busemann, who was at the Illinois Insitute of Technology and the University of Southern California during the forties, working out his beautiful theories of geometry in the nondifferentiable setting.

If we wish to look beyond the general Bourbaki trend to explain the unfashionableness of differential geometry in the forties, we must attribute it at least in part to the *kind* of geometry that was current at the major institutions in the thirties. Graustein at Harvard, Eisenhart in Princeton, and Lane at Chicago were not proving the sort of theorems that were destined to be memorable and influential. Even more, the subject as a whole seemed to have run out of steam after a surge of fundamental work, including that of Ricci and Levi–Civita at the turn of the century. There was the much-lamented "debauch of indices," covering or substituting for geometric content. Furthermore, there was a sense of isolation from the rest of mathematics.

The turnaround can be traced to a series of developments that served first to renew some of geometry's ties with other fields, and then gradually to move geometry more and more toward center stage.

First, and perhaps most important, came the development of a global theory, relating geometry to topology. The work of Allendoerfer, Myers, and Synge already mentioned was almost all in that direction as was that of Hopf, Cohn–Vossen, Preissmann, and much of Blaschke's. One culmination was Chern's work on the general Gauss–Bonnet theorem and on characteristic classes. Still other geometry/topology links arose out of Bochner's vanishing theorems in the late forties, which in turn had a major impact on algebraic geometry through Kodaira. Then in the fifties came Rauch's comparison theorem and all the results flowing out of that, in particular the sphere theorems of Berger and Klingenberg. Both topological and algebraic components were present in work on Lie groups and their quotients — homogeneous and symmetric spaces — much of which grew out of Cartan's fundamental work. Bott and Samelson presented Marston Morse in 1958 with a singularly appropriate 65th birthday present consisting of a beautiful application of Morse theory to the study of symmetric spaces. Like Bott and Samelson, Milnor would be viewed as primarily a topologist, but with significant geometric interests.

The links between geometry and topology were the central focus of two series of lectures by Heinz Hopf during visits to the United States: the first at New York University in 1946 and the second at Stanford University in 1956. Both were written up informally as lecture notes. The Stanford notes, devoted to the global theory of surfaces, went through several "editions" and were circulated for years as an underground classic, providing for many their introduction to differential geometry in the large. In 1983, they were finally published officially, along with the NYU notes, when Springer–Verlag was looking for something special to appear as Volume 1000 in their series, "Lecture Notes in Mathematics."

In another direction, there were the connections with partial differential equations as exemplified by the work of Philip Hartman and Louis Nirenberg.

But for a single most decisive factor contributing to the rebirth of geometry in America, I would propose the move of Chern to the United States from China in the late forties.

Shiing-shen Chern began his studies in China. He went to Hamburg to work with Blaschke from 1934 to 1936, receiving his doctorate in 1936. He then spent a year in Paris with Élie Cartan before returning to China in 1937. He remained there until the end of 1948, except for two years, 1943–1945, at the Institute for Advanced Study in Princeton. During those two years he did some of his most important work, including his intrinsic proof of the Gauss–Bonnet theorem, and his fundamental paper on characteristic classes, referred to earlier.

In 1949, Chern joined the mathematics department at the University of Chicago. There ensued the first mass production of high-caliber geometry Ph.D.s in United States history, starting with Nomizu in 1953. But Chern's influence was far wider than that. It spread to MIT via Singer, who was a student at Chicago, not at the time a geometer. Singer attended Chern's lectures and then developed his own course at MIT. There he made several converts, including Warren Ambrose and Barrett O'Neill, neither of whom started out in geometry. It was through Chern that André Weil was led to his contributions to the theory of characteristic classes. During the fifties, Chern wrote joint papers with Spanier, Kuiper, Hartman, Wintner, Lashof, Hirzebruch, and Serre. These collaborations accelerated the movement I referred to earlier in which differential geometry was gradually integrated with surrounding areas of mathematics: algebraic topology, algebraic geometry, and partial differential equations.

The same period witnessed one of the more anomalous and yet not insignificant features of the transformation and revitalization of geometry: the notation wars. There was fairly uniform disaffection for the classical coordinate-based notation for vectors, tensors, and forms. It worked well enough for surfaces, where there were fewer indices to manipulate, and where the existence of special types of coordinates allowed significant simplifications. But in higher dimensions the notation could itself be a deterrent to approaching a geometric problem. When Cartan introduced his method of moving frames, it was seen by many as the perfect tool. It was wholeheartedly adopted and promoted by Chern, who took full advantage of its flexibility and advantages, for example in moving up and down from a manifold to a frame bundle or tangent bundle, with perhaps a slight sleight of hand in employing the same "ω_{ij}" in the two contexts.

But the Cartan notation had its own drawbacks. It by no means eliminated the indices. It still involved making arbitrary choices of local frame fields, forming expressions based on those choices, and then checking behavior under changes of frame field. Finally, one of its strengths — its relative ease of manipulation in computation involving covariant or exterior derivatives — had a negative side in a more distant relationship to the underlying geometry. The much-maligned old-fashioned terminology of gradient, curl, and divergence were thought by many to have been rendered obsolete, as they could all be subsumed as special cases of the exterior derivative acting on forms of varying degrees. However, they had (and still have) the advantage of direct geometric meaning and illuminating physical interpretations. Somewhat in this spirit, a new notation was invented by Koszul and apparently first introduced in print by Nomizu in the mid-fifties. The fundamental operation is the covariant derivative of a vector field with respect to a given tangent vector at a point. The notation is totally independent of local coordinates or frame fields and is free of indices. It was quickly adopted by a large number of geometers. However, it too had its drawbacks, including considerable awkwardness for certain types of computations. Thus, unlike most earlier battles over notation, such as the one in calculus where Leibniz' notation won an almost total victory over Newton's, the outcome here has been a standoff. Just as one formerly had to learn two modern languages and one dead language, an aspiring geometer needs now to learn two modern systems of notation (Cartan and Koszul) and one dead one (coordinates) to be able to read 20th century papers and books. It may be worth adding that it is not always "just" a question of notation, since even the content of a theorem may be affected: proving the existence of a certain kind of frame field is not equivalent to proving the existence of certain local coordinates.

Before leaving the fifties I should mention one more somewhat isolated, but important result, which had resonances far beyond its immediate consequences. That was John Nash's embedding theorem. For the first time one knew that the class of Riemannian manifolds coincided with the class of submanifolds of Euclidean space with the induced metric. The proof was a *tour de force* of original ideas, seemingly coming out of nowhere.

The sixties began with Chern's move from Chicago to Berkeley, which gradually became a central focus of geometric activity. The decade ended with the emergence of a new generation of first-rate geometers, including Alan Weinstein from Berkeley, Jeff Cheeger from Princeton, and Blaine Lawson from Stanford. In the interim, a powerful force adding to the momentum of the subject was the appearance of a new generation of books offering modern presentations and viewpoints, and making use of one of the newer notations. They almost immediately supplanted the older classics of Eisenhart vintage. Among them were the hardcover texts by Helgason, Kobayashi and Nomizu, Bishop and Crittenden, and Sternberg, as well as the no-less-significant softcover lecture notes by Hicks, Berger, Gromoll–Klingenberg–Meyer, and Milnor's notes on Morse Theory. The decade of feverish bookwriting came to a fitting end at 3:30 a.m. on July 6, 1970, when Michael Spivak finished the preface to Volume II of his *Comprehensive Introduction to Differential Geometry.* In that remarkable book, Spivak takes the reader step-by-step from the origins of differential geometry in the 18th century, through the fundamental papers of Gauss and Riemann, the contributions of Bianchi and Ricci, and into the thicket of the various concepts of a "connection," as seen from the points of view of Levi–Civita, Cartan, Ehresmann, Koszul, and others. In the process, he gives a constructive proof of the invariance of differential geometry under changes of notation by taking one theorem: *a Riemannian manifold with vanishing Riemann curvature tensor is locally isometric to euclidean space,* (referred to as "The Test Case") and providing seven different proofs, from each of seven different viewpoints or notations.

There is no doubt that the spate of new books helped make differential geometry more accessible and interesting to students. But even more important for the health and growth of the subject were the spectacular successes on the research front. The most celebrated was the Atiyah–Singer index theorem — a grand synthesis of analysis, topology, and geometry leading, in particular, to a new way of viewing the Gauss–Bonnet theorem: not as an isolated result, but as one instance of a larger scheme of things.

Other papers were less noted for specific results than for their seminal nature, in some cases laying the foundation for whole new areas of study. Among them are:

1. The 1960 paper on normal and integral currents by Federer and Fleming, which led to the creation of geometric measure theory and to the definitive book on the subject by Federer in 1969.
2. Eells and Sampson's 1964 paper on harmonic mappings of Riemannian manifolds. Although the notion of a harmonic map was not new, this paper was the starting point for the whole future development of the subject.
3. Palais and Smale's 1964 paper on a generalized Morse Theory. This paper, together with others around the same time by each of the authors and earlier ones by Eells, laid the foundation for the study of infinite dimensional manifolds. Lang's book on differentiable manifolds adopted a similar viewpoint and was also influential. The subject was called "global analysis," and found significant applications in the seventies.
4. Kobayashi's 1967 paper introducing an invariant pseudodistance on complex manifolds. Unlike the elaborate machinery used to set up the basis for geometric measure theory and for global analysis, the fundamental idea here is quite elementary and seems almost simple-minded, somewhat like the notion of cobordism. However, the implications have been profound, the latest being unsuspected connections

with Diophantine analysis, described in the 1987 book of Lang on complex hyperbolic spaces.

5. Also in 1967, McKean and Singer's paper on curvature and the eigenvalues of the Laplacian. It played a big role in the subsequent development of that subject.
6. Mostow's rigidity theorem of 1968. It was the first of a whole series in which "the topology determines the geometry." The proof uses and extends Gehring's basic results on higher-dimensional quasiconformal mappings.
7. Also in 1968, Simons' fundamental study of minimal varieties in Riemannian manifolds. This is the first serious account of the subject in full generality, and is chiefly responsible for moving the field of minimal surfaces from a somewhat marginal position to a more central one in differential geometry. There are a number of interesting results in the paper, but the most notable concerns Bernstein's Theorem: *if an n-dimensional minimal hypersurface S in* $\mathbb{R}^{n+1}$ *has a one-to-one projection onto a hyperplane, then S is itself a hyperplane.* Combining some of the results in his paper with earlier developments using geometric measure theory, Simons proved Bernstein's Theorem for dimensions $n \leq 7$. The following year, in 1969, Bombieri, de Giorgi, and Giusti finished the story in a startling fashion: Bernstein's Theorem is false for $n > 7$. Perhaps the only comparable example of a dimensionally-dependent discontinuity in behavior was the founding fact of differential topology: Milnor's exotic 7-sphere, discovered a decade before. There have been various attempts to link the two phenomena, but none have been totally convincing.

I need hardly add that there was a lot more notable work in differential geometry than the sample I have described here. Some old problems were being settled — such as Blaschke's conjecture, by Leon Green, in 1963, and the topology of positively curved complete manifolds by Gromoll and Meyer in 1969 — at the same time as new areas were opening up and a new range of questions being posed.

But it was in the seventies that the field of differential geometry came into full blossom. For the first time, there were whole groups of geometers, rather than one or two isolated individuals, at several universities, most notably Berkeley, SUNY at Stony Brook, and the University of Pennsylvania. It would be hard to even begin to describe the scope of new accomplishments. But it is worth noting that the decade started with Thurston and Yau both doing their graduate work at Berkeley. Thurston went on to Princeton where he inaugurated his monumental study of hyperbolic geometry, and Yau went to Stanford, where his accomplishments included the solution of the Calabi conjecture, a part of the Smith conjecture (together with Meeks, a student of

Lawson at Berkeley) and the positive mass conjecture in relativity (together with R. Schoen — a joint student of Yau and Leon Simon at Stanford).

By the 1980s, the mathematical world was finally ready to award its first Fields Medal ever in differential geometry. Not only one, but two: to both Thurston and Yau.

Another major boost for geometry in America during the seventies was the presence of Gromov at Stony Brook from 1974 to 1980. During that period he wrote his fundamental papers on almost flat manifolds and on bounds for topological types of a manifold with certain curvature and volume constraints. In addition, he did important work on a variety of topics including isoperimetric inequalities, smooth ergodic theory, and scalar curvature (with Lawson). In 1980, he shared with Yau the Veblen prize of the American Mathematical Society. The full title, incidentally, is the "Oswald Veblen Prize in Geometry." It was set up after Veblen's death in 1960. The first seven recipients were all in topology, and it was not till 1976, with Simons and Thurston, that work in geometry proper was deemed award-worthy.

One sign of the burgeoning health of geometry in the eighties can be seen in the growth of regular geometry conferences, such as the Pacific Northwest Geometry Seminar, held three times a year on the West Coast, and the annual Geometry Festival in the East, whose attendance has been growing exponentially. The 1988 Geometry Festival, held at Chapel Hill, North Carolina, had one striking feature: a large proportion of the talks dealt with the construction of specific examples. There was a general feeling, explicitly expressed by Gromoll, that in the past one had been fairly free in making conjectures based on very little concrete evidence, whereas now for the first time we were building a solid basis for our conjectures in the form of examples of manifolds with prescribed topology, curvature of one sort or another, and possibly other geometric constraints, such as diameter or volume bounds. Among the talks at Chapel Hill was one by Gang Tian about his work with Yau on obtaining complete Kähler–Einstein metrics with prescribed Ricci curvature for various noncompact manifolds, and another by Nicolaos Kapouleas presenting his construction of compact and complete surfaces of constant mean curvature in $\mathbb{R}^3$ with prescribed topology. In both of these cases, the proofs involved highly sophisticated uses of partial differential equations.

The eighties had already produced other renowned examples. In 1986, Wente produced his torus of constant mean curvature immersed in $\mathbb{R}^3$, thus answering a question posed by Heinz Hopf in 1951, whether any compact surfaces other than the sphere could be immersed in $\mathbb{R}^3$ with constant mean curvature. (Hopf had proved that starting with a sphere, any such immersion would have to have the standard sphere as its image, and A.D. Alexandrov had shown that a higher genus surface could not be *embedded* in $\mathbb{R}^3$ with constant mean curvature.)

Can we conclude that the pendulum of fashion has now come full swing, from the Bourbaki ideas of generality and structure in the fifties, to the concrete, specific and intuitive in the eighties? And to the degree that it may be true, what does it portend for the future? One of the big unknowns is the impact that computers in general, and computer graphics in particular, will have on the direction and accomplishment of geometric research. The most striking example to date has been the discovery of new families of complete embedded minimal surfaces, with computer graphics playing a significant role. The story (and the pictures) can be found in an article by David Hoffman in the 1987 *Mathematical Intelligencer.* More recently, Hoffman has collaborated with a group of polymer scientists in examining various periodic minimal surfaces and surfaces of constant mean curvature as models for certain interfaces recently revealed by electron microscope photographs. This work appears as the cover article of the August 18, 1988 issue of the journal *Nature.*

Thus, in time for the AMS centennial, differential geometry has recovered not only its links with other parts of mathematics, but its roots in physical reality. It has also entered the realms opened up by the new computer technology. There are now active groups using computer graphics at the University of Massachusetts at Amherst, the University of California at Santa Cruz, Brown University, and Princeton, as well as the new Geometry Supercomputer project, whose goal is to provide a number of mathematicians with high resolution systems, all linked to each other and to a supercomputer at the University of Minnesota. Whether the outcome will be a series of exciting and fundamental new developments, or just a flurry of special cases leading to a renewed cry for a Bourbaki-type clarification and cleansing remains to be seen. That judgement will no doubt be made by the time of the AMS sesquicentennial celebration in 2038.

Postscript. The "geometry" in the title may seem to promise more than the text delivers. I have in fact dealt only with one facet: differential geometry. I did not restrict the title, because I believe that other parts of geometry enjoyed a similar renaissance, but I leave it to others to fill in the details. Even in differential geometry, I do not feel I have done justice to the whole field, but have concentrated on what I know best. In order to compensate at least in part for my own limited knowledge and perspective, I have consulted with a number of people who have offered additional background, comments and suggestions. They are Garrett Birkhoff, Eugene Calabi, Jeff Cheeger, Irving Kaplansky, Blaine Lawson, Cathleen Morawetz, Barrett O'Neill, Halsey Royden, Hans Samelson, James Simons, Isadore Singer, and George Whitehead. My thanks to all of them.

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